Int. Jr. of Mathematical Sciences & Applications
Vol. 7, No. 2, (July-December, 2017)
Copyright: Mind Reader Publications
ISSN No: 2230-9888
www.ijmsa.yolasite.com

Fractional Integration of the Product of Two Multivariable Aleph-Functions and a General Class of Polynomials I

$F.Y.\ AYANT^1$

1 Teacher in High School , France
E-mail : fredericayant@gmail.com

Abstract : A significantly large number of earlier works on the subject of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, summation of series, etc.). The main object of the present paper is to study and develop the Saigo operators. First, we establish two results that give the images of the product of two multivariable Aleph-functions and a general class of multivariable polynomials in Saigo operators. On account of the general nature of the Saigo operators, multivariable Aleph-functions and a general class multivariable polynomials a large number of new and Known Theorems involving Riemann-Liouville and Erdelyi-Kober fractional integral operators and several special functions notably the Aleph-functions of two variables and one variable and the I-functions of two variables and one variable.

Keywords: General class of multivariable polynomial, Saigo operator, multivariable Aleph-function, multivariable H-function, Aleph-function of two variables, I-function of two variables Aleph-function, I-function.

2010 Mathematics Subject Classification. 33C60, 82C31

1.Introduction and preliminaries.

The fractional integral operator involving various special functions has found significant importance and applications in various subfields of applicable mathematical analysis. Since last four decades, a number of workers like Love [15], McBride [17], Kalla [5,6], Kalla and Saxena [7,8], Saxena et al. [23], Saigo [19-20], Kilbas [9], Kilbas and Sebastian [11] and Kiryakova [13,14] have studied in depth the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Samko, Kilbas and Marichev [22], Miller and Ross [18]; Kiryakova [13,14], Kilbas, Srivastava and Trujillo [12] and Debnath and Bhatta [3]. A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [20,21], has been introduced by Marichev [16] (see details in Samko et al. [22] and also see Kilbas and Saigo [10])as follows:
Let α,β,η be complex numbers and $x>0$, then the generalized fractional integral operators (the Saigo operators [20]) involving Gaussian hypergeometric function are defined by the following equations:

$$I_{0^+}^{\alpha,\beta,\eta}f(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)\mathrm{d}t(Re(\alpha))>0) \tag{1.1}$$

and

$$I_{-}^{\alpha,\beta,\eta}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty t^{-\alpha-\beta}(x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)\mathrm{d}t(Re(\alpha))>0) \tag{1.2}$$

When $\beta=-\alpha$, equations (1.1) and (1.2) reduce to the following classical Riemann–Liouville fractional integral operator (see Samko et al. [22], p. 94, (5.1), (5.3)):

$$I_{0^+}^{\alpha,-\alpha,\eta}f(x)=I_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\mathrm{d}t(x>0) \tag{1.3}$$

$$I_{0^-}^{\alpha,-\alpha,\eta}f(x)=I_{0-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty(x-t)^{\alpha-1}f(t)\mathrm{d}t(x>0) \tag{1.4}$$

Again, if $\beta=0$ Equations (1.1) and (1.2) reduce to the following Erdelyi–Kober fractional integral operator (see Samko et al. [22], p.322, Eqns. (18.5), (18.6)):

$$I_{0^+}^{\alpha,0,\eta}f(x)=I_{\alpha,\eta}^{+}f(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^\eta f(t)\mathrm{d}t(x>0) \tag{1.5}$$

and

$$I_{-}^{\alpha,0,\eta}f(x)=K_{\alpha,\eta}^{-}f(x)=\frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(x-t)^{\alpha-1}t^{-\alpha-\eta}f(t)\mathrm{d}t(x>0) \tag{1.6}$$

Recently, Gupta et al. [4] have obtained the images of the product of two H-functions in Saigo operator given by (1.1) and (1.2) and thereby generalized several important results obtained earlier by Kilbas, Kilbas and Sebastian and Saxena et al. as mentioned in this paper cited above. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of this paper is to obtain two results that give the Theorems of the product of two multivariable Aleph-functions and a general class of multivariable polynomials [26] in Saigo operators.

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [26] , itself is an a generalisation of G and H-functions of several variables defined by Srivastava et Panda [30,31]. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function of $r-$ variables throughout our present study and will be defined and represented as follows (see Ayant [2]).

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \cdots\cdots\cdots\cdots , \\ \cdot & \\ \mathrm{z}_r & \end{array}\right.$

$$[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:\ [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;$$
$$\cdot \qquad \cdot$$
$$[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:[(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;$$

$$\left.\begin{array}{c} [(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.7}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.8}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.9}$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,$$

with $k=1,\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)}$ (1.10)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,,min(\,|z_1|,\cdots,|z_r|\,)\to\infty$$

where $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

We shall note $\aleph(z_1,\cdots,z_r)$ = $\aleph_1(z_1,\cdots,z_r)$ and

$$\aleph(z'_1,\cdots,z'_s)=\aleph^{0,n':m'_1,n'_1,\cdots,m'_s,n'_s}_{p'_i,q'_i,\iota_i;r':p'_{i^{(1)}},q'_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;p'_{i^{(s)}},q'_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}\left(\begin{array}{c}\mathrm{z}'_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_s\end{array}\ \middle|\ \begin{array}{c}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r')})_{1,n'}],\\ \cdot\\ \cdots\cdots\cdots\cdots\cdots\cdots ,\end{array}\right.$$

$$[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r')})_{n'+1,p'_i}]:[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,n'_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p'^{(1)}_i}];\cdots;$$

$$[\iota_i(v_{ji};\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(r')})_{m'+1,q'_i}]:[(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,m'_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q'^{(1)}_i}];\cdots;$$

