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Fractional Integration of the Product of Two Multivariable I-Functions and a General Class of Polynomials I

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Abstract : A significantly large number of earlier works on the subject of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, summation of series, etc.). The main object of the present paper is to study and develop the Saigo operators. First, we establish two results that give the images of the product of two multivariable I-functions defined by Prathima et al [17] and a general class of multivariable polynomials in Saigo operators. On account of the general nature of the Saigo operators, multivariable I-functions and a general class multivariable polynomials a large number of new and Known Images involving Riemann-Liouville and Erdelyi-Kober fractional integral operators and several special functions notably the I-function of two variables and one variable follow as special cases of our main findings. Results given by Kilbas, Kilbas and Sebastian, Saxena et al. and Gupta et al., follow as special cases of our findings.

Keywords: General class of multivariable polynomial, Saigo operator, multivariable I-function, multivariable H-function, I-function of two variables, I-function of one variable.

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1.Introduction and preliminaries.

The fractional integral operator involving various special functions has found significant importance and applications in various subfields of applicable mathematical analysis. Since last four decades, a number of workers like Love [12], McBride [14], Kalla [3,4], Kalla and Saxena [5,6], Saxena et al. [24], Saigo [21-22], Kilbas [7], Kilbas and Sebastian [9] and Kiryakova [10,11] have studied in depth the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Samko, Kilbas and Marichev [23], Miller and Ross [15]; Kiryakova [10,11], Kilbas, Srivastava and Trujillo [9] and Debnath and Bhatta [1]. A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [18-20], has been introduced by Marichev [13] (see details in Samko et al. [23] and also see Kilbas and Saigo [8])as follows:
Let α,β,η be complex numbers and $x>0$, then the generalized fractional integral operators (the Saigo operators [21]) involving Gaussian hypergeometric function are defined by the following equations:

$$I_{0^+}^{\alpha,\beta,\eta}f(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)\mathrm{d}t(Re(\alpha))>0) \tag{1.1}$$

and

$$I_{-}^{\alpha,\beta,\eta}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty t^{-\alpha-\beta}(x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)\mathrm{d}t(Re(\alpha))>0) \tag{1.2}$$

When $\beta=-\alpha$, equations (1.1) and (1.2) reduce to the following classical Riemann–Liouville fractional integral operator (see Samko et al. [24], p. 94, (5.1), (5.3)):

$$I_{0^+}^{\alpha,-\alpha,\eta}f(x)=I_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\mathrm{d}t(x>0) \tag{1.3}$$

$$I_{0^-}^{\alpha,-\alpha,\eta}f(x)=I_{0-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty(x-t)^{\alpha-1}f(t)\mathrm{d}t(x>0) \tag{1.4}$$

Again, if $\beta=0$ Equations (1.1) and (1.2) reduce to the following Erdelyi–Kober fractional integral operator (see Samko et al. [24], p.322, Eqns. (18.5), (18.6)):

$$I_{0^+}^{\alpha,0,\eta}f(x)=I_{\alpha,\eta}^{+}f(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^{\eta}f(t)\mathrm{d}t(x>0) \quad (1.5)$$

and

$$I_{-}^{\alpha,0,\eta}f(x)=K_{\alpha,\eta}^{-}f(x)=\frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty}(x-t)^{\alpha-1}t^{-\alpha-\eta}f(t)\mathrm{d}t(x>0) \quad (1.6)$$

Recently, Gupta et al. [1] have obtained the images of the product of two H-functions in Saigo operator given by (1.1) and (1.2) and thereby generalized several important results obtained earlier by Kilbas, Kilbas and Sebastian and Saxena et al. as mentioned in this paper cited above. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of this paper is to obtain two results that give the Theorems of the product of two multivariable I-functions defined by Prathima et al [17] and a general class of multivariable polynomials [26] in Saigo operators.
The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I_{p,q:p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r}\end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.7)$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \quad (1.8)$$

and

$$\theta_i(s_i)=\frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \quad (1.9)$$

For more details, see Prathima et al [17].

