Int Jr. of Mathematical Sciences & Applications
Vol. 11, No. 1, (January-June, 2021)

ISSN No: 2230-9888
www.journalshub.com

On Bicomplex Mobius Maps

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Abstract

In the present paper, we have introduced the concept of bicomplex mobius maps with its idempotent decomposition. We have also discussed the fixed points and Lipschitz condition for bicomplex mobius maps.

Keywords: Bicomplex Numbers, Complex Clifford Algebras, Bicomplex Meromorphic Functions, Bicomplex Riemann Sphere, Bicomplex mobius maps,fixed points, Lipschitz conditions.

1 Preliminaries

As in [13] (see also [5] and [6]), the algebra of bicomplex numbers

$$\mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\} \tag{1.1}$$

is the space isomorphic to $\mathbb{R}^4$ via the map

$$z_1 + z_2\mathbf{i_2} = x_0 + x_1\mathbf{i_1} + x_2\mathbf{i_2} + x_3\mathbf{j} \to (x_0, x_1, x_2, x_3) \in \mathbb{R}^4,$$

and the multiplication is defined using the following rules:

$$\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1, \qquad \mathbf{i_1}\mathbf{i_2} = \mathbf{i_2}\mathbf{i_1} = \mathbf{j} \quad \text{so that} \quad \mathbf{j}^2 = 1.$$

Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$. Hence, it is easy to see that the multiplication of two bicomplex numbers is commutative. In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are **unique** among the **complex Clifford algebras** (see [4, 8] and [15]) in that they are commutative but not division algebra. Also, since the map $z_1 + z_2\mathbf{i_2} \to (z_1, z_2)$ gives a natural isomorphism between the $\mathbb{C}$-vector spaces $\mathbb{T}$ and $\mathbb{C}^2$, we have $\mathbb{T} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. That is, we can view the algebra $\mathbb{T}$ as the complexified $\mathbb{C}(\mathbf{i_1})$ exactly the way $\mathbb{C}$ is complexified $\mathbb{R}$. In particular, in the equation (1.1), if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers (see, e.g. [13], [18]):

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

The two projection maps $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{T} \longrightarrow \mathbb{C}(\mathbf{i_1})$ defined by

$$\mathcal{P}_1(z_1 + z_2\mathbf{i_2}) = z_1 - z_2\mathbf{i_1} \qquad \text{and} \qquad \mathcal{P}_2(z_1 + z_2\mathbf{i_2}) = z_1 + z_2\mathbf{i_1}, \tag{1.2}$$

are used extensively in the sequel.

The complex (square) norm $CN(w)$ of the bicomplex number w is the complex number $z_1{}^2 + z_2{}^2$; writing $w^* = z_1 - z_2\mathbf{i_2}$, we see that $CN(w) = ww^*$. Then a bicomplex number $w = z_1 + z_2\mathbf{i_2}$ is invertible if and only if $CN(w) \neq 0$. Precisely,

$$w^{-1} = \frac{w^*}{CN(w)}. \tag{1.3}$$

The set of units in the algebra $\mathbb{T}$ forms a multiplicative group which we shall denote by $\mathbb{T}_*$ (see [2]). Unlike the algebra $\mathbb{C}$, the bicomplex algebra $\mathbb{T}$ has zero divisors given by

$$\mathcal{NC} = \{w \in \mathbb{T} : CN(w) = 0\} = \{z(1 \pm \mathbf{j}) |\ z \in \mathbb{C}(\mathbf{i_1})\}, \tag{1.4}$$

which we may call the *null-cone*. Note that, using orthogonal idempotents

$$\mathbf{e_1} = \frac{1+\mathbf{j}}{2}, \qquad \mathbf{e_2} = \frac{1-\mathbf{j}}{2}, \ \text{ in } \mathcal{NC},$$

each bicomplex number $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ can be expressed uniquely as

$$w = \mathcal{P}_1(w)\mathbf{e_1} + \mathcal{P}_2(w)\mathbf{e_2}, \tag{1.5}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are projection maps defined in (1.2). This representation of $\mathbb{T}$ as $\mathbb{C} \oplus \mathbb{C}$ helps to do addition, multiplication and division term-by-term. With this representation we can directly express $|w|_j$ as

$$|w|_j := |\mathcal{P}_1(w)|\mathbf{e_1} + |\mathcal{P}_2(w)|\mathbf{e_2}$$

and will be referred to as the **j-*modulus*** of $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ (see [13]).