$$\left.\begin{array}{c}[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,n'_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n'_s+1,P_i^{(s)}}]\\ \cdot\\ [(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,m'_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m'_s+1,Q_i^{(s)}}]\end{array}\right)=\frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k){z'_k}^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (1.11)$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=n'+1}^{p'_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{q'_i}\Gamma(1-v_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]} \quad (1.12)$$

and $$\phi_k(t_k)=\frac{\prod_{j=1}^{m'_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{n'_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=m'_k+1}^{q'_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=n'_k+1}^{p'_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \quad (1.13)$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_k|<\frac{1}{2}B_i^{(k)}\pi$, where

$$B_i^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} - \iota_i \sum_{j=n'+1}^{p'_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{q'_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{n'_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m'_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0,$$

with $k = 1, \cdots, s, i = 1, \cdots, r' , i^{(k)} = 1, \cdots, r^{(k)}$ (1.14)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}) , max(|z_1|, \cdots, |z_s|) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}) , min(|z_1|, \cdots, |z_s|) \rightarrow \infty$$

where $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m'_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n'_k$$

We shall note $\aleph(z'_1, \cdots, z'_s)$ = $\aleph_2(z'_1, \cdots, z'_s)$

The generalized polynomials of multivariables defined by Srivastava [29], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1, \cdots, y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}$$

$$A[N_1, K_1; \cdots; N_v, K_v] y_1^{K_1} \cdots y_v^{K_v} \tag{1.15}$$

where $\mathfrak{M}_1, \cdots, \mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_v, K_v]$ are arbitrary constants, real or complex.

We shall note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots; N_v, K_v]$

2. Lemma

Lemma 1 (Kilbas and Sebastien ([11], page 871, (15)-(18))

$$\left(I_{0^+}^{\alpha,\beta,\eta} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)} x^{\mu-\beta-1} \tag{2.1}$$

where $\alpha, \beta, \eta \in \mathbb{C}, Re(\alpha) > 0$ and $Re(\mu) > max\{0, Re(\beta - \eta)\}$

in particular, if $\beta = -\alpha$ and $\beta = 0$ in (2.1), we have respectively

$$\left(I_{0^+}^{\alpha} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)} x^{\mu-\beta-1}, Re(\alpha) > 0, Re(\mu) > 0, \tag{2.2}$$

$$\left(I_{\eta,\alpha}^{+} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)} x^{\mu-1}, Re(\alpha) > 0, Re(\mu) > -Re(\eta). \tag{2.3}$$

Lemma 2 (Kilbas and Sebastien ([11], page 872, (21)-(24))

$$\left(I_{0-}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x)=\frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)}x^{\mu-\beta-1} \tag{2.4}$$

where $\alpha,\beta,\eta\in\mathbb{C}, Re(\alpha)>0$ and $Re(\mu)<1+min\{Re(\beta),Re(\eta)\}$

in particular, if $\beta=-\alpha$ and $\beta=0$ in (2.4), we have respectively

$$\left(I_{-}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)}x^{\mu-\beta-1}, 1-Re(\mu)>Re(\alpha)>0 \tag{2.5}$$

$$\left(K_{\eta,\alpha}^{-}t^{\mu-1}\right)(x)=\frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)}x^{\mu-1}, Re(\mu)<Re(\eta)+1. \tag{2.6}$$

3. Main results

We have the following result

Theorem 1

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}\aleph_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}\aleph_{U_{33}:W}^{0,\mathbf{n}+n'+3;V}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A},\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B},\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right) \tag{3.1}$$

where

$$V=m_1,n_1;\cdots;m_r,n_r\ ;m'_1,n'_1;\cdots;m'_s,n'_s;1,0 \tag{3.2}$$

$$W=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)};p'_{i^{(1)}},q'_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;p'_{i^{(r)}},q'_{i^{(r)}},\iota_{i^{(s)}};r^{(s)};0,1 \tag{3.3}$$

and

$$U_{33}=3,3;p_i,q_i,\tau_i;R\ ;p'_i,q'_i,\iota_i;r' \tag{3.4}$$

$$A=\quad\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1\right),\ \left(1-\mu-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right),$$

$$\left(1-\mu-\eta+\beta-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right) \tag{3.5}$$

$$\mathbf{A}=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},0,\cdots,0,0)_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)},0,\cdots,0,0)_{n+1,p_i}\},$$

$$\{(u_j;0,\cdots,0,\mu_j^{(1)},\cdots,\mu_j^{(s)},0)_{1,n'}\},\{\iota_i(u_{ji};0,\cdots,0,\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)},0)_{n'+1,p'_i}\}:$$

$$\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\}$$

$$;\{(a_j^{(1)};\alpha_j^{(1)})_{1,n'_1},\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p'_{i^{(1)}}}\};\cdots;\{(a_j^{(s)};\alpha_j^{(s)})_{1,n'_s},\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n'_s+1,p'_{i^{(s)}}}\} \tag{3.6}$$

$$B=\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0\right),\left(1-\mu+\beta-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right),$$

$$\left(1-\mu-\eta-\alpha-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right) \tag{3.7}$$

$$\mathbf{B}=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)},0,\cdots,0,0)_{m+1,q_i}\},\{\iota_i(v_{ji};0,\cdots,0,\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(s)},0)_{m'+1,q'_i}\}:$$

$$\{(d'^{(1)}_j;\delta'^{(1)}_j)_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\cdots;\{(d'^{(r)}_j;\delta'^{(r)}_j)_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\};$$

$$\{(b_j^{(1)};\beta_j^{(1)})_{1,m'_1},\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q'_{i^{(1)}}}\};\cdots;\{(b_j^{(s)};\beta_j^{(s)})_{1,m'_s},\iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m'_s+1,q'_{i^{(s)}}}\};(0,1) \tag{3.8}$$