The integral (2.1) converges absolutely if

$|arg(z_k)|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r$ where

$$\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}>0 \quad (1.10)$$

We shall note $I(z_1,\cdots,z_r) = I_1(z_1,\cdots,z_r)$

Consider the second multivariable I-function.

$$I(z'_1,\cdots,z'_s) = I^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s}_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}\left(\begin{array}{c|c} \mathrm{z'}_1 & (\mathrm{a'}_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(s)}_j;A'_j)_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z'}_s & (\mathrm{b'}_j;\beta'^{(1)}_j,\cdots,\beta'^{(s)}_j;B'_j)_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c'}^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,\gamma'^{(s)}_j;C'^{(s)}_j)_{1,p'_s} \\ \\ (\mathrm{d'}^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)_{1,q'_s} \end{array}\right) = \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\psi(t_1,\cdots,t_s)\prod_{i=1}^{s}\xi_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (1.11)$$

where $\psi(t_1,\cdots,t_s), \xi_i(s_i), i=1,\cdots,s$ are given by :

$$\psi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'}\Gamma^{A'_j}\left(1-a'_j+\sum_{i=1}^{s}\alpha'^{(i)}_j t_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A'_j}\left(a'_j-\sum_{i=1}^{s}\alpha'^{(i)}_j t_j\right)\prod_{j=1}^{q'}\Gamma^{B'_j}\left(1-b'_j+\sum_{i=1}^{s}\beta'^{(i)}_j t_j\right)} \quad (1.12)$$

and

$$\xi_i(s_i) = \frac{\prod_{j=1}^{n'_i}\Gamma^{C'^{(i)}_j}\left(1-c'^{(i)}_j+\gamma'^{(i)}_j t_i\right)\prod_{j=1}^{m'_i}\Gamma^{D'^{(i)}_j}\left(d'^{(i)}_j-\delta'^{(i)}_j t_i\right)}{\prod_{j=n'_i+1}^{p'_i}\Gamma^{C'^{(i)}_j}\left(c'^{(i)}_j-\gamma'^{(i)}_j t_i\right)\prod_{j=m'_i+1}^{q'_i}\Gamma^{D'^{(i)}_j}\left(1-d'^{(i)}_j+\delta'^{(i)}_j t_i\right)} \quad (1.13)$$

For more details, see Prathima et al [17].

$|arg(z'_k)| < \frac{1}{2}\Delta'_k\pi, k=1,\cdots,s$ where

$$\Delta'_k = -\sum_{j=n'+1}^{p'}A'_j\alpha'^{(k)}_j - \sum_{j=1}^{q'}B'_j\beta'^{(k)}_j + \sum_{j=1}^{m'_k}D'^{(k)}_j\delta'^{(k)}_j - \sum_{j=m'_k+1}^{q'_k}D'^{(k)}_j\delta'^{(k)}_j + \sum_{j=1}^{n'_k}C'^{(k)}_j\gamma'^{(k)}_j - \sum_{j=n'_k+1}^{p'_k}C'^{(k)}_j\gamma'^{(k)}_j > 0 \quad (1.14)$$

We shall note $I(z'_1,\cdots,z'_s) = I_2(z'_1,\cdots,z'_s)$

Remark : The multivariable I-function is an extension of the multivariable H-function defined by Srivastava and Panda [28,29].

Srivastava and Garg [26] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(L;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \quad (1.15)$$

The coefficients are $B[L;R_1,\ldots,R_u]$ arbitrary constants, real or complex.

We shall note $b_u = \frac{(-L)_{F_1L_1+\cdots+F_uL_u}B(L;L_1,\cdots,L_u)}{L_1!\cdots L_u!}$

2. Lemma

Lemma 1 (Kilbas and Sebastien ([9], page 871, (15)-(18))

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)}x^{\mu-\beta-1} \quad (2.1)$$

where $\alpha,\beta,\eta\in\mathbb{C}, Re(\alpha)>0$ and $Re(\mu)>max\{0,Re(\beta-\eta)\}$

in particular, if $\beta=-\alpha$ and $\beta=0$ in (2.1), we have respectively

$$\left(I_{0^+}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}x^{\mu-\beta-1}, Re(\alpha)>0, Re(\mu)>0, \tag{2.2}$$

$$\left(I_{\eta,\alpha}^{+}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)}x^{\mu-1}, Re(\alpha)>0, Re(\mu)>-Re(\eta). \tag{2.3}$$

Lemma 2 (Kilbas and Sebastien ([9], page 872, (21)-(24))

$$\left(I_{0^-}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x)=\frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)}x^{\mu-\beta-1} \tag{2.4}$$

where $\alpha,\beta,\eta\in\mathbb{C}, Re(\alpha)>0$ and $Re(\mu)<1+min\{Re(\beta),Re(\eta)\}$

in particular, if $\beta=-\alpha$ and $\beta=0$ in (2.4), we have respectively

$$\left(I_{-}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)}x^{\mu-\beta-1}, 1-Re(\mu)>Re(\alpha)>0 \tag{2.5}$$

$$\left(K_{\eta,\alpha}^{-}t^{\mu-1}\right)(x)=\frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)}x^{\mu-1}, Re(\mu)<Re(\eta)+1. \tag{2.6}$$