Definition 1 *Let X_1 and X_2 be subsets of $\mathbb{C}(\mathbf{i_1})$. Then the following set*

$$X_1 \times_e X_2 := \{w = z_1 + z_2\mathbf{i_2} \in \mathbb{T} \ : \ \mathcal{P}_1(w) \in X_1 \ \ and \ \ \mathcal{P}_2(w) \in X_2\}$$

is called a $\mathbb{T}$***-cartesian set*** *determined by X_1 and X_2, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are projections as defined in Eqn.*(1.2).

It is easy to see that if X_1 and X_2 are domains (open and connected) of $\mathbb{C}(\mathbf{i_1})$ then $X_1 \times_e X_2$ is also a domain of $\mathbb{T}$. We define the "discus" with center $a = a_1 + a_2\mathbf{i_2}$ of radius r_1 and r_2 of $\mathbb{T}$ as follows [9]:

$$\begin{aligned} D(a; r_1, r_2) &= B^1(a_1 - a_2\mathbf{i_1}, r_1) \times_e B^1(a_1 + a_2\mathbf{i_1}, r_2) \\ &= \{w_1\mathbf{e_1} + w_2\mathbf{e_2} \ : \ |w_1 - (a_1 - a_2\mathbf{i_1})| < r_1, |w_2 - (a_1 + a_2\mathbf{i_1})| < r_2\}, \end{aligned}$$

where $B^n(z, r)$ is an open ball with center $z \in \mathbb{C}^n(\mathbf{i_1})$ and radius $r > 0$. In the particular case where $r = r_1 = r_2$, $D(a; r, r)$ will be called the $\mathbb{T}$-disc with center a and radius r. In particular, we define

$$\overline{D}(a; r_1, r_2) := \overline{B^1(a_1 - a_2\mathbf{i_1}, r_1)} \times_e \overline{B^1(a_1 + a_2\mathbf{i_1}, r_2)} \subset \overline{D(a; r_1, r_2)}.$$

We remark that $D(0; r, r)$ is, in fact, the **Lie Ball** (see [1]) of radius r in $\mathbb{T}$.

Further, the projections as defined in *Eqn.*(1.2), help to understand bicomplex holomorphic functions in terms of the following Ringleb's Decomposition Lemma [10].

Theorem 1 *Let $\Omega \subset \mathbb{T}$ be an open set. A function $f : \Omega \longrightarrow \mathbb{T}$ is $\mathbb{T}$-holomorphic on Ω if and only if the two natural functions $f_{e1} : \mathcal{P}_1(\Omega) \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : \mathcal{P}_2(\Omega) \longrightarrow \mathbb{C}(\mathbf{i_1})$ are holomorphic, and*

$$f(w) = f_{e1}(\mathcal{P}_1(w))\mathbf{e_1} + f_{e2}(\mathcal{P}_2(w))\mathbf{e_2}, \ \forall \ w = z_1 + z_2\mathbf{i_2} \in \Omega,$$

The Ringleb's Lemma for bicomplex meromorphic functions is as follows [5].

