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Selberg integral involving a general sequence of functions, class of polynomials, multivariable A-function and multivariable Gimel-function

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ABSTRACT
The aim of the present paper is to evaluate a Selberg integral. This integral involves the product of function $R_n^{(\alpha,\beta)}$ with multivariable gimel-function. a multivariable A-function and general class of multivariable polynomials. Further, the argument of this integral are quite general in nature. A number of other integrals can also be obtained as special cases of our integral thus unifying several simpler integrals lying scattered in the literature.

Keywords:Multivariable Gimel-function , general sequence of functions $R_n^{(\alpha,\beta)}$, Selberg integral, class of multivariable polynomials, multivariable A-function.

1.Introduction.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,n_r},$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{n_r+1,p_r}:\ [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}}$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_r}:\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}}$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}}\\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,n^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{n^{(r)}+1,q_i^{(r)}}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k (k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([4] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re\left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [3].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [8].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [7].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [12,13].

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})]_{1,n_{r-1}},$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})]_{n_{r-1}+1,p_{i_{r-1}}} \tag{1.5}$$

$$\mathbf{A}=\ [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,n_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{\mathfrak{n}+1,p_{i_r}} \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C^{(1)}]_j)_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}};\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})]_{1,q_{i_{r-1}}} \tag{1.8}$$

$$\mathbf{B}=\ [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_{i_r}} \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}} \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}$$

(1.12)Agarwal and Chaubey ([1], p. 1155) have introduced and studied a sequence of functions which will be defined and represented in the following slightly modified manner

Agarwal and Chaubey [1], Raizada [9] and several others have studied a general sequence of functions. In the present document we shall study the following useful series formula for a general sequence of functions.

$$R_n^{(\alpha,\beta)}[x;E,F,g,h;p,q;\gamma;\delta;e^{-sx^r}]=\sum_{w,v,u,t,e,k_1,k_2,}\psi(w,v,u,t,e,k_1,k_2)x^R \tag{1.12}$$

$$\text{where}\sum_{w,v,u,t,e,k_1,k_2}=\sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{c=0}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \tag{1.13}$$

and the infinite series on the right side (2.1) is absolutely convergent, $R=ln+qv+pt+rw+k_1r+k_2q$

$$\text{and}\quad\psi(w,v,u,t,e,k_1,k_2)=\ \frac{(-)^{t+w+k_2}(-v)_u(-t)_e(\alpha)_t l^n}{w!v!u!t!e!K_nk_1!k_2!}\ \frac{s^{w+k_1}F^{\gamma n-t}}{(1-\alpha-t)_e}(\alpha-\gamma n)_e$$

$$(-\beta-\delta n)_v g^{v+k_2}h^{\delta n-v-k_2}(v-\delta n)_{k_2}E^t\left(\frac{pe+rw+\lambda+qn}{l}\right)_n \tag{1.14}$$

where K_n is a sequence of constants.

By suitably specializing the parameters involving in (2.1), a general sequence of function reduced to generalized polynomials set studied by Raizada [4], a class of polynomials introduced by Fujiwara [5] and several others authors.

It may be pointed out that $R_n^{(\alpha,\beta)}(x)$ unifies and extends a large number of named classical polynomials and other polynomials studied by several research workers.

The serie representation of the multivariable A-function is given by Gautam [6] as

$$A[u_1,\cdots,u_v]=A^{0,\lambda:(\alpha',\beta');\cdots;(\alpha^{(v)},\beta^{(v)})}_{A,C:(M',N');\cdots;(M^{(v)},N^{(v)})}\left(\begin{array}{c|c} \mathrm{u}_1 & [(\mathrm{g}_j);\gamma',\cdots,\gamma^{(v)}]_{1,A}: \\ \cdot & \cdots \\ \cdot & \\ \cdot & [(\mathrm{f}_j);\xi',\cdots,\xi^{(v)}]_{1,C}: \\ \mathrm{u}_v & \end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{q}^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(v)},\eta^{(v)})_{1,M^{(v)}} \\ \cdot \\ \cdot \\ (\mathrm{p}^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(v)},\epsilon^{(v)})_{1,N^{(v)}}\end{array}\right)=\sum_{M_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon^i_{m_i}g_i!} \tag{1.15}$$

where

$$\phi_1=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-g_j+\sum_{i=1}^{v}\gamma_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\lambda'+1}^{A}\Gamma\left(g_j-\sum_{i=1}^{v}\gamma_j^{(i)}U_i\right)\prod_{j=1}^{C}\Gamma\left(1-f_j+\sum_{i=1}^{v}\xi_j^{(i)}\eta_{G_i,g_i}\right)} \tag{1.16}$$