3. Main results

We have the following result

Theorem 1

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}b_uc_1^{R_1}\cdots c_u^{R_u}b^{-\sum_{j=1}^u\delta_jR_j}x^{\sum_{j=1}^u\lambda_jR_j}I_{p+p'+3,q+q'+3;Y}^{0,n+n'+3;X}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A},\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B},\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right) \tag{3.1}$$

where

$$X=m_1,n_1;\cdots;m_r,n_r;m'_1,n'_1;\cdots;m'_s,n'_s;1,0 \tag{3.2}$$

$$Y = p_1, q_1; \cdots ; p_r, q_r; p'_1, q'_1; \cdots ; p'_s, q'_s; 0, 1 \tag{3.3}$$

$$A = \left(1 - \upsilon - \sum_{j=1}^{u} \delta_j R_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 1; 1\right), \left(1 - \mu - \sum_{j=1}^{u} \lambda_j R_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1\right),$$

$$\left(1 - \mu - \eta + \beta - \sum_{j=1}^{u} \lambda_j R_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1\right) \tag{3.4}$$

$$\mathbf{A} = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}, 0, \cdots, 0, 0; A_j)_{1,p}, (a'_j; 0, \cdots, 0, {\alpha'_j}^{(1)}, \cdots, {\alpha'_j}^{(s)}, 0; A'_j)_{1,p'} :$$

$$(\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots ; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r}; ({c'_j}^{(1)}, {\gamma'_j}^{(1)}; {C'_j}^{(1)})_{1,p'_1}; \cdots ; ({c'_j}^{(r)}, \gamma_j^{(s)}; {C'_j}^{(s)})_{1,p'_s} \tag{3.5}$$

$$B = \left(1 - \upsilon - \sum_{j=1}^{u} \delta_j R_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0; 1\right), \left(1 - \mu + \beta - \sum_{j=1}^{u} \lambda_j R_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1\right)$$

$$\left(1 - \mu - \eta - \alpha - \sum_{j=1}^{u} \lambda_j R_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1\right) \tag{3.6}$$

$$\mathbf{B} = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}, 0 \cdots, 0, 0; B_j)_{1,q}, (b'_j; 0, \cdots, 0, {\beta'_j}^{(1)}, \cdots, {\beta'_j}^{(s)}, 0; B'_j)_{1,q'} :$$

$$(\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots ; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r}; (\mathrm{d'}_j^{(1)}, {\delta'_j}^{(1)}; {D'_j}^{(1)})_{1,q'_1}; \cdots ; ({d'_j}^{(s)}, {\delta'_j}^{(s)}; {D'_j}^{(s)})_{1,q'_s}; (0, 1; 1) \tag{3.7}$$

Provided that

$a, b, c, \alpha, \beta, \eta, \mu, \upsilon, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \cdots, u; i = 1, \cdots, r; j = 1, \cdots, s$

$\lambda_k, \sigma_i, \sigma'_j > 0, k = 1, \cdots, u; i = 1, \cdots, r; j = 1, \cdots, s$

$|arg(z_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, r$ where Δ_k is defined by (1.10).

$|arg(z'_k)| < \frac{1}{2}\Delta'_k \pi, k = 1, \cdots, s$ where Δ'_k is defined by (1.14).

$$Re\left[\mu + \sum_{i=1}^{r} \sigma_i \min_{1 \leqslant l \leqslant m_i} \frac{d_l^{(i)}}{\delta_l^{(i)}} + \sum_{j=1}^{s} \sigma'_j \min_{1 \leqslant l' \leqslant m'_j} \frac{{d'}_{l'}^{(j)}}{{\delta'}_{l'}^{(j)}}\right] > max\{0, Re(\beta - \eta)\}$$

$$Re\left[\upsilon + \sum_{i=1}^{r} \omega_i \min_{1 \leqslant l \leqslant m_i} \frac{d_l^{(i)}}{\delta_l^{(i)}} + \sum_{j=1}^{s} \omega'_j \min_{1 \leqslant l' \leqslant m'_j} \frac{{d'}_{l'}^{(j)}}{{\delta'}_{l'}^{(j)}}\right] > max\{0, Re(\beta - \eta)\} \text{ and } \left|\frac{a}{b}x\right| < 1.$$

Proof

To prove (3.1), we first express the class of multivariable polynomials $S_L^{h_1, \cdots, h_u}[.]$ in series with the help of (1.15), the multivariable I-functions in terms of Mellin-Barnes type contour integrals with the help of (1.7) and (1.11) respectively.