Theorem 2 *Let $\Omega \subset \mathbb{T}$ be an open set. A function $f : \Omega \longrightarrow \mathbb{T}$ is bicomplex meromorphic on Ω if and only if the two natural functions $f_{e1} : \mathcal{P}_1(\Omega) \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : \mathcal{P}_2(\Omega) \longrightarrow \mathbb{C}(\mathbf{i_1})$ are meromorphic, and*

$$f(w) = f_{e1}(\mathcal{P}_1(w))\mathbf{e_1} + f_{e2}(\mathcal{P}_2(w))\mathbf{e_2}, \ \forall \ w = z_1 + z_2\mathbf{i_2} \in \Omega.$$

Definition 2 *Let $f : \Omega \longrightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set $\Omega \subset \mathbb{T}$, and let $f_{e1} : \mathcal{P}_1(\Omega) \longrightarrow \mathbb{C}(\mathbf{i_1})$ and $f_{e2} : \mathcal{P}_2(\Omega) \longrightarrow \mathbb{C}(\mathbf{i_1})$ be the natural maps. Then we say that $w = \mathcal{P}_1(w)\mathbf{e_1} + \mathcal{P}_2(w)\mathbf{e_2} \in \Omega$ is a (strong)* ***pole*** *for the bicomplex meromorphic function*

$$f(w) = f_{e1}\mathcal{P}_1(w)\mathbf{e_1} + f_{e1}\mathcal{P}_2(w)\mathbf{e_2}$$

if $\mathcal{P}_1(w)$ (and) or $\mathcal{P}_2(w)$ is a pole for f_{e1} or f_{e2}, respectively.

Remark 1 *Poles of bicomplex meromorphic functions are not isolated singularities.*

It is also easy to obtain the following characterization of poles.

Proposition 1 *Let $f : X \longrightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set $\Omega \subset \mathbb{T}$. If $w_0 \in \Omega$ then w_0 is a pole of f if and only if*

$$\lim_{w \to w_0} |f(w)| = \infty.$$

2 The Extended Bicomplex Plane $\overline{\mathbb{T}}$

Since the range of bicomplex meromorphic function lies beyond the bicomplex plane, we need the **extended bicomplex plane** to study the bicomplex meromorphic functions. Further, it would help to study the limit points of unbounded sets in bicomplex plane. We obtain this extended bicomplex plane by using extended $\mathbb{C}(\mathbf{i_1})$-plane.

For, we may consider the set

$$\begin{aligned}
\overline{\mathbb{C}(\mathbf{i_1})} \times_e \overline{\mathbb{C}(\mathbf{i_1})} &= (\mathbb{C}(\mathbf{i_1}) \cup \{\infty\}) \times_e (\mathbb{C}(\mathbf{i_1}) \cup \{\infty\}) \\
&= (\mathbb{C}(\mathbf{i_1}) \times_e \mathbb{C}(\mathbf{i_1})) \cup (\mathbb{C}(\mathbf{i_1}) \times_e \{\infty\}) \cup (\{\infty\} \times_e \mathbb{C}(\mathbf{i_1})) \cup \{\infty\} \\
&= \mathbb{T} \cup I_\infty,
\end{aligned}$$

writing I_∞ for the set $(\mathbb{C}(\mathbf{i_1}) \times_e \{\infty\}) \cup (\{\infty\} \times_e \mathbb{C}(\mathbf{i_1})) \cup \{\infty\}$. Clearly, any unbounded sequence in $\mathbb{T}$ will have a limit point in I_∞.

Definition 3 *The set $\overline{\mathbb{T}} = \overline{\mathbb{C}(\mathbf{i_1})} \times_e \overline{\mathbb{C}(\mathbf{i_1})}$ is called the **extended bicomplex plane**. That is,*

$$\overline{\mathbb{T}} = \mathbb{T} \cup I_\infty, \quad \text{with} \quad I_\infty = \{w \in \overline{\mathbb{T}} : \|w\| = \infty\}.$$

It is of significant importance to observe that formation of the extended bicomplex plane $\overline{\mathbb{T}}$ requires us to add an infinity set viz. I_∞, which we may call the **bicomplex infinity set**.