$$\phi_i=\frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}}\Gamma\left(p_j^{(i)}-\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{\beta^{(i)}}\Gamma\left(1-q_j^{(i)}+\eta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}}\Gamma\left(1-p_j^{(i)}+\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=\beta^{(i)}+1}^{M^{(i)}}\Gamma\left(q_j^{(i)}-\eta_j^{(i)}\eta_{G_i,g_i}\right)},i=1,\cdots,v \tag{1.17}$$

and

$$\eta_{G_i,g_i}=\frac{p_{m_i}^{(i)}+g_i}{\epsilon_{m_i}^{(i)}},i=1,\cdots,v \tag{1.18}$$

which is valid under the following conditions :

$$\epsilon_{m_i}^{(i)}[p_j^{(i)}+p_i']\neq\epsilon_j^{(i)}[p_{m_i}+g_i] \tag{1.19}$$

and

$$u_i\neq 0,\sum_{j=1}^{A}\gamma_j^{(i)}-\sum_{j=1}^{C}\xi_j^{(i)}+\sum_{j=1}^{M^{(i)}}\eta_j^{(i)}-\sum_{j=1}^{N^{(i)}}\epsilon_j^{(i)}<0,i=1,\cdots,v \tag{1.20}$$

Here $\lambda,A,C,\alpha_i,\beta_i,m_i,n_i\in\mathbb{N}^*;i=1,\cdots,v;f_j,g_j,p_j^{(i)},q_j^{(i)},\gamma_j^{(i)},\xi_j^{(i)},\eta_j^{(i)},\epsilon_j^{(i)}\in\mathbb{C}$

The generalized polynomials defined by Srivastava [11], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}$$

$$A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \tag{1.21}$$

Where $M_1, \cdots, M_u$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_u, K_u]$ are arbitrary constants, real or complex.

We shall note

$$A_u = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_u)_{M_u K_u}}{K_u!} A[N_1, K_1; \cdots; N_u, K_u] \tag{1.22}$$

2. Required integrals.

We note $S(a,b,c)$, the Selberg integral , see Andrew G.G and Askey R ([2], page 402) by :

Lemma 1.

$$S(a,b,c) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1} \prod_{1 \leqslant j<k \leqslant n} |x_j - x_k|^{2c} \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

$$\prod_{j=1}^{n-1} \frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{2.1}$$

provided $Re(a) > 0, Re(b) > 0, \ Re(c) > Max\left\{-\frac{1}{n}, -\frac{Re(a)}{n-1}, -\frac{Re(b)}{n-1}\right\}$

We consider the new integral , see Andrew G.G and Askey R ([2], page 402) defined by :

Lemma 2.

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1} \prod_{1 \leqslant j<k \leqslant n} |x_j - x_k|^{2c} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \prod_{i=1}^k \frac{a+(n-i)c}{a+b+(2n-1-i)c} S(a,b,c) \tag{2.2}$$

provided $Re(a) > 0, Re(b) > 0, \ Re(c) > Max\left\{-\frac{1}{n}, -\frac{Re(a)}{n-1}, -\frac{Re(b)}{n-1}\right\}$ and $k \leqslant n$

3. Main integral.

Let $X_{u,v,w}(x_1, \cdots, x_n) = \prod_{i=1}^n x_i^u (1-x_i)^v \prod_{1 \leqslant j<k \leqslant n} |x_j - x_k|^{2w}$

We have the following integral

Theorem.

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1} \prod_{1 \leqslant j<k \leqslant n} |x_j - x_k|^{2c} \, R_n^{(\alpha',\beta')}\left(zX_{\alpha,\beta,\gamma}(x_1, \cdots, x_n)\right)$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\begin{pmatrix} \mathrm{z}_1'' X_{\alpha_1,\beta_1,\gamma_1}(x_1,\cdots,x_n) \\ \cdot \\ \cdot\ \cdot \\ \mathrm{z}_u'' X_{\alpha_u,\beta_u,\gamma_u}(x_1,\cdots,x_n) \end{pmatrix} A \begin{pmatrix} \mathrm{z}_1' X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n) \\ \cdot \\ \cdot\ \cdot \\ \mathrm{z}_v' X_{\delta_v,\psi_v,\phi_v}(x_1,\cdots,x_n) \end{pmatrix}$$