Now interchange the order of summations and two multiple Mellin-Barnes contour integrals, respectively and taking the fractional integral operator inside (which is permissible under the stated conditions) and make simplifications. Next, we express the terms $(b-ax)^{-\upsilon-\sum_{k=1}^{u}\delta_k R_k-\sum_{j=1}^{r}\omega_i s_i-\sum_{j=1}^{s}\omega'_j t_j}$ in the terms of Mellin-Barnes contour integral (Srivastava et al [27], page 18, (2.6.3) ; page 10, (2.1.1)) and after algebric manipulations, we obtain

$$\text{L.H.S. of (3.1)} = b^{-\upsilon}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} b_u\, c_1^{R_1}\cdots c_u^{R_u}\, b^{-\sum_{j=1}^{u}\delta_j R_j}\,\frac{1}{(2i\pi)^{r+s+1}}\int_{L_1}\cdots\int_{L_r}\int_{L'_1}\cdots\int_{L'_s}\psi(s_1,\cdots,s_r)$$

$$\zeta(t_1,\cdots,t_s)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\prod_{i=1}^{s}\xi_i(t_i)z_i'^{t_i}b^{-\sum_{j=1}^{r}\omega_i s_i-\sum_{j=1}^{s}\omega'_j t_j}$$

$$\int_L\frac{\Gamma(\upsilon+\sum_{k=1}^{u}\delta_k R_k+\sum_{i=1}^{r}\omega_i s_i+\sum_{j=1}^{s}\omega'_j t_j+u)}{\Gamma(\upsilon+\sum_{k=1}^{u}\delta_k R_k+\sum_{i=1}^{r}\omega_i s_i+\sum_{j=1}^{s}\omega'_j t_j)\Gamma(1+u)}\left(-\frac{a}{b}\right)^u$$

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\mu+\sum_{k=1}^{u}\lambda_k R_k+\sum_{i=1}^{r}\sigma_i s_i+\sum_{j=1}^{s}\sigma_j t_j+u-1}\right)(x)\mathrm{d}u\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{3.8}$$

Now using the Lemma 1 and interpreting the $(r+s+1)-$Mellin-barnes contour integral of (3.8) in multivariable I-function of $(r+s+1)-$variables, we obtain the desired result.

If we put $\beta=-\alpha$ in Theorem 1, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.3) and using (2.2):

Corollary 1

$$\left\{I_{0+}^{\alpha}\left(t^{\mu-1}(b-at)^{-\upsilon}\,S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}b_uc_1^{R_1}\cdots c_u^{R_u}b^{-\sum_{j=1}^{u}\delta_jR_j}x^{\sum_{j=1}^{u}\lambda_jR_j}I_{p+p'+2,q+q'+2;Y}^{0,n+n'+2;X}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A'},\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B'},\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right) \tag{3.9}$$

where

$$A'=\left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1;1\right),\ \left(1-\mu-\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.10}$$

$$B'=\left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0;1\right),\ \left(1-\mu-\alpha-\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1\right) \tag{3.11}$$

with the same validity conditions that (3.1).

If $\beta = 0$ in Theorem 1, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.5) and using (2.3) :

$$\left\{I_{\eta,\alpha}^{+}\left(t^{\mu-1}(b-at)^{-\upsilon}\,S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} b_u c_1^{R_1}\cdots c_u^{R_u} b^{-\sum_{j=1}^{u}\delta_jR_j}x^{\sum_{j=1}^{u}\lambda_jR_j} I_{p+p'+2,q+q'+2;Y}^{0,n+n'+2;X}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A}'',\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B}'',\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right) \tag{3.12}$$

where

$$A'' = \left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1;1\right),\ \left(1-\mu-\eta-\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.13}$$

$$B'' = \left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0;1\right),\ \left(1-\mu-\alpha-\eta-\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.14}$$

Provided that

$a,b,c,\alpha,\beta,\eta,\mu,\upsilon,\delta_k,\omega_i,\omega'_j\in\mathbb{C}, k=1,\cdots,u; i=1,\cdots,r; j=1,\cdots,s$

$\lambda_k,\sigma_i,\sigma'_j>0, k=1,\cdots,u; i=1,\cdots,r; j=1,\cdots,s$

$|arg(z_k)|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r$ where Δ_k is defined by (1.10).