We need some definitions in order to give a characterization of this set.

Definition 4 *An element $w \in I_\infty$ is said to be a $\mathcal{P}_1$-infinity ($\mathcal{P}_2$-infinity) element if $\mathcal{P}_1(w) = \infty$ ($\mathcal{P}_2(w) = \infty$) and $\mathcal{P}_2(w) \neq \infty$ ($\mathcal{P}_1(w) \neq \infty$).*

Definition 5 *The set of all $\mathcal{P}_1$-infinity elements is called I_1**-infinity set**. It is denoted by $I_{1,\infty}$. Therefore,*

$$I_{1,\infty} = \{w \in \overline{\mathbb{T}} : \mathcal{P}_1(w) = \infty,\ \mathcal{P}_2(w) \neq \infty\}.$$

*Similarly we can define the I_2**-infinity set** as:*

$$I_{2,\infty} = \{w \in \overline{\mathbb{T}} : \mathcal{P}_1(w) \neq \infty,\ \mathcal{P}_2(w) = \infty\}.$$

We now construct the following two new sets:

$$I_\infty^- = I_{1,\infty} \cup I_{2,\infty}, \qquad I_0^- = I_{1,0} \cup I_{2,0}, \tag{2.1}$$

so that $I_\infty = I_\infty^- \cup \{\infty\}$ and $\mathcal{NC} = I_0^- \cup \{0\}$.

With these definitions, each element in the null-cone has an inverse in I_∞ and vice versa. One can easily check that the elements of the set I_∞^- do not satisfy all the properties as satisfied by the $\mathbb{C}(\mathbf{i_1})$-infinity but the element $\infty = \infty\mathbf{e_1} + \infty\mathbf{e_2}$ does. We may call the set I_∞^-, the **weak bicomplex infinity set** and the element $\infty = \infty\mathbf{e_1} + \infty\mathbf{e_2}$, the **strong infinity**. This nature of the set I_∞ generates the idea of weak and strong poles for bicomplex meromorphic functions (see [6]). Now, in order to work in the extended bicomplex plane, it is desirable to have a geometric model wherein the elements of $\overline{\mathbb{T}}$ have a concrete representative so as to treat the points of I_∞ as good as any other point of $\overline{\mathbb{T}}$. To obtain such a model, one can use the usual stereographic projections of $\overline{\mathbb{C}(\mathbf{i_1})}$ as two components in the idempotant decomposition to get a one-to-one and onto correspondence between the points of $S \times S$, where S is the unit sphere in $\mathbb{R}^3$, and $\overline{\mathbb{T}}$. Hence, we can visualize the extended bicomplex plane directly in $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. With this representation, we call $\overline{\mathbb{T}}$ the **bicomplex Riemann sphere**.

Observe that what is done above is basically a compactification of $\mathbb{C}^2$, using bicomplex setting. That is, suitable points at infinity are added to $\mathbb{T}$ to get the extended bicomplex plane $\overline{\mathbb{T}}$. In higher dimensions such compactifications are well known under the name, conformal compactifications. In fact, such compactifications are obtained as homogeneous spaces of Lie groups (see [2] and [3]).

2.1 The Chordal Metric on $\overline{\mathbb{T}}$

Proposition 2 *If $\chi : \overline{\mathbb{C}(\mathbf{i_1})} \times \overline{\mathbb{C}(\mathbf{i_1})} \longrightarrow \mathbb{R}$ be the chordal metric on $\overline{\mathbb{C}(\mathbf{i_1})}$. Then the mapping $\chi_e : \overline{\mathbb{T}} \times \overline{\mathbb{T}} \longrightarrow \mathbb{R}$ defined as:*

$$\chi_e\,(z, w) = \sqrt{\frac{\chi^2(\mathcal{P}_1(z), \mathcal{P}_1(w)) + \chi^2(\mathcal{P}_2(z), \mathcal{P}_2(w))}{2}} \tag{2.2}$$

is a metric on $\overline{\mathbb{T}}$.