Provided that

$a,b,c,\alpha,\beta,\eta,\mu,\upsilon,\delta_k,\omega_i,\omega'_j\in\mathbb{C},k=1,\cdots,v;i=1,\cdots,r;j=1,\cdots,s$

$\lambda_k,\sigma_i,\sigma'_j>0,k=1,\cdots,v;i=1,\cdots,r;j=1,\cdots,s$

$|argz_i|<\dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.10)

$|argz'_i|<\dfrac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.14)

$$Re\left[\mu+\sum_{i=1}^{r}\sigma_i\min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}}+\sum_{j=1}^{s}\sigma'_j\min_{1\leqslant l'\leqslant m'_j}\frac{b_{l'}^{(j)}}{\beta_{l'}^{(j)}}\right]>max\{0,Re(\beta-\eta)\}$$

$$Re\left[\upsilon+\sum_{i=1}^{r}\omega_i\min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}}+\sum_{j=1}^{s}\omega'_j\min_{1\leqslant l'\leqslant m'_j}\frac{b_{l'}^{(j)}}{\beta_{l'}^{(j)}}\right]>max\{0,Re(\beta-\eta)\}\text{ and }\left|\frac{a}{b}x\right|<1.$$

Proof

To prove (3.1), we first express the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ in series with the help of (1.15), the multivariable Aleph-functions in terms of Mellin-Barnes type contour integrals with the help of (1.7) and (1.11) respectively. Now interchange the order of summations and two multiple Mellin-Barnes contour integrals, respectively and taking the fractional integral operator inside (which is permissible under the stated conditions) and make simplifications. Next, we express the terms $(b-ax)^{-\upsilon-\sum_{k=1}^{v}\delta_k K_k-\sum_{j=1}^{r}\omega_i s_i-\sum_{j=1}^{s}\omega'_j t_j}$ in the terms of Mellin-Barnes contour integral (Srivastava et al [28], page 18, (2.6.3) ; page 10, (2.1.1)) and after algebric manipulations, we obtain

$$\text{L.H.S. of (3.1)} = b^{-\upsilon}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v\, c_1^{K_1}\cdots c_v^{K_v}\, b^{-\sum_{j=1}^{v}\delta_j K_j}\frac{1}{(2i\pi)^{r+s+1}}\int_{L_1}\cdots\int_{L_r}\int_{L'_1}\cdots\int_{L'_s}\psi(s_1,\cdots,s_r)$$

$$\zeta(t_1,\cdots,t_s)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\prod_{i=1}^{s}\phi_i(t_i)z_i'^{t_i}b^{-\sum_{j=1}^{r}\omega_i s_i-\sum_{j=1}^{s}\omega'_j t_j}$$

$$\int_L\frac{\Gamma(\upsilon+\sum_{k=1}^{v}\delta_k K_k+\sum_{i=1}^{r}\omega_i s_i+\sum_{j=1}^{s}\omega'_j t_j+u)}{\Gamma(\upsilon+\sum_{k=1}^{v}\delta_k K_k+\sum_{i=1}^{r}\omega_i s_i+\sum_{j=1}^{s}\omega'_j t_j)\Gamma(1+u)}\left(-\frac{a}{b}\right)^u$$

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\mu+\sum_{k=1}^{v}\lambda_k K_k+\sum_{i=1}^{r}\sigma_i s_i+\sum_{j=1}^{s}\sigma_j t_j+u-1}\right)(x)\mathrm{d}u\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{3.9}$$

Now using the Lemma 1 and interpreting the $(r+s+1)-$Mellin-barnes contour integral of (3.8) in multivariable Aleph-function of $(r+s+1)-$variables, we obtain the desired result.

If we put $\beta=-\alpha$ in Theorem 1, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.3) and using (2.2):

Corollary 1

$$\left\{I_{0+}^{\alpha}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}\aleph_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}\aleph_{U_{22}:W}^{0,\mathbf{n}+n'+2;V}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A'},\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B'},\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right) \tag{3.10}$$

where

$$A'=\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1\right),\left(1-\mu-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right) \tag{3.11}$$

$$B' = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0\right), \left(1 - \mu - \alpha - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right) \tag{3.12}$$

and

$U_{22} = 2, 2; p_i, q_i, \tau_i; R\,; p'_i, q'_i, \iota_i; r'$

with the same validity conditions that (3.1).

If $\beta = 0$ in Theorem 1, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.5) and using (2.3) :

Corollary 2

$$\left\{I_{\eta,\alpha}^{+}\left(t^{\mu-1}(b-at)^{-\upsilon} S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{c}_1 t^{\lambda_1}(b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{c}_v t^{\lambda_v}(b-at)^{-\delta_v} \end{pmatrix} \aleph_1 \begin{pmatrix} \mathrm{z}_1 t^{\sigma_1}(b-at)^{-\omega_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r t^{\sigma_r}(b-at)^{-\omega_r} \end{pmatrix} \aleph_2 \begin{pmatrix} \mathrm{z}'_1 t^{\sigma'_1}(b-at)^{-\omega'_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_s t^{\sigma'_s}(b-at)^{-\omega'_s} \end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon} x^{\mu-\beta-1} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v c_1^{K_1} \cdots c_v^{K_v} b^{-\sum_{j=1}^{v} \delta_j K_j} x^{\sum_{j=1}^{v} \lambda_j K_j} \aleph_{U_{22}:W}^{0,\mathbf{n}+n'+2;V} \left(\begin{array}{c|c} \mathrm{z}_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A}'', \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r \frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot \\ \mathrm{z}'_1 \frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s \frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B}'', \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \tag{3.13}$$

where

$$A'' = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 1\right), \left(1 - \mu - \eta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right) \tag{3.14}$$

$$B'' = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0\right), \left(1 - \mu - \alpha - \eta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right)$$

and

$$U_{22} = 2, 2; p_i, q_i, \tau_i; R\,; p'_i, q'_i, \iota_i; r' \tag{3.15}$$

Provided that

$a, b, c, \alpha, \beta, \eta, \mu, \upsilon, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \cdots, v; i = 1, \cdots, r; j = 1, \cdots, s$