$|arg(z'_k)|<\frac{1}{2}\Delta'_k\pi, k=1,\cdots,s$ where Δ'_k is defined by (1.14).

$$Re\left[\mu+\sum_{i=1}^{r}\sigma_i\min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}}+\sum_{j=1}^{s}\sigma'_j\min_{1\leqslant l'\leqslant m'_j}\frac{d_{l'}^{\prime(j)}}{\delta_{l'}^{\prime(j)}}\right]>max\{0,Re(-\eta)\}$$

$$Re\left[\upsilon+\sum_{i=1}^{r}\omega_i\min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}}+\sum_{j=1}^{s}\omega'_j\min_{1\leqslant l'\leqslant m'_j}\frac{d_{l'}^{\prime(j)}}{\delta_{l'}^{\prime(j)}}\right]>max\{0,Re(-\eta)\}\text{ and }\left|\frac{a}{b}x\right|<1.$$

Theorem 2

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} b_u c_1^{R_1}\cdots c_u^{R_u} b^{-\sum_{j=1}^u\delta_jR_j}x^{\sum_{j=1}^u\lambda_jR_j} I_{p+p'+3,q+q'+3;Y}^{0,n+n'+3;X}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A},\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B},\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right) \tag{3.15}$$

where

$$\mathbb{A}=\left(1-\upsilon-\sum_{j=1}^u\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1;1\right),\ \left(\mu-\eta+\sum_{j=1}^u\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right),$$

$$\left(-\beta+\mu+\sum_{j=1}^u\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.16}$$

$$\mathbb{B}=\left(1-\upsilon-\sum_{j=1}^u\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0;1\right),\left(-\alpha-\beta-\eta+\mu+\sum_{j=1}^u\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right),$$

$$\left(\mu+\sum_{j=1}^u\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.17}$$

Provided that

$a,b,c,\alpha,\beta,\eta,\mu,\upsilon,\delta_k,\omega_i,\omega'_j\in\mathbb{C},k=1,\cdots,u;i=1,\cdots,r;j=1,\cdots,s$

$\lambda_k,\sigma_i,\sigma'_j>0,k=1,\cdots,u;i=1,\cdots,r;j=1,\cdots,s$

$|arg(z_k)|<\frac{1}{2}\Delta_k\pi,k=1,\cdots,r$ where Δ_k is defined by (1.10).

$|arg(z'_k)|<\frac{1}{2}\Delta'_k\pi,k=1,\cdots,s$ where Δ'_k is defined by (1.14).

$$Re\left[\mu-\sum_{i=1}^r\sigma_i\min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}}-\sum_{j=1}^s\sigma'_j\min_{1\leqslant l'\leqslant m'_j}\frac{d'^{(j)}_{l'}}{\delta'^{(j)}_{l'}}\right]<1+min\{Re(\beta),Re(\eta)\}$$

$$Re\left[\upsilon-\sum_{i=1}^{r}\omega_i\min_{1\leqslant l\leqslant m_i}\frac{d_l^{(i)}}{\delta_l^{(i)}}-\sum_{j=1}^{s}\omega'_j\min_{1\leqslant l'\leqslant m'_j}\frac{d'^{(j)}_{l'}}{\delta'^{(j)}_{l'}}\right]<1+min\{Re(\beta),Re(\eta)\}\text{ and }\left|\frac{a}{b}x\right|<1.$$

The proof of the Theorem 2 is similar that Theorem 1 (use the Lemma 2).

If we put $\beta=-\alpha$ in Theorem 2, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.4) and using (2.5):

Corollary 3

$$\left\{I_-^{\alpha}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}b_uc_1^{R_1}\cdots c_u^{R_u}b^{-\sum_{j=1}^{u}\delta_jR_j}x^{\sum_{j=1}^{u}\lambda_jR_j}I_{p+p'+2,q+q'+2;Y}^{0,n+n'+2;X}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}',\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B}',\ \mathbf{B}\\ -\frac{ax}{b} & \end{array}\right)\tag{3.18}$$

where

$$\mathbb{A}'=\left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1;1\right),\ \left(\alpha+\mu+\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right)\tag{3.21}$$

$$\mathbb{B}'=\left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0;1\right),\ \left(\mu+\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right)\tag{3.22}$$

with the same validity conditions that (3.15).