$$\daleth \begin{pmatrix} \mathrm{z}_1 X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n) \\ \cdot \\ \cdot\ \cdot \\ \mathrm{z}_r X_{\epsilon_r,\eta_r,\zeta_r}(x_1,\cdots,x_n) \end{pmatrix} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \sum_{w,v,u,t,e,k_1,k_2,} \psi(w,v,u,t,e,k_1,k_2) z^R$$

$$\sum_{M_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{m_i}^{i}g_i!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}A_u\, z_1''^{K_1}\cdots z_u''^{K_u}$$

$$\daleth^{U;0,n_r+3n+2k:V}_{X;p_{i_r}+3n+2k,q_{i_r}+n+1+2k,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\,\mathrm{A}_1,\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\,\mathbf{B},\,\mathrm{B}_1:B \end{array}\right) \tag{3.1}$$

where

$$A_1=\left[1-a-\alpha R-\sum_{i=1}^{u}K_i\alpha_i-\sum_{i=1}^{v}\eta_{G_i,g_i}\delta_i-j\left(c+\gamma R+\sum_{i=1}^{u}K_i\gamma_i+\sum_{i=1}^{v}\eta_{G_i,g_i}\phi_i\right);\epsilon_1+j\zeta_1,\cdots,\epsilon_r+j\zeta_r;1\right]_{0,n-1},$$

$$\left[1-b-\beta R-\sum_{i=1}^{u}K_i\beta_i-\sum_{i=1}^{v}\eta_{G_i,g_i}\psi_i-j\left(c+\gamma R+\sum_{i=1}^{u}K_i\gamma_i+\sum_{i=1}^{v}\eta_{G_i,g_i}\eta_i\right);\eta_1+j\zeta_1,\cdots,\eta_r+j\zeta_r;1\right]_{0,n-1},$$

$$\left[-(j+1)\left(c+\gamma R+\sum_{i=1}^{u}K_i\gamma_i+\sum_{i=1}^{v}\eta_{G_i,g_i}\psi_i\right);(j+1)\zeta_1,\cdots,(j+1)\zeta_r;1\right]_{0,n-1},$$

$$\left[-a-\alpha R-\sum_{j=1}^{u}K_j\alpha_j-\sum_{k=1}^{v}\eta_{G_k,g_k}\delta_k-(n-i)\left(c+\gamma R+\sum_{j=1}^{u}K_j\gamma_j+\sum_{k=1}^{v}\eta_{G_k,g_k}\phi_k\right);\epsilon_1+(n-i)\zeta_1,\cdots,\epsilon_r+(n-i)\zeta_r;1\right]_{1,k},$$

$$[1-a-\alpha R-\sum_{j=1}^{u}K_j\alpha_j-\sum_{k=1}^{v}\eta_{G_k,g_k}\delta_k-b-R\beta-\sum_{j=1}^{u}K_j\beta_j-\sum_{k=1}^{v}\eta_{G_k,g_k}\psi_k\ -(2n-i-1)$$

$$(c+\gamma R+\sum_{j=1}^{u}K_j\gamma_j+\sum_{k=1}^{v}\eta_{G_k,g_k}\phi_k;\epsilon_1+\eta_1+(2n-i-1)\zeta_1,\cdots,\epsilon_r+\eta_r+(2n-i-1)\zeta_r;1]_{1,k} \tag{3.2}$$

$$B_1=\left[-c-\gamma R-\sum_{i=1}^{u}K_i\gamma_i-\sum_{i=1}^{v}\eta_{G_i,g_i}\phi_i;\zeta_1,\cdots,\zeta_r;n\right],$$

$$[1-a-b-(\alpha+\beta)R-\sum_{i=1}^{u}K_i(\alpha_i+\beta_i)-\sum_{i=1}^{v}\eta_{G_i,g_i}(\delta_i+\psi_i)-(n-1+j)$$

$$(c+\gamma R+\sum_{i=1}^{u}K_i\gamma_i+\sum_{i=1}^{v}\eta_{G_i,g_i}\phi_i;\epsilon_1+\eta_1+j\zeta_1,\cdots,\epsilon_r+\eta_r+j\zeta_r;1)]_{0,n-1},$$

$$\left[1-a-\alpha R-\sum_{j=1}^{u}K_j\alpha_j-\sum_{k=1}^{v}\eta_{G_k,g_k}\delta_k)-(n-i)\left(c+\gamma R+\sum_{j=1}^{u}K_j\gamma_j+\sum_{k=1}^{v}\eta_{G_k,g_k}\phi_k\right);\epsilon_1+(n-i)\zeta_1,\cdots,\epsilon_r+(n-i)\zeta_r;1\right]_{1,k},$$

$$[-a-b-(\alpha+\beta)R-\sum_{j=1}^{u}K_j(\alpha_j+\beta_j)-\sum_{k=1}^{v}\eta_{G_k,g_k}(\delta_k+\psi_k)-(2n-1-i)$$

$$(c+\gamma R+\sum_{j=1}^{u}K_j\gamma_j+\sum_{k=1}^{v}\eta_{G_k,g_k}\phi_k;\epsilon_1+\eta_1+(2n-1-i)\zeta_1,\cdots,\epsilon_r+\eta_r+(2n-1-i)\zeta_r;1)]_{1,k} \tag{3.3}$$