$\lambda_k, \sigma_i, \sigma'_j > 0, k = 1, \cdots, v; i = 1, \cdots, r; j = 1, \cdots, s$

$|argz_i| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.10)

$|argz'_i| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.14)

$$Re\left[\mu + \sum_{i=1}^{r}\sigma_i \min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}} + \sum_{j=1}^{s}\sigma'_j \min_{1\leqslant l'\leqslant m'_j}\frac{b_{l'}^{(j)}}{\beta_{l'}^{(j)}}\right] > max\{0, Re(-\eta)\}$$

$$Re\left[\upsilon + \sum_{i=1}^{r}\omega_i \min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}} + \sum_{j=1}^{s}\omega'_j \min_{1\leqslant l'\leqslant m'_j}\frac{b_{l'}^{(j)}}{\beta_{l'}^{(j)}}\right] > max\{0, Re(-\eta)\} \text{ and } \left|\frac{a}{b}x\right| < 1.$$

Theorem 2

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{c}_1 t^{\lambda_1}(b-at)^{-\delta_1} \\ . \\ . \\ . \\ \mathrm{c}_v t^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1\begin{pmatrix} \mathrm{z}_1 t^{\sigma_1}(b-at)^{-\omega_1} \\ . \\ . \\ . \\ \mathrm{z}_r t^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}\aleph_2\begin{pmatrix} \mathrm{z}'_1 t^{\sigma'_1}(b-at)^{-\omega'_1} \\ . \\ . \\ . \\ \mathrm{z}'_s t^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v\, c_1^{K_1}\cdots c_v^{K_v} b^{-\sum_{j=1}^{v}\delta_j K_j} x^{\sum_{j=1}^{v}\lambda_j K_j}\aleph_{U_{33}:W}^{0,\mathbf{n}+n'+3;V}\left(\begin{array}{c|c} \mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A},\ \mathbf{A} \\ . & . \\ . & . \\ . & . \\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & . \\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & . \\ . & . \\ . & . \\ . & . \\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B},\ \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \tag{3.16}$$

where

$$\mathbb{A} = \left(1-\upsilon-\sum_{j=1}^{v}\delta_j K_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1\right),\ \left(\mu-\eta+\sum_{j=1}^{v}\lambda_j K_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right),$$

$$\left(-\beta+\mu+\sum_{j=1}^{v}\lambda_j K_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right) \tag{3.17}$$

$$\mathbb{B} = \left(1-\upsilon-\sum_{j=1}^{v}\delta_j K_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0\right),\ \left(-\alpha-\beta-\eta+\mu+\sum_{j=1}^{v}\lambda_j K_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right),$$

$$\left(\mu+\sum_{j=1}^{v}\lambda_j K_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right) \tag{3.18}$$

and

$$U_{33} = 3,3; p_i, q_i, \tau_i; R\, ; p'_i, q'_i, \iota_i; r' \tag{3.19}$$

Provided that

$a, b, c, \alpha, \beta, \eta, \mu, \upsilon, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \cdots, v; i = 1, \cdots, r; j = 1, \cdots, s$

$\lambda_k, \sigma_i, \sigma'_j > 0, k = 1, \cdots, v; i = 1, \cdots, r; j = 1, \cdots, s$

$|argz_i| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.10)

$|argz'_i| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.14)

$$Re\left[\mu - \sum_{i=1}^{r} \sigma_i \min_{1\leqslant l\leqslant m_i} \frac{d_l^{(i)}}{\delta_l^{(i)}} - \sum_{j=1}^{s} \sigma'_j \min_{1\leqslant l'\leqslant m'_j} \frac{b_{l'}^{(j)}}{\beta_{l'}^{(j)}}\right] < 1 + min\{Re(\beta), Re(\eta)\}$$

$$Re\left[\upsilon - \sum_{i=1}^{r} \omega_i \min_{1\leqslant l\leqslant m_i} \frac{d_l^{(i)}}{\delta_l^{(i)}} - \sum_{j=1}^{s} \omega'_j \min_{1\leqslant l'\leqslant m'_j} \frac{b_{l'}^{(j)}}{\beta_{l'}^{(j)}}\right] < 1 + min\{Re(\beta), Re(\eta)\} \text{ and } \left|\frac{a}{b}x\right| < 1.$$

The proof of the Theorem 2 is similar that Theorem 1 (use the Lemma 2).