We call this metric χ_e on $\overline{\mathbb{T}}$ the **bicomplex chordal metric**. The virtue of the bicomplex chordal metric is that it allows $w \in I_\infty$ to be treated like any other point. Hence, we are able now to analyse the behavior of the bicomplex meromorphic functions in the extended bicomplex plane, especially on the set I_∞.

Remark 2 *As for the* $\mathbf{j}-$*modulus, let us define*

$$\chi_{\mathbf{j}}(z,w) := \chi(\mathcal{P}_1(z),\mathcal{P}_1(w))\mathbf{e_1} + \chi(\mathcal{P}_2(z),\mathcal{P}_2(w))\mathbf{e_2} \tag{2.3}$$

in the extended hyperbolic numbers. Then

$$\mathrm{Re}({\chi_{\mathbf{j}}}^2(z,w)) = {\chi_e}^2(z,w)$$

and thus we have

$$\chi_e(z,w) = \sqrt{\mathrm{Re}({\chi_{\mathbf{j}}}^2(z,w))} \tag{2.4}$$

where

$$\chi_{\mathbf{j}}(z,w) = \frac{|z-w|_{\mathbf{j}}}{\sqrt{1+|z|_{\mathbf{j}}}\sqrt{1+|w|_{\mathbf{j}}}} \text{ if } z,w \in \mathbb{T}. \tag{2.5}$$

Some of the important properties of the bicomplex chordal metric are discussed in the following results.

Theorem 3 *If* $z = z_1\mathbf{e_1} + z_2\mathbf{e_2}$ *and* $w = w_1\mathbf{e_1} + w_2\mathbf{e_2}$ *are any two elements in the extended bicomplex plane and* χ_e *is the bicomplex chordal metric on* $\overline{\mathbb{T}}$*. Then,*

1. $\chi_e(z,w) \leq 1$*;*
2. $\chi_e(0,\infty) = 1$ *which shows that* 0 *and* ∞ *are the farthest points on* $S \times S$*;*
3. $\chi_e(z,w) = \frac{1}{\sqrt{2}}\chi(z_1,\infty)$ *if* $\mathcal{P}_2(z) = \mathcal{P}_2(w) = 0$ *and* $\mathcal{P}_1(w) = \infty$*;*
4. $\chi_e(z,w) = \frac{1}{\sqrt{2}}\chi(z_1,w_1)$ *if* $\mathcal{P}_2(z) = \mathcal{P}_2(w) = \infty$*;*
5. $\chi_e(z,\infty) = \frac{1}{\sqrt{2}}\chi(z_2,\infty)$ *if* $\mathcal{P}_1(z) = \infty$*;*
6. $\chi_e(z,w) = \chi_e(z^{-1},w^{-1})$*;*
7. $\chi_e(z,w) = \chi(z,w)$ *if* $z,w \in \overline{\mathbb{C}(\mathbf{i_1})}$*;*
8. $\chi_e(z,w) \leq \|z-w\|$ *if* $z,w \in \mathbb{T}$*;*
9. $\chi_e(z,w)$ *is a continuous function on* $\mathbb{T}$*.*

The notion of continuity with respect to the bicomplex chordal metric is given in the following definition.

Definition 6 *A function f is* ***bispherically continuous*** *at a point $w_0 \in \mathbb{T}$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\chi_e(f(w), f(w_0)) < \epsilon,$$

whenever $\|w - w_0\| < \delta$.

In the case of **bicomplex meromorphic functions** we have the following result.

Theorem 4 *If $f(w)$ is a bicomplex meromorphic function in a domain $E \subset \mathbb{T}$, then f is bispherically continuous in E.*

Remark 3 *Since*

$$\chi_e\left(f(w), f(w_0)\right) \leq \left\| f(w) - f(w_0) \right\|,$$

we see that equicontinuity with respect of the euclidean metric implies bispherical equicontinuity.

