

Subclasses of p-valent Functions involving Modified q-Sigmoid

Olubunmi A. Fadipe-Joseph^{1,*}, O. D. Olawumi², U. A. Ezeafulukwe³ Catherine N. Ejieji⁴ and E. O. Titiloye⁵

¹ Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria,
famelov@unilorin.edu.ng; famelov@gmail.com

² Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria
ssdolawumi2@gmail.com

³ Department of Mathematics, University of Nigeria, Nsukka, Nigeria,
uzoamaka.ezeafulukwe@unn.edu.ng

⁴ Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria
ejieji.cn@unilorin.edu.ng

⁵ Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria
eotitiloye@gmail.com

Abstract

In this work, two classes of multivalent functions were defined. The initial coefficient bounds for the subclasses of multivalent functions involving modified q-sigmoid were investigated. The work was concluded by establishing the Fekete-Szegö functional and Hankel determinant for the classes of functions defined. The results generalised some earlier ones in literature.

Mathematics Subject Classification: 30C45.

Keywords: Analytic functions, Multivalent function, Subordination, q-sigmoid functions, Fekete-Szegö functional and Hankel determinant.

1 Introduction and Preliminaries

Let A_p represent the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (1.1)$$

which are analytic and p-valent in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, $p \in \mathbb{N} \sim \{1\}$, $z \in \mathbb{U}$.

Note: $A_1 = A$

Let m and M be analytic in the unit disk \mathbb{U} . A function m is said to be subordinate to M , written as $m \prec M$ or $m(z) \prec M(z)$, if there exists a function ω , analytic in \mathbb{U} , with $\omega(0) = 0$, $|\omega(z)| < 1$, and $m(z) = M(\omega(z))$ for $z \in \mathbb{U}$.

The set P is the set of all functions of the form

$$f(z)1 + p_1 z + p_2 z^2 + \dots + p_n z^n + \dots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

that are regular in \mathbb{U} and such that for $z \in \mathbb{U}$

$$\operatorname{Re}(f(z)) > 0$$

Any function in P is called a function with positive real part in \mathbb{U} . See [4]

Lemma 1.1. [3] If $\omega(z) = b_1 z + b_2 z^2 + \dots$, $b_1 \neq 0$ is analytic and satisfy $|\omega(z)| < 1$ in the unit disk \mathbb{U} , then for each $0 < r < 1$, $|\omega'(z)| < 1$ and $|\omega(re^{i\theta})| < 1$ unless $\omega(z) = e^{i\sigma} z$ for some real number σ .

*Olubunmi A. Fadipe-Joseph

Lemma 1.2. [3] Let $\omega \in \Omega = \{\omega \in A : |\omega(z)| \leq |z|, (z \in \mathbb{U})\}$.
 If $\omega \in \Omega$, $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ ($z \in \mathbb{U}$), then

$$|c_k| \leq 1, \quad k \in \mathbb{N}, \quad |c_2| \leq 1 - |c_1|^2$$

and

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\} \quad (\mu \in \mathbb{C}).$$

The result is sharp. The functions

$$\omega(z) = z, \quad \omega_a(z) = z \frac{z+a}{1+\bar{a}z} \quad (z \in \mathbb{U}, |a| < 1)$$

are extremal functions.

Special function theory was developed by several researchers in the nineteen century. However in the twentieth century, other fields have employed the theory of special functions.

Sigmoid function which is defined as

$$h(z) = \frac{1}{1+e^{-z}} \quad (1.2)$$

is an example of activation function which is a special function. Many researchers have established results connecting sigmoid function and geometric function theory. For details see [1],[3],[5],[6],[7],[8],[9],[10] and [11]

Lemma 1.3. [2] Let $h(z)$ be a sigmoid function and

$$G(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{k=1}^{\infty} \frac{(-1)^k}{[k]!} z^k \right]^m$$

then $G(z) \in P$, where $G(z)$ is a modified sigmoid function.

Definition 1.4. [1] A q-analogue of the ordinary exponential function $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ is defined by

$$e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{[k]q!}$$

Definition 1.5. [1] A q-Sigmoid function is defined by

$$G_q(z) = \frac{1}{1+e_q^{-z}} \quad (1.3)$$

Definition 1.6. [1] A modified q-sigmoid is defined by

$$\gamma_{q,m,k}(z) = \frac{2}{1+e_q^{-z}} = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{k=1}^{\infty} \frac{(-1)^k}{[k]_q!} z^k \right]^m \right) \quad (1.4)$$

Definition 1.7. For $b \in \mathbb{C}$. Let the class $M_{p,\lambda}(b, \gamma_{q,m,k}(z))$ denote the subclass of A_p satisfying

$$p + \frac{1}{b} \left[\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda zf'(z) + (1-\lambda)f(z)} - p \right] \prec p\gamma_{q,m,k}(z)$$

for $0 \leq \lambda \leq 1$ and $\gamma_{q,m,k}(z)$ is as defined in (1.5).

Definition 1.8. For $b \in \mathbb{C}$. Let the class $G_{p,\lambda}(b, \gamma_{q,m,k}(z))$ denote the subclass of A_p satisfying

$$p + \frac{1}{b} \left[\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - p \right] \prec p\gamma_{q,m,k}(z)$$

for $0 \leq \lambda \leq 1$ and $\gamma_{q,m,k}(z)$ is as defined in (1.5).

