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THE STRUCTURE OF α−LIMIT SETS FOR CONTINUOUS MAPS

Dr. B. Sankara Rao

Assistant Professor, Adikavai Nannaya University, Rajahmundry

V.V.S. Ramachandram

Associate Professor, B.V.C.College of Engineering, Rajahmundry.

ABSTRACT

In this paper we discuss the structure of alpha limit sets of points in a compact space**.**

KEY WORDS: Transitive map, strongly invariant set, weakly incompressible set, Negative orbit, α–limit set.

Mathematics Subject Classification: Primary 54H20, Secondary 37E35.

1. INTRODUCTION

Limit sets play an important role in understanding the dynamics of a system. To understand the dynamical properties of a system it is necessary to know the behavior of the trajectories of any point in the space under the iteration of f . For this different tools are used. In this paper we focus on the concept of α–limit sets of points in the space . The aim of this paper is to study the structure of α–limit set of points in a compact space.

2. PRELIMINARIES

Definition 2.1: A dynamical system is an ordered pair (X, f) , where X is a nonempty set and $f : X \to X$ is a continuous map.

Definition 2.2: A continuous map $f : X \to X$ is *transitive* if for every pair of open sets U, V in X there exists a positive integer k , such that $f^k(U) \cap V \neq \varnothing$.

Definition 2.3: A set $A \subseteq X$ is *weakly incompressible* if $\overline{f(U)} \cap (A \setminus U) \neq \varnothing$ for any proper nonempty open set, $U \subseteq A$. Equivalently, $D \cap \overline{f(A \setminus D)} \neq \varnothing$ for a closed set D.

Definition 2.4: The set $A \subseteq X$ is *strongly invariant* if $f(A) = A$.

Definition 2.5: The α-*limit set* of a point x in X is the set of elements y in X for which there exists a sequence of negative integers $\{n_k\}$ tending to $-\infty$ such that $f^{n_k}(x) \to y$. Equivalently, for a continuous function $f : X \to X$, the α-*limit set* of the point x_0 in X with respect to the mapping f is defined as, $\alpha(x_0, f) = \bigcap_{n \in N} \overline{\{f^{-k}(x_0) : k > n\}}$.

3. RESULTS

We start this section with the following basic result.

Theorem 3.1: If X is a compact space then, for any $x_0 \in X$ the set $A = \alpha(x_0, f)$ is closed, nonempty and strongly invariant.

Proof : By the definition of α–limit set, it is the intersection of closed subsets of X. So it is closed.
Since X is compact, it is a non-empty subset of X. Let $a \in A$.There is a sequence $\{n_k\}$ such that

$f^{n_k}(x_0) \to a$ as $n_k \to -\infty$. So, we have $a = \lim_{n_k \to -\infty} f^{n_k}(x_0) = f\left(\lim_{n_k \to -\infty}\left(f^{n_k-1}(x_0)\right)\right) = f(b)$

where $b = \lim_{n_k \to -\infty}\left(f^{n_k-1}(x_0)\right) \in A$. This gives $a \in f(A)$ and hence $A \subseteq f(A)$.

On the other hand, if $x \in f(A)$ then $x = f(a)$ for some $a \in A$.

As $a \in A$, we have $a = \lim_{n \to -\infty} f^n(x_0)$ giving $f(a) = f\left(\lim_{n \to -\infty}\left(f^n(x_0)\right)\right)$.

Thus $f(a) = \lim_{n_k \to -\infty}\left(f^{n_k+1}(x_0)\right)$ giving $f(a) = \lim_{m \to -\infty}\left(f^m(x_0)\right)$ so that $x \in A$.

Theorem 3.2: If $X = \alpha(x_0, f)$ for some $x_0 \in X$, then f is transitive.

Proof: Suppose $X = \alpha(x_0, f)$ for some $x_0 \in X$. Let U and V be two nonempty open sets in X.

For some $m, n > 0$, $f^{-m}(x_0) \in U$, $f^{-n}(x_0) \in V$ so that $f^{-n}\left(f^{-m}(x_0)\right) \in f^{-n}(U)$.
So, $f^{-m}(y) \in f^{-n}(U)$ where $f^{-n}(x_0) = y$ and hence $y \in f^{m-n}(U)$.

This shows that $f^k(U) \cap V \neq \varnothing$ and soothe mapping f is transitive.

Theorem 3.3: If f is an open and transitive mapping, then $X = \overline{Orb^-(x)}$, for some $x \in X$.

Proof: For every $n \in N$, by the compactness of the space X, it is covered by finitely many open balls of radius $\frac{1}{n}$. Let the family of open sets be $\{U_n\}$. Put $G_k = \bigcup_{n \in N} f^n(U_k)$.
Then for each k, G_k is an open set and it is dense in X. Since X is compact, by Baire's Category theorem, $G = \bigcap_{k \in N} G_k$ is also dense in X. So, there exists $x \in X$ such that $x \in G$. Thus $x \in G_k$ for each k.
This implies for each k, there exists $n \in N$ such that $x \in f^n(U_k)$.
Hence for some $z \in U_k$, $x = f^n(z)$ which shows that $z \in \{f^{-n}(x)\}$. Thus, $z \in U_k \cap Orb^-(x)$ for every k, proving the theorem.