If we put $\beta=0$ in Theorem 2, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (1.4) and using (2.6):

Corollary 4

$$\left\{K_{\eta,\alpha}^{-}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_st^{\sigma'_s}(b-at)^{-\omega'_s}\end{pmatrix}\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} b_u c_1^{R_1}\cdots c_u^{R_u} b^{-\sum_{j=1}^{u}\delta_jR_j} x^{\sum_{j=1}^{u}\lambda_jR_j} I_{p+p'+2,q+q'+2;Y}^{0,n+n'+2;X}\left(\begin{array}{c|c} \mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}'',\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot \\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B}'',\ \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \tag{3.23}$$

where

$$\mathbb{A}'' = \left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,1;1\right),\ \left(-\alpha-\eta+\mu+\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.24}$$

$$\mathbb{B}'' = \left(1-\upsilon-\sum_{j=1}^{u}\delta_jR_j;\omega_1,\cdots,\omega_r,\omega'_1,\cdots,\omega'_s,0;1\right),\ \left(\mu-\alpha-\eta+\sum_{j=1}^{u}\lambda_jR_j;\sigma_1,\cdots,\sigma_r,\sigma'_1,\cdots,\sigma'_s,1;1\right) \tag{3.25}$$

with the same validity conditions that (3.15).

Remark : We have the same relations with the multivariable H-functions.

4. Particular cases

The generalized fractional integral operator Theorem 1 and Theorem 2 established here are unified in nature and act as key formulae. Thus the product of general class of polynomials involved in Images 1 and 2 reduces to a large spectrum of polynomials listed by Srivastava and Singh ([30], pp. 158–161), and so from Theorem1 and Theorem 2 we can further obtain various fractional integral results involving a number of simpler polynomials. Again, the multivariable I-function occurring in these Theorems can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of I-function of two variables defined by Rathie et al [19], I-function of one variable defined by Rathie [18]. For example

a) Srivastava-Daoust polynomial [25]

$$\text{If}\ \ B(L;R_1,\cdots,R_u) = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}} \tag{4.1}$$

we have

Corollary 5

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} \mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{array}\right)I_1\left(\begin{array}{c} \mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_rt^{\sigma_r}(b-at)^{-\omega_r}\end{array}\right)\right.\right.$$

$$I_2\left(\begin{array}{c} z'_1 t^{\sigma'_1}(b-at)^{-\omega'_1} \\ \cdot \\ \cdot \\ \cdot \\ z'_s t^{\sigma'_s}(b-at)^{-\omega'_s} \end{array}\right)\Bigg)\Bigg\}(x) = b^{-\upsilon} x^{\mu-\beta-1} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} b'_u c_1^{R_1}\cdots c_u^{R_u}$$

$$b^{-\sum_{j=1}^{u}\delta_j R_j} x^{\sum_{j=1}^{u}\lambda_j R_j} I^{0,n+n'+3;X}_{p+p'+3,q+q'+3;Y}\left(\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot \\ z'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathrm{B},\ \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \tag{4.2}$$

where $b'_u = \dfrac{(-L)_{h_1R_1+\cdots+h_uR_u} B(L;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[L;R_1,\ldots,R_u]$ is defined by (4.1)

The validity conditions are the same that (3.1).

Corollary 6

$$\Big\{I^{\alpha,\beta,\eta}_{0^-}\left(t^{\mu-1}(b-at)^{-\upsilon} F^{1+\bar{A}:B';\cdots;B^{(u)}}_{\bar{C}:D';\cdots;D^{(u)}}\left(\begin{array}{c} c_1 t^{\lambda_1}(b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ c_u t^{\lambda_u}(b-at)^{-\delta_u} \end{array}\right) I_1\left(\begin{array}{c} z_1 t^{\sigma_1}(b-at)^{-\omega_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r t^{\sigma_r}(b-at)^{-\omega_r} \end{array}\right)\right.$$

$$I_2\left(\begin{array}{c} z'_1 t^{\sigma'_1}(b-at)^{-\omega'_1} \\ \cdot \\ \cdot \\ \cdot \\ z'_s t^{\sigma'_s}(b-at)^{-\omega'_s} \end{array}\right)\Bigg)\Bigg\}(x) = b^{-\upsilon} x^{\mu-\beta-1} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} b'_u c_1^{R_1}\cdots c_u^{R_u}$$

$$b^{-\sum_{j=1}^{u}\delta_j R_j} x^{\sum_{j=1}^{u}\lambda_j R_j} I^{0,n+n'+3;X}_{p+p'+3,q+q'+3;Y}\left(\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \cdot \\ z'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z'_s\frac{x^{\sigma'_s}}{b^{\omega'_s}} & \mathbb{B},\ \mathbf{B} \\ -\frac{ax}{b} & \end{array}\right) \tag{4.3}$$

The validity conditions are the same that (3.15).

b) I-function of two variables.

If $r = 2$, then the multivariable I-functions reduce in to I-function of two variables. We have.