If we put $\beta = -\alpha$ in Theorem 2, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.4) and using (2.5):

Corollary 3

$$\left\{I_-^{\alpha}\left(t^{\mu-1}(b-at)^{-\upsilon}\, S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{c}_1 t^{\lambda_1}(b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{c}_v t^{\lambda_v}(b-at)^{-\delta_v} \end{pmatrix} \aleph_1 \begin{pmatrix} \mathrm{z}_1 t^{\sigma_1}(b-at)^{-\omega_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r t^{\sigma_r}(b-at)^{-\omega_r} \end{pmatrix} \aleph_2 \begin{pmatrix} \mathrm{z}'_1 t^{\sigma'_1}(b-at)^{-\omega'_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_s t^{\sigma'_s}(b-at)^{-\omega'_s} \end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon} x^{\mu+\alpha-1} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v\, c_1^{K_1} \cdots c_v^{K_v} b^{-\sum_{j=1}^{v}\delta_j K_j} x^{\sum_{j=1}^{v}\lambda_j K_j} \aleph_{U_{22}:W}^{0,\mathbf{n}+n'+2;V} \left(\begin{array}{c|c} \mathrm{z}_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}',\, \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r \frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot \\ \mathrm{z}'_1 \frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s \frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B}',\, \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \tag{3.20}$$

where

$$\mathbb{A}' = \left(1-\upsilon-\sum_{j=1}^{v}\delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 1\right),\ \left(\alpha+\mu+\sum_{j=1}^{v}\lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right) \tag{3.21}$$

$$\mathbb{B}' = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0\right), \left(\mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right) \quad (3.22)$$

with the same validity conditions that (3.15).

If we put $\beta = 0$ in Theorem 2, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.4) and using (2.6):

Corollary 4

$$\left\{K^{-}_{\eta,\alpha}\left(t^{\mu-1}(b-at)^{-\upsilon} S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v}\begin{pmatrix} \mathrm{c}_1 t^{\lambda_1}(b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{c}_v t^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix} \aleph_1 \begin{pmatrix} \mathrm{z}_1 t^{\sigma_1}(b-at)^{-\omega_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r t^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix} \aleph_2 \begin{pmatrix} \mathrm{z}'_1 t^{\sigma'_1}(b-at)^{-\omega'_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_s t^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon} x^{\mu-1} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v\, c_1^{K_1} \cdots c_v^{K_v} b^{-\sum_{j=1}^{v} \delta_j K_j} x^{\sum_{j=1}^{v} \lambda_j K_j} \aleph^{0,\mathbf{n}+n'+2;V}_{U_{22}:W}\left(\begin{array}{c|c} \mathrm{z}_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}'', \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r \frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot \\ \mathrm{z}'_1 \frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s \frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B}'', \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \quad (3.23)$$

where

$$\mathbb{A}'' = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 1\right), \left(-\alpha - \eta + \mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right) \quad (3.24)$$

$$\mathbb{B}'' = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0\right), \left(\mu - \alpha - \eta + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1\right) \quad (3.25)$$

with the same validity conditions that (3.15).

Remark : If the multivariable Aleph-functions reduce in to multivariable H-functions defined by Srivastava and Panda [30,31] and $S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v} \to \prod_{k=1}^{v} S^{\mathfrak{M}_k}_{N_k}$, we obtain the result of Agarwal [1].

4. Particular cases

The generalized fractional integral operator Theorems 1 and 2 established here are unified in nature and act as key formulae. Thus the product of general class of polynomials involved in Theorems 1 and 2 reduces to a large spectrum of polynomials listed by Srivastava and Singh ([32], pp. 158–161), and so from Images 1 and 2 we can further obtain various fractional integral results involving a number of simpler polynomials. Again, the multivariable Aleph-function occurring in these Theorems can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of Aleph-functions of two variable defined by Sharma [25], the I-functions of two variables defined by Sharma and Mishra [27], the Aleph-functions of one variable [33,34] and I-functions of one variable [24].

a) Aleph-function of two variables

If $r = s = 2$, the multivariables Aleph-functions reduce in to Aleph-functions of two variables. We obtain.

Corollary 5

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2t^{\sigma_2}(b-at)^{-\omega_2}\end{pmatrix}\aleph_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2t^{\sigma'_2}(b-at)^{-\omega'_2}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}\aleph_{U_{33}:W_2}^{0,\mathbf{n}+n'+3;V_2}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A}_2,\mathbf{A}_2\\ \cdot & \cdot\\ \mathrm{z}_2\frac{x^{\sigma_2}}{b^{\omega_2}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_2\frac{x^{\sigma'_2}}{b^{\omega'_2}} & \cdot\\ -\frac{ax}{b} & \mathrm{B}_2,\mathbf{B}_2\end{array}\right) \tag{4.1}$$

The validity conditions are the same that (3.1) with $r = s = 2$ and

$$A_2 = \left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\omega_2,\omega'_1,\omega'_2,1\right),\left(1-\mu-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1\right),$$

$$\left(1-\mu-\eta+\beta-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1\right) \tag{4.2}$$

$$B_2 = \left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\omega_2,\omega'_1,\omega'_2,0\right),\left(1-\mu+\beta-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1\right)$$

$$\left(1-\mu-\eta-\alpha-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1\right) \tag{4.3}$$

Corollary 6

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2t^{\sigma_2}(b-at)^{-\omega_2}\end{pmatrix}\aleph_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2t^{\sigma'_2}(b-at)^{-\omega'_2}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon} x^{\mu-\beta-1} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]} a_v \, c_1^{K_1} \cdots c_v^{K_v} b^{-\sum_{j=1}^{v} \delta_j K_j} x^{\sum_{j=1}^{v} \lambda_j K_j} \aleph_{U_{33}:W_2}^{0,\mathbf{n}+n'+3;V_2} \left(\begin{array}{c|c} \mathrm{z}_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}_2, \, \mathbf{A}_2 \\ \cdot & \cdot \\ \mathrm{z}_2 \frac{x^{\sigma_2}}{b^{\omega_2}} & \cdot \\ \mathrm{z}_1' \frac{x^{\sigma_1'}}{b^{\omega_1'}} & \cdot \\ \cdot & \cdot \\ \mathrm{z}_2' \frac{x^{\sigma_2'}}{b^{\omega_2'}} & \cdot \\ -\frac{ax}{b} & \mathbb{B}_2, \, \mathbf{B}_2 \end{array} \right) \tag{4.4}$$