Now we are ready to prove an important property of alpha limit sets.

Theorem 3.4: The set $A=\alpha(x_0, f)$ is weakly incompressible.

Proof: Let $M\subset A=\alpha(x_0, f)$ be a nonempty closed set in X such that $M\cap\overline{f(A\setminus M)}=\varnothing$. By the normality of the space X, there exists two open sets U and V such that $M\subset U, \overline{f(A\setminus M)}\subset V$ and $\overline{U}\cap\overline{V}=\varnothing$. So, $f(A\setminus M)\subset V$ and $(A\setminus M)\subset f^{-1}(V)$.

By the continuity of the function f, W is a nonempty open set and $f(\overline{W})=\overline{f(W)}=\overline{V}$.
So, $f(\overline{W})\cap\overline{U}=\varnothing$. Since $A\subset W\cup U$, there exists an integer k_o such that $f^{-n}(x_o)\in W$ for infinitely many $n\geq k_0$ and $f^{-m}(x_o)\in U$ for infinitely many $m\geq k_0$.

This implies that $f^{-n_i}(x_0)\in W, f^{-n_i+1}(x_0)\in U$ for infinitely many $n\geq k_o$.

Hence, there exists a sequence of negative integers $\{n_i\}$ such that $f^{-n_i}(x_0)\in W, f^{-n_i+1}(x_0)\in U$.
Thus, $y=\underset{i\to\infty}{Lt}\left(f^{-n_i}(x_0)\right)\in\overline{W}$ and $f(y)=f\left(\underset{i\to\infty}{Lt}\left(f^{-n_i}(x_0)\right)\right)\in\overline{U}$.

Therefore $f(y)\in\overline{U}\cap f(\overline{W})$ which is a contradiction.

Remark 1:

Let X and Y be compact spaces and $g:Y\to Y, \phi:Y\to X, \tau:X\to X$ are continuous functions such that $\phi og=\tau o\phi$ .Then $\phi og^n=\tau^n o\phi$ for all $n>1$.

Remark 2:

Let X and Y be compact spaces and $g:Y\to Y, \phi:Y\to X, \tau:X\to X$ are continuous functions such that $\phi og=\tau o\phi$. If $\phi:Y\to X, \tau:X\to X$ are invertible then $\phi og^n=\tau^n o\phi$ for all negative integers n.

Using the above remarks we can prove the following theorem.

Theorem 3.5 : Let X and Y be compact spaces and $g:Y\to Y, \tau:X\to X, \phi:Y\to X$ are continuous functions such that $\phi og=\tau o\phi$. Then $\alpha(\phi(x),\tau)=\phi(\alpha(x,g))$ for every $x\in Y$.

Proof: Let $y\in\alpha(\phi(x),\tau)$. There exists a sequence $\{n_k\}$ of negative integers such that $y=\lim_{k\to\infty}\tau^{n_k}(\phi(x))$. By the above remark, $y=\lim_{k\to\infty}(\phi og^{n_k})(x)$ giving $y=\lim_{k\to\infty}\phi(g^{n_k}(x))$. The remaining proof follows by the continuity of the function ϕ.

Conversely, suppose $z\in\phi(\alpha(x,g))$.There is an element $p\in\alpha(x,g)$ such that $z=\phi(p)$.

There exists a decreasing sequence $\{m_k\}$ of negative integers such that $p = \lim_{k\to\infty} g^{m_k}(x)$.

Now, $\phi(p) = \lim_{k\to\infty} \phi\left(g^{m_k}(x)\right) = \lim_{k\to\infty} \tau^{m_k}\left(\phi(x)\right)$ which gives $z \in \alpha\left(\phi(x), \tau\right)$.

Theorem 3.6 : $\alpha(x, f) = \bigcup_{j=0}^{n-1} \alpha\left(f^{-j}(x), f^{-n+j}\right)$, for $x \in X$.

Proof: Let $y \in \alpha(x, f)$. So, $y = \lim_{n\to\infty}\left(f^{-n}(x)\right) = \lim_{n\to\infty} f^{-n+k}\left(f^{-k}(x)\right)$

This shows that $y \in \alpha\left(f^{-k}(x), f^{-n+k}\right)$ and hence $y \in \bigcup_{j=0}^{n-1} \alpha\left(f^{-j}(x), f^{-n+k}\right)$.

On the other hand, if $z \in \bigcup_{j=0}^{n-1} \alpha\left(f^{-j}(x), f^{-n+k}\right)$ then $z \in \alpha\left(f^{-p}(x), f^{-n+p}\right)$ for $0 \le p \le n-1$.

This implies $z = \lim_{n\to\infty} f^{-n+p}\left(f^{-p}(x)\right)$ giving $z = \lim_{n\to\infty}\left(f^{-n}(x)\right)$.

CONCLUSION:

In this paper we established some basic properties of α–limit sets. We proved that the α–limit set is strongly invariant and weakly incompressible. We also proved that for an open, transitive map the entire space is the negative orbit of some point in the space.

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