Corollary 7

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2t^{\sigma_2}(b-at)^{-\omega_2}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2t^{\sigma'_2}(b-at)^{-\omega'_2}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}b_uc_1^{R_1}\cdots c_u^{R_u}b^{-\sum_{j=1}^u\delta_jR_j}x^{\sum_{j=1}^u\lambda_jR_j}I_{p+p'+3,q+q'+3;Y_2}^{0,n+n'+3;X_2}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathrm{A}_2,\mathbf{A}_2\\ \cdot & \cdot\\ \mathrm{z}_2\frac{x^{\sigma_2}}{b^{\omega_2}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_2\frac{x^{\sigma'_2}}{b^{\omega'_2}} & \cdot\\ -\frac{ax}{b} & \mathrm{B}_2,\mathbf{B}_2\end{array}\right) \tag{4.4}$$

The validity conditions are the same that (3.1) with $r=s=2$ and

$$A_2=\left(1-\upsilon-\sum_{j=1}^u\delta_jR_j;\omega_1,\omega_2,\omega'_1,\omega'_2,1;1\right),\left(1-\mu-\sum_{j=1}^u\lambda_jR_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1;1\right),$$

$$\left(1-\mu-\eta+\beta-\sum_{j=1}^u\lambda_jR_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1;1\right) \tag{4.5}$$

$$B_2=\left(1-\upsilon-\sum_{j=1}^u\delta_jR_j;\omega_1,\omega_2,\omega'_1,\omega'_2,0;1\right),\left(1-\mu+\beta-\sum_{j=1}^u\lambda_jR_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1;1\right)$$

$$\left(1-\mu-\eta-\alpha-\sum_{j=1}^u\lambda_jR_j;\sigma_1,\sigma_2,\sigma'_1,\sigma'_2,1;1\right) \tag{4.6}$$

Corollary 8

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_ut^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\begin{pmatrix}\mathrm{z}_1t^{\sigma_1}(b-at)^{-\omega_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2t^{\sigma_2}(b-at)^{-\omega_2}\end{pmatrix}I_2\begin{pmatrix}\mathrm{z}'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2t^{\sigma'_2}(b-at)^{-\omega'_2}\end{pmatrix}\right)\right\}(x)$$

$$=b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}b_uc_1^{R_1}\cdots c_u^{R_u}b^{-\sum_{j=1}^u\delta_jR_j}x^{\sum_{j=1}^u\lambda_jR_j}I_{p+p'+3,q+q'+3;Y_2}^{0,n+n'+3;X_2}\left(\begin{array}{c|c}\mathrm{z}_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \mathbb{A}_2,\ \mathbf{A}_2\\ \cdot & \cdot\\ \mathrm{z}_2\frac{x^{\sigma_2}}{b^{\omega_2}} & \cdot\\ \mathrm{z}'_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_2\frac{x^{\sigma'_2}}{b^{\omega'_2}} & \cdot\\ -\frac{ax}{b} & \mathbb{B}_2,\ \mathbf{B}_2\end{array}\right) \tag{4.7}$$

The validity conditions are the same that (3.15) with $r=s=2$.

where $X_2, Y_2, \mathbf{A}_2, \mathbf{B}_2$ are equals to $X, Y, \mathbf{A}, \mathbf{B}$ respectively for $r=s=2$ and

$$\mathbb{A}_2 = \left(1-\upsilon-\sum_{j=1}^{u}\delta_j R_j;\omega_1,\omega_2,\omega_1',\omega_2',1;1\right), \left(\mu-\eta+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_2,\sigma_1',\sigma_2',1;1\right),$$

$$\left(-\beta+\mu+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_2,\sigma_1',\sigma_2',1;1\right) \tag{4.8}$$

$$\mathbb{B}_2 = \left(1-\upsilon-\sum_{j=1}^{u}\delta_j R_j;\omega_1,\omega_2,\omega_1',\omega_2',0;1\right), \left(-\alpha-\beta-\eta+\mu+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_2,\sigma_1',\sigma_2',1;1\right),$$

$$\left(\mu+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_2,\sigma_1',\sigma_2',1;1\right) \tag{4.9}$$

c) I-function of one variable

If $r=s=1$, the multivariables I-functions reduce in to I-functions of one variable. We obtain.