The validity conditions are the same that (3.15) with $r = s = 2$.

where $V_2, W_2, \mathbf{A}_2, \mathbf{B}_2$ are equals to $V, W, \mathbf{A}, \mathbf{B}$ respectively for $r = s = 2$ and

$$\mathbb{A}_2 = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j ; \omega_1, \omega_2, \omega_1', \omega_2', 1 \right), \left(\mu - \eta + \sum_{j=1}^{v} \lambda_j K_j ; \sigma_1, \sigma_2, \sigma_1', \sigma_2', 1 \right),$$

$$\left(-\beta + \mu + \sum_{j=1}^{v} \lambda_j K_j ; \sigma_1, \sigma_2, \sigma_1', \sigma_2', 1 \right) \tag{4.5}$$

$$\mathbb{B}_2 = \left(1 - \upsilon - \sum_{j=1}^{v} \delta_j K_j ; \omega_1, \omega_2, \omega_1', \omega_2', 0 \right), \left(-\alpha - \beta - \eta + \mu + \sum_{j=1}^{v} \lambda_j K_j ; \sigma_1, \sigma_2, \sigma_1', \sigma_2', 1 \right),$$

$$\left(\mu + \sum_{j=1}^{v} \lambda_j K_j ; \sigma_1, \sigma_2, \sigma_1', \sigma_2', 1 \right) \tag{4.6}$$

b) I-function of two variables

If $r = s = 2; \tau_i, \tau_{i'}, \tau_{i''}, \iota_i, \iota_{i'}, \iota_{i''} \to 1$, the multivariable Aleph-functions reduce in to I-functions of two variables. We have.

Corollary 7

$$\left\{ I_{0^+}^{\alpha,\beta,\eta} \left(t^{\mu-1} (b-at)^{-\upsilon} S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}} \begin{pmatrix} \mathrm{c}_1 t^{\lambda_1} (b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{c}_v t^{\lambda_v} (b-at)^{-\delta_v} \end{pmatrix} I_1 \begin{pmatrix} \mathrm{z}_1 t^{\sigma_1} (b-at)^{-\omega_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_2 t^{\sigma_2} (b-at)^{-\omega_2} \end{pmatrix} I_2 \begin{pmatrix} \mathrm{z}_1' t^{\sigma_1'} (b-at)^{-\omega_1'} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_2' t^{\sigma_2'} (b-at)^{-\omega_2'} \end{pmatrix} \right) \right\} (x)$$

$$= b^{-\upsilon} x^{\mu-\beta-1} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]} a_v \, c_1^{K_1} \cdots c_v^{K_v} b^{-\sum_{j=1}^{v} \delta_j K_j} x^{\sum_{j=1}^{v} \lambda_j K_j} I_{U_{33}:W_2}^{0,\mathbf{n}+n'+3;V_2} \left(\begin{array}{c|c} \mathrm{z}_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A}_2, \mathbf{A}_2' \\ \cdot & \cdot \\ \mathrm{z}_2 \frac{x^{\sigma_2}}{b^{\omega_2}} & \cdot \\ \mathrm{z}_1' \frac{x^{\sigma_1'}}{b^{\omega_1'}} & \cdot \\ \cdot & \cdot \\ \mathrm{z}_2' \frac{x^{\sigma_2'}}{b^{\omega_2'}} & \cdot \\ -\frac{ax}{b} & \mathrm{B}_2, \mathbf{B}_2' \end{array} \right) \tag{4.7}$$

The validity conditions are the same that (3.1) with $r = s = 2; \tau_i, \tau_{i'}, \tau_{i''}, \iota_i, \iota_{i'}, \iota_{i''} \to 1$

Corollary 8

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2t^{\sigma_2}(b-at)^{-\omega_2}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2t^{\sigma'_2}(b-at)^{-\omega'_2}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}I_{U_{33}:W_2}^{0,\mathbf{n}+n'+3;V_2}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}_2,\ \mathbf{A'}_2\\ \cdot & \cdot\\ \mathrm{z}_2\frac{x^{\sigma_2}}{b^{\omega_2}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_2\frac{x^{\sigma'_2}}{b^{\omega'_2}} & \cdot\\ -\frac{ax}{b} & \mathbb{B}_2,\ \mathbf{B'}_2\end{array}\right) \tag{4.8}$$

The validity conditions are the same that (3.15) with $r=s=2;\tau_i,\tau_{i'},\tau_{i''},\iota_i,\iota_{i'},\iota_{i''}\rightarrow 1.$

where $V_2,W_2,\mathbf{A}'_2,\mathbf{B}'_2$ are equals to $V,W,\mathbf{A},\mathbf{B}$ respectively for $r=s=2;\tau_i,\tau_{i'},\tau_{i''},\iota_i,\iota_{i'},\iota_{i''}\rightarrow 1.$

c) Aleph-function of one variable

If $r=s=1$, the multivariables Aleph-functions reduce in to Aleph-functions of one variable. We obtain.