3 Bicomplex Mobius Maps

Definition 7 *The bicomplex mobius maps are the bicomplex rational functions of the form*

$$\mathbf{T}(w) = \frac{aw + b}{cw + d}, \; a, b, c, d \in \mathbb{T}, \; ad - bc \notin \mathcal{NC}.$$

We can represent every bicomplex mobius map in the idempotent form as follws:

$$\mathbf{T}(w) = \frac{aw + b}{cw + d} = \mathbf{T}_{\mathbf{e_1}}(w_1) + \mathbf{T}_{\mathbf{e_2}}(w_2) \tag{3.1}$$

Where $w = w_1\mathbf{e_1} + w_2\mathbf{e_2}$ and $\mathbf{T}_{\mathbf{e_i}}(w_i) = \frac{a_i w_i + b_i}{c_i w_i + d_i} \;\; for\ i = 1, 2$. Thus

$$\mathcal{P}_1(\mathbf{T}(w)) = \mathbf{T}_{\mathbf{e_1}}(w_1) \text{ and } \mathcal{P}_2(\mathbf{T}(w)) = \mathbf{T}_{\mathbf{e_1}}(w_2)$$

On the basis of above definition we have the following theorem

Theorem 5 *$\mathbf{T}(w)$ is the bicomplex mobius map if and only if $\mathcal{P}_1(\mathbf{T}(w))$ and $\mathcal{P}_2(\mathbf{T}(w))$ are the mobius maps in $\mathbb{C}(\mathbf{i_1})$.*

This theorem facilitates to understand the structure of bicomplex mobius maps.

4 Analiticity of bicomplex mobius maps

It is interesting to know that the bicomplex mobius map $\mathbf{T}(w) = \frac{aw+b}{cw+d}$ is differentiable for all $w \in \mathbb{T}$ except for the case when $cw + d \notin \mathcal{NC}$ and we have

$$\mathbf{T}^{'}(w) = \frac{ad - bc}{(cw+d)^2}, \quad cw + d \notin \mathcal{NC}$$

Therefore the bicomplex mobius map $\mathbf{T}(w)$ is bicomplex holomorphic on $\mathbb{T} \backslash \{w : cw + d \in \mathcal{NC}\}$

5 Fixed points of a bicomplex mobius maps

Definition 8 *A point w_0 is siad to be a fixed point of the mobius map $\mathbf{T}(w)$ if $\mathbf{T}(w_0) = w_0$*

Example 1 *The bicomplex mobius map $\mathbf{T}(w) = w - 1$ has no fixed point in $I_\infty^- \supset \mathbb{T}$ where as it has only one fixed point namely ∞*

Theorem 6 *If w is the fixed point of the mobius map $\mathbf{T}(w)$ then $\mathcal{P}_i(w)$ is the fixed point of $\mathcal{P}_i(\mathbf{T}(w))$ for $i = 1, 2$*

Now we discuss the fixed points of bicomplex Mobius maps
We consider the map $\mathbf{T} : \overline{\mathbb{T}} \to \overline{\mathbb{T}}$ as

$$\mathbf{T}(w) = \frac{aw + b}{cw + d}$$

such that $\mathbf{T}(w) \neq w$ identically Then fixed points of bicomplex mobius maps can be determined by using 3.1.
If $c = 0$ then,

$$\mathbf{T}_{\mathbf{e_i}}(w_i) = \frac{a_i w_i + b_i}{d_i}$$

have two fixed points namely ∞ and $\frac{b_i}{d_i - a_i}$ for $i = 1, 2$. So $\mathbf{T}$ have one fixed points in $\mathbb{T}$ and four fixed points in $\overline{\mathbb{T}}$.
Now if $c \neq 0$, we have three cases:
(1). $c \neq 0$ such that $\mathcal{P}_1(c) = 0$ and $\mathcal{P}_2(c) \neq 0$. Then $\mathbf{T}_{\mathbf{e_1}}(w_1)$ has two fixed points namely ∞ and $\frac{b_1}{d_1 - a_1}$ and $\mathbf{T}_{\mathbf{e_2}}(w_2)$ has atmost two finite fixed points. So $\mathbf{T}$ has atmost two fixed points in $\mathbb{T}$ and atmost four fixed points in $\overline{\mathbb{T}}$.