The others notations are the same that section I.

provided

$\alpha_i,\beta_i,\gamma_i>0(i=1,\cdots,u);\delta_i,\phi_i,\psi_i>0(i=1,\cdots,v);\epsilon_i,\eta_i,\zeta_i>0(i=1,\cdots,r);R(a),R(b),\alpha,\beta,\gamma>0$

$u_i\neq 0,\sum_{j=1}^{A}\gamma_j^{(i)}-\sum_{j=1}^{C}\xi_j^{(i)}+\sum_{j=1}^{M^{(i)}}\eta_j^{(i)}-\sum_{j=1}^{N^{(i)}}\epsilon_j^{(i)}<0,i=1,\cdots,v$

$$A'=\left(a+R\alpha+\sum_{i=1}^{u}K_i\alpha_i+\sum_{j=1}^{v}\delta_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}\epsilon_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$B'=\left(b+R\beta+\sum_{i=1}^{u}K_i\beta_i+\sum_{j=1}^{v}\psi_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}\eta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$C'=\left(c+R\gamma+\sum_{i=1}^{u}K_i\gamma_i+\sum_{j=1}^{v}\phi_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}\zeta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>Max\left\{-\frac{1}{n},-\frac{A'}{n-1},-\frac{B'}{n-1}\right\}$$

$|\arg(z_rX_{\epsilon_r,\eta_r,\zeta_r}(x_1,\cdots,x_n)|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Proof

Expressing the sequence of functions in series with the help of (1.12) , expressing the class of multivariable polynomials $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[.]$ in series with the help of (1.21) and the multivariable A-function defined by Gautam and Goyal . [6] in series with the help of (1.15), expressing the multivariable Gimel-function by this multiple integrals contour with the help of (1.1), Interchanging the order of summations and integrations, which is justified under the conditions mentioned above, we have (say I)

$$I=\sum_{w,v,u,t,e,k_1,k_2,}\psi(w,v,u,t,e,k_1,k_2)z^R\sum_{M_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{m_i}^{i}g_i!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}A_u\,z_1''^{K_1}\cdots z_u''^{K_u}$$

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\left[\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a+\alpha+\sum_{i=1}^{u}K_i\alpha_i+\sum_{j=1}^{v}\eta_{G_j,g_j}\delta_j+\sum_{i=1}^{r}\epsilon_is_i-1}\right.$$

$$\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{\,2c+2\gamma+2\sum_{i=1}^{u}K_i\gamma_i+2\sum_{j=1}^{v}\phi_i\eta_{G_j,g_j}+2\sum_{i=1}^{r}\zeta_is_i-1}$$

$$\left.(1-b_i)^{b+\beta+\sum_{i=1}^{u}K_i\beta_i+\sum_{j=1}^{v}\psi_i\eta_{G_j,g_j}+\sum_{i=1}^{r}\eta_is_i-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

Now using the lemma 2 and finally, reinterpreting the multiple Mellin-Barnes integrals contour in terms of, multivariable Gimel-function, we obtain the desired result.

Remarks :
We obtain the same Selberg integrals about the functions cited in the section I, the general class of polynomials introduced and studied by Srivastava [10].

$$S_V^U(x) = \sum_{\eta=0}^{[V/U]} \frac{(-V)_{U\eta} A_{V,\eta}}{\eta!} x^\eta \tag{3.5}$$

where $V = 0, 1, \cdots$ and U is an arbitrary positive integer. The coefficients $A_{V,\eta}(V, \eta \geqslant 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{V,\eta}$, $S_V^U(x)$ yields a number of known polynomials, these include the Jacobi polynomials, Laguerre polynomials and others polynomials ([14], p. 158-161.)

4. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of the multiple selberg integrals involving general class of multivariable polynomials, the multivariable A-function, the multivariable Gimel-function and a sequence of functions with general arguments utilized in this study, we can obtain a large variety of single, double and multiple integrals. Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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