Corollary 9

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1\left(\ \mathrm{z}\ \mathrm{t}^{\sigma}(b-at)^{-\omega}\ \right)\aleph_2\left(\ \mathrm{z'}\ \mathrm{t}^{\sigma'}(b-at)^{-\omega'}\ \right)\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}\aleph_{3,3:W_1}^{0,3;V_1}\left(\begin{array}{c|c}\mathrm{z}\ \frac{x^{\sigma}}{b^{\omega}} & \mathrm{A}_1,\mathbf{A}_1\\ \mathrm{z'}\ \frac{x^{\sigma'}}{b^{\omega'}} & \cdot\\ -\frac{ax}{b} & \mathrm{B}_1,\mathbf{B}_1\end{array}\right) \tag{4.9}$$

The validity conditions are the same that (3.1) with $r=s=1$ and

$$A_1=\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\omega'_1,1\right),\left(1-\mu-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma'_1,1\right),$$

$$\left(1-\mu-\eta+\beta-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma'_1,1\right) \tag{4.10}$$

$$B_1=\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\omega'_1,0\right),\left(1-\mu+\beta-\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma'_1,1\right)$$

$$\left(1-\mu-\eta-\alpha-\sum_{j=1}^{v}\lambda_j K_j;\sigma_1,\sigma_1',1\right) \tag{4.11}$$

$$\mathbf{A}_1=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\} \;;\{(a_j^{(1)};\alpha_j^{(1)})_{1,n_1'}\;,\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n_1'+1,p'_{i^{(1)}}}\} \tag{4.12}$$

$$\mathbf{B}_1=\{(d_j'^{(1)};\delta_j'^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}\;;\{(b_j^{(1)};\beta_j^{(1)})_{1,m_1'},\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m_1'+1,q'_{i^{(1)}}}\};(0,1) \tag{4.13}$$

Corollary 10

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}\aleph_1(\;\mathrm{z}\ \mathrm{t}^{\sigma}(b-at)^{-\omega}\;)\aleph_2(\;\mathrm{z'}\ \mathrm{t}^{\sigma'}(b-at)^{-\omega'}\;)\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}\aleph_{3,3:W_1}^{0,3;V_1}\left(\begin{array}{c|c}\mathrm{z}\ \frac{x^{\sigma}}{b^{\omega}} & \mathbb{A}_1,\ \mathbf{A}_1\\ \mathrm{z'}\ \frac{x^{\sigma'}}{b^{\omega'}} & \cdot\\ -\frac{ax}{b} & \cdot\\ & \mathbb{B}_1,\ \mathbf{B}_1\end{array}\right) \tag{4.14}$$

The validity conditions are the same that (3.15) with $r=s=1$, where V_1,W_1 are equals to V,W respectively for $r=s=1$ and

$$\mathbb{A}_1=\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\omega_1',1\right),\left(\mu-\beta+\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_1',1\right),\left(\mu-\eta+\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_1',1\right) \tag{4.15}$$

$$\mathbb{B}_1=\left(1-\upsilon-\sum_{j=1}^{v}\delta_jK_j;\omega_1,\omega_1',0\right),\left(-\alpha-\beta-\eta+\mu+\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_1',1\right),\left(\mu+\sum_{j=1}^{v}\lambda_jK_j;\sigma_1,\sigma_1',1\right) \tag{4.16}$$

b) I-function of one variable

If $r=s=1;\tau_{i^{(1)}},\iota_{i^{(1)}}\to 1$, the multivariable Aleph-functions reduce in to I-functions of one variable. We have.

Corollary 11

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v}\end{pmatrix}I_1(\;\mathrm{z}\ \mathrm{t}^{\sigma}(b-at)^{-\omega}\;)I_2(\;\mathrm{z'}\ \mathrm{t}^{\sigma'}(b-at)^{-\omega'}\;)\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}I_{3,3:W_1}^{0,3;V_1}\left(\begin{array}{c|c}\mathrm{z}\ \frac{x^{\sigma}}{b^{\omega}} & \mathrm{A}_1,\mathbf{A}_1'\\ \mathrm{z'}\ \frac{x^{\sigma'}}{b^{\omega'}} & \cdot\\ -\frac{ax}{b} & \cdot\\ & \mathrm{B}_1,\mathbf{B}_1'\end{array}\right) \tag{4.17}$$

The validity conditions are the same that (3.1) with $r=s=1;\tau_{i^{(1)}},\iota_{i^{(1)}}\to 1$.

Corollary 12

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1} \\ . \\ . \\ . \\ \mathrm{c}_vt^{\lambda_v}(b-at)^{-\delta_v} \end{pmatrix} I_1\left(\text{ z } \mathrm{t}^{\sigma}(b-at)^{-\omega} \right) I_2\left(\text{ z' } \mathrm{t}^{\sigma'}(b-at)^{-\omega'} \right)\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v\, c_1^{K_1}\cdots c_v^{K_v}b^{-\sum_{j=1}^{v}\delta_jK_j}x^{\sum_{j=1}^{v}\lambda_jK_j}I_{3,3:W_1}^{0,3;V_1}\left(\begin{array}{c|c} \text{z } \frac{x^{\sigma}}{b^{\omega}} & \mathbb{A}_1,\ \mathbf{A}_1' \\ \text{z' } \frac{x^{\sigma'}}{b^{\omega'}} & . \\ & . \\ -\frac{ax}{b} & \mathbb{B}_1,\ \mathbf{B}_1' \end{array}\right) \tag{4.18}$$

The validity conditions are the same that (3.15) with $r=s=1;\tau_{i^{(1)}},\iota_{i^{(1)}}\rightarrow 1$

where V_1,W_1 are equals to V,W respectively for $r=s=1;\tau_{i^{(1)}},\iota_{i^{(1)}}\rightarrow 1$ and $\mathbf{A}_1',\mathbf{B}_1'$ are equals to $\mathbf{A}_1,\mathbf{B}_1$ respectively for $\tau_{i^{(1)}},\iota_{i^{(1)}}\rightarrow 1$.