Corollary 9

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix} \mathrm{c}_1 t^{\lambda_1}(b-at)^{-\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{c}_u t^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix} I_1\left(\ \mathrm{z}\ \mathrm{t}^{\sigma}(b-at)^{-\omega}\ \right) I_2\left(\ \mathrm{z'}\ \mathrm{t}^{\sigma'}(b-at)^{-\omega'}\ \right)\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} b_u c_1^{R_1}\cdots c_u^{R_u} b^{-\sum_{j=1}^{u}\delta_j R_j} x^{\sum_{j=1}^{u}\lambda_j R_j} I_{3,3;Y_1}^{0,3;X_1}\left(\begin{array}{c|c} \mathrm{z}\ \frac{x^{\sigma}}{b^{\omega}} & \mathrm{A}_1, \mathbf{A}_1 \\ \mathrm{z'}\ \frac{x^{\sigma'}}{b^{\omega'}} & \cdot \\ -\frac{ax}{b} & \cdot \\ & \mathrm{B}_1, \mathbf{B}_1 \end{array}\right) \tag{4.10}$$

The validity conditions are the same that (3.1) with $r=s=1$ and

$$A_1 = \left(1-\upsilon-\sum_{j=1}^{u}\delta_j R_j;\omega_1,\omega_1',1;1\right), \left(1-\mu-\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right),$$

$$\left(1-\mu-\eta+\beta-\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right) \tag{4.11}$$

$$B_1 = \left(1-\upsilon-\sum_{j=1}^{u}\delta_j R_j;\omega_1,\omega_1',0;1\right),\left(1-\mu+\beta-\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right),$$

$$\left(1-\mu-\eta-\alpha-\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1\right) \tag{4.12}$$

and

$$\mathbf{A}_1 = (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};(c_j'^{(1)},\gamma_j'^{(1)};C_j'^{(1)})_{1,p_1'} \tag{4.13}$$

$$\mathbf{B}_1 = (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};(\mathrm{d'}_j^{(1)},\delta_j'^{(1)};D_j'^{(1)})_{1,q_1'};(0,1;1) \tag{4.14}$$

Corollary 10

$$\left\{I_{0^-}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\upsilon}S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{c}_1 t^{\lambda_1}(b-at)^{-\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{c}_u t^{\lambda_u}(b-at)^{-\delta_u}\end{pmatrix}I_1\left(\ \mathrm{z}\ \mathrm{t}^{\sigma}(b-at)^{-\omega}\ \right)I_2\left(\ \mathrm{z'}\ \mathrm{t}^{\sigma'}(b-at)^{-\omega'}\ \right)\right)\right\}(x)$$

$$= b^{-\upsilon}x^{\mu-\beta-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} b_u c_1^{R_1}\cdots c_u^{R_u} b^{-\sum_{j=1}^{u}\delta_j R_j} x^{\sum_{j=1}^{u}\lambda_j R_j} I_{3,3;Y_1}^{0,3;X_1}\left(\begin{array}{c|c}\mathrm{z}\ \frac{x^{\sigma}}{b^{\omega}} & \mathbb{A}_1,\ \mathbf{A}_1\\ \mathrm{z'}\ \frac{x^{\sigma'}}{b^{\omega'}} & \cdot\\ -\frac{ax}{b} & \mathbb{B}_1,\ \mathbf{B}_1\end{array}\right) \tag{4.15}$$

The validity conditions are the same that (3.15) with $r=s=1$, where X_1,Y_1 are equals to X,Y for $r=s=1$ and

$$\mathbb{A}_1 = \left(1-\upsilon-\sum_{j=1}^{u}\delta_j R_j;\omega_1,\omega_1',1;1\right),\left(\mu-\beta+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right),\left(\mu-\eta+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right) \tag{4.16}$$

$$\mathbb{B}_1 = \left(1-\upsilon-\sum_{j=1}^{u}\delta_j R_j;\omega_1,\omega_1',0;1\right),\left(-\alpha-\beta-\eta+\mu+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right),$$

$$\left(\mu+\sum_{j=1}^{u}\lambda_j R_j;\sigma_1,\sigma_1',1;1\right) \tag{4.17}$$

Remark: By the similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by prathima et al [17] and a class of multivariable polynomials defined by Srivastava and Garg [26].

5. Conclusion

In this paper, we have obtained the Theorems of the generalized fractional integral operators given by Saigo. The images have been developed in terms of the product of the two multivariables Aleph-function and a general class of multivariable polynomials in a compact and elegant form with the help of Saigo operators. Most of the results obtained in this paper are useful in deriving certain composition formulas involving Riemann–Liouville, Erdelyi–Kober fractional calculus operators and multivariable Aleph-functions. The findings of this paper provide an extension of the results given earlier by Kilbas, Kilbas and Saigo, Kilbas and Sebastain, Saxena et al. and Gupta et al. as mentioned earlier.

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