(2). $c \neq 0$ such that $\mathcal{P}_1(c) \neq 0$ and $\mathcal{P}_2(c) = 0$. This case is similar as in case (1).

(3) $c \neq 0$ such that $\mathcal{P}_1(c) \neq 0$ and $\mathcal{P}_1(c) \neq 0$. Then $\mathbf{T}_{\mathbf{e_i}}(w_i)$ has atmost two finite fixed points for $i = 1, 2$. So $\mathbf{T}$ has atmost four finite fixed points in $\mathbb{T}$ and no fixed point in I_∞

Now we have the following theorems

Theorem 7 *The bicomplex mobius maps* $\mathbf{T}(w) = \frac{aw+b}{cw+d}$ *has atmost four fixed points in* $\mathbb{T}$ *if* $c \notin \mathcal{NC}$

Theorem 8 *The bicomplex mobius maps* $\mathbf{T}(w) = \frac{aw+b}{cw+d}$ *has no fixed points in* I_∞ *if* $c \notin \mathcal{NC}$

Theorem 9 *Any bicomplex mobius maps* $\mathbf{T}(w) = \frac{aw+b}{cw+d}$ *has atmost four fixed points in* $\overline{\mathbb{T}}$

Theorem 10 *The bicomplex mobius maps of the form* $\mathbf{T}(w) = \frac{aw+b}{d}$ *has one fixed point in* $\mathbb{T}$ *and three fixed points in* I_∞.

Theorem 11 *Every bicomplex mobius maps which has three fixed points such that the difference of any two points does not belong to the Null cone is the identity map.*

6 Bicomplex mobius maps satisfy Lipschitz condition

Theorem 12 *Each bicomplex mobius map is a Lipschitz map w.r.t bicomplex chordal metric*

Proof: We consider the bicomplex mobius map

$$\mathbf{T}(w) = \frac{aw+b}{cw+d}$$

Since the mobius maps in $\mathbb{C}(\mathbf{i_1})$ satisfy the lipschitz condition, so $\mathbf{T}_{\mathbf{e_1}}(w_1)$ and $\mathbf{T}_{\mathbf{e_2}}(w_2)$ both satisfy the lipschitz condition and let g_1 and g_2 be their lipschitz constants respectivily.

$$\begin{aligned}\chi_e\left(\mathbf{T}(w), \mathbf{T}(u)\right) &= \sqrt{\frac{\chi^2(\mathbf{T}_{\mathbf{e_1}}(w_1), \mathbf{T}_{\mathbf{e_1}}(u_1)) + \chi^2(\mathbf{T}_{\mathbf{e_2}}(w_2), \mathbf{T}_{\mathbf{e_2}}(u_2)}{2}} \\ &\leq \sqrt{\frac{\|g_1\|^4 \chi^2(w_1, u_1) + \|g_2\|^4 \chi^2(w_2, u_2)}{2}}\end{aligned}$$

$$\leq \|g\|^2 \sqrt{\frac{\chi^2(w_1, u_1) + \chi^2(w_2, u_2)}{2}}$$

$$= \|g\|^2 \chi_e\,(w, u)$$

Where $\|g\| = max\{\|g_1\|, \|g_2\|\}$. Thus bicomplex mobius map is a Lipschitz map w.r.t bicomplex chordal metric.

This result opens the scope for further study of bicomplex mobius maps.

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