Remark: By the similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions and a class of multivariable polynomials defined by Srivastava [29].

5. Conclusion

In this paper, we have obtained the Theorems of the generalized fractional integral operators given by Saigo. The images have been developed in terms of the product of the two multivariables Aleph-function and a general class of multivariable polynomials in a compact and elegant form with the help of Saigo operators. Most of the results obtained in this paper are useful in deriving certain composition formulas involving Riemann–Liouville, Erdelyi–Kober fractional calculus operators and multivariable Aleph-functions. The findings of this paper provide an extension of the results given earlier by Kilbas, Kilbas and Saigo, Kilbas and Sebastain, Saxena et al. and Gupta et al. as mentioned earlier.

References

[1] P. Agarwal, Fractional integration of the product of two multivariable H-functions and a general class of polynomials, Advances in Applied Mathematics and Approximation Theory, Springer (2013) , Chapter 23 , 359-374.

[2] F.Y. Ayant, Generalized finite integral involving the multiple logarithm-function, a general class of polynomials,the multivariable Aleph-function,the multivariable I-function I, International Journal of Mathematics Trends of Technology (IJMTT), 48(1) (2017), 6-14.

[3]L.Debnath and D. Bhatta, Integral Transforms and Their Applications, Chapman and Hall/CRC Press, Boca Raton FL, 2006.

[4] K. C. Gupta, K. Gupta and A. Gupta, Generalized fractional integration of the product of two H-functions. J. Rajasthan Acad. Phy. Sci., 9(3), 203–212(2010).

[5] S. L. Kalla, Integral operators involving Fox's H-function I, Acta Mexicana Cienc. Tecn. 3, 117–122, (1969).

[6] S. L. Kalla, Integral operators involving Fox's H-function II, Acta Mexicana Cienc. Tecn. 7, 72–79, (1969).

[7] S. L. Kalla and R. K. Saxena, Integral operators involving hypergeometric functions, Math. Z. 108, 231–234, (1969)

[8] S. L. Kalla and R. K. Saxena, Integral operators involving hypergeometric functions II, Univ. Nac. Tucuman, Rev. Ser., A24, 31–36, (1974).

[9] A. A. Kilbas, Fractional calculus of the generalized Wright function, Fract.Calc.Appl.Anal.8 (2), 113–126, (2005).

[10] A. A. Kilbas and M. Saigo, H-transforms, theory and applications, Chapman & Hall/CRC, Press, Boca Raton, FL, 2004.

[11] A. A. Kilbas and N. Sebastain, Generalized fractional integration of Bessel function of first kind, Integral transform and Spec. Funct. 19(12), 869–883,(2008).

[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, 204(North-Holland Mathematics), Elsevier, 540, 2006.

[13] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman Scientific & Tech., Essex, 1994.

[14] V. Kiryakova, A brief story about the operators of the generalized fractional calculus, Fract. Calc. Appl. Anal. 11 (2), 203–220, (2008).

[15] E. R. Love, Some integral equations involving hypergeometric functions, Proc. Edin. Math. Soc. 15 (3), 169–198, (1967).

[16] O. I. Marichev, Volterra equation of Mellin convolution type with a Horn function in the kernel (In Russian). Izv. AN BSSR Ser. Fiz.-Mat. Nauk 1, 128–129, (1974).

[17] A. C. McBride, Fractional powers of a class of ordinary differential operators, Proc. London, Math. Soc. (III) 45, 519–546, (1982).

[18] K. S. Miller and B. Ross An Introduction to the Fractional Calculus and Differential Equations, A Wiley Interscience Publication, John Wiley and Sons Inc., New York, 1993.

[19] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Univ. 11, 135–143, (1978).

[20] M. Saigo, A certain boundary value problem for the Euler-Darboux equation I, Math.Japonica, 24 (4), 377–385, (1979).

[21] M. Saigo, A certain boundary value problem for the Euler-Darboux equation II, Math. Japonica 25 (2), 211–220, (1980).

[22] S. Samko, A.Kilbas and O. Marichev Fractional Integrals and Derivatives. Theory and Applications, Gordon & Breach Sci. Publ., New York, 1993.

[23] R. K. Saxena, J. Ram and D. L. Suthar, Fractional calculus of generalized Mittag-Leffler functions, J. India Acad. Math.(1),165–172,(2009).

[24] V.P. Saxena, Formal solution of certain new pair of dual integral equations involving H-function, Proc. Nat. Acad. Sci. IndiaSect., (2001), A51, 366–375.

[25] K. Sharma, On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences 3(1) (2014), 1-13.

[26] C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math , 20(2) (1994), 113-116.

[27] C.K. Sharma C.K. and P.L. Mishra, On the I-function of two variables and its properties. Acta Ciencia Indica Math,17 (1991), 667-672.

[28] H. M. Srivastava, K.C. Gupta and S. P. Goyal, The H-function of One and Two Variables with Applications, South Asian Publications, New Delhi, Madras, 1982.

[29] H.M. Srivastava, A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), 183-191.

[30] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975), 119-137.

[31] H.M. Srivastava and R.Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.

[32] H. M. Srivastava and N. P. Singh,The integration n of certain products of the multivariable H-function with a general class of polynomials, Rend. Circ. Mat. Palermo, 32, 157–187, (1983).

[33] N. Südland, B. Baumann and T.F. Nonnenmacher, Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998), 401-402.

[34] N.Sudland, B. Baumann and T.F. Nannenmacher, Fractional drift-less Fokker-Planck equation with power law diffusion coefficients, in V.G. Gangha, E.W. Mayr, W.G. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing (CASC Konstanz 2001), Springer, Berlin, 2001, 513–525