In this work, the coefficient estimates for a class subordinated to modified q-sigmoid is established.

2 Initial Coefficients

The first few coefficient estimates for the classes of $M_{p,\lambda}(b, \gamma_{q,m,k}(z))$ and $G_{p,\lambda}(b, \gamma_{q,m,k}(z))$ were obtained as follows.

Theorem 2.1. If $f(z)$ of the form (1.1) belongs to $M_{p,\lambda}(b, \gamma_{q,m,k}(z))$ then

$$|a_{1+p}| \leq \frac{|b|p(1+\lambda(p-1))}{2[1]_q!(1+p\lambda)}$$

$$|a_{2+p}| \leq \frac{|b|p(p\lambda+1-\lambda)}{2(1+\lambda(1+p))} \left| \frac{1}{2[1]_q!} + A + \frac{bp}{4([1]_q!)^2} \right|$$

$$|a_{3+p}| \leq \frac{|b|p((p\lambda+1-\lambda))}{3(1+2\lambda+p\lambda)} \left| \frac{1}{2[1]_q!} + 2A + B + \frac{bp}{(2[1]_q!)^2} + \frac{Abp}{2[1]_q!} + \frac{bp}{4[1]_q!} \left(\frac{1}{2[1]_q!} + A + \frac{bp}{(2[1]_q!)^2} \right) \right|$$

where

$$A = \left(\frac{[2]_q! - 2([1]_q!)^2}{4([1]_q!)^2[2]_q!} \right)$$

$$B = \left(\frac{4([1]_q!)^3[2]_q! - 4([1]_q!)^2[3]_q! + [2]_q![3]_q!}{8([1]_q!)^3[2]_q![3]_q!} \right)$$

Proof:

Suppose $f(z) \in M_{p,\lambda}(b, \gamma_{q,m,k}(z))$, then by definition and simple calculation, we have

$$\begin{aligned} \frac{1}{b} [(1+p\lambda)a_{1+p}z^{1+p} + (2+2\lambda+2p\lambda)a_{2+p}z^{2+p} + (3+6\lambda+3p\lambda)a_{3+p}z^{3+p} + \dots] &= \frac{(p\lambda+1-\lambda)pc_1}{2[1]_q!} z^{p+1} \\ &+ \left[\frac{(p\lambda+1-\lambda)pc_2}{2[1]_q!} + A(p\lambda+1-\lambda)pc_1^2 + \frac{pc_1(1+p\lambda)a_{1+p}}{2[1]_q!} \right] z^{p+2} + \left[\frac{(p\lambda+1-\lambda)pc_3}{2[1]_q!} + 2A(p\lambda+1-\lambda)pc_1c_2 + \right. \\ &\quad \left. B(p\lambda+1-\lambda)pc_1^3 + \frac{pc_2(1+p\lambda)a_{1+p}}{2[1]_q!} + Apc_1^2(1+p\lambda)a_{1+p} + \frac{pc_1(1+\lambda+p\lambda)a_{2+p}}{2[1]_q!} \right] z^{p+3} + \dots \end{aligned} \quad (2.1)$$

Comparing coefficients of z^{p+1}, z^{p+2} and z^{p+3} in (1.1), we obtain $a_{1+p}, a_{2+p}, a_{3+p}$

$$a_{1+p} = \frac{(p\lambda+1-\lambda)bp c_1}{(1+p\lambda)2[1]_q!} \quad (2.2)$$

$$a_{2+p} = \frac{bp(p\lambda+1-\lambda)}{(2+2\lambda+2p\lambda)} \left[\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2} \right] \quad (2.3)$$

$$\begin{aligned} a_{3+p} &= \frac{bp(p\lambda+1-\lambda)}{(3+6\lambda+3p\lambda)} \left[\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3 + \frac{bpc_1c_2}{(2[1]_q!)^2} + A \frac{bpc_1^3}{2[1]_q!} + \right. \\ &\quad \left. \frac{bpc_1(1+\lambda+p\lambda)}{(2+2\lambda+2p\lambda)2[1]_q!} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2} \right) \right] \end{aligned} \quad (2.4)$$

Corollary 2.2. If $f(z) \in M_{1,\lambda}(b, \gamma_{q,m,k}(z))$, then

$$|a_2| \leq \left| \frac{b}{2[1]_q!(\lambda+1)} \right|$$

$$|a_3| \leq \frac{|b|}{2(2\lambda+1)} \left| \frac{1}{2[1]_q!} + A + \frac{b}{4([1]_q!)^2} \right|$$

$$|a_4| \leq \frac{|b|}{3(1+3\lambda)} \left| \frac{1}{2[1]_q!} + 2A + B + \frac{b}{(2[1]_q!)^2} + A \frac{b}{2[1]_q!} + \frac{b}{4[1]_q!} \left(\frac{1}{2[1]_q!} + A + \frac{b}{(2[1]_q!)^2} \right) \right|$$

Theorem 2.3. If $f(z)$ of the form (1.1) belongs to $G_{p,\lambda}(b, \gamma_{q,m,k}(z))$ then

$$|a_{1+p}| \leq \frac{|b|p}{2[1]_q!(1+p\lambda+\lambda p^2)}$$

$$|a_{2+p}| \leq \frac{|b|p}{(2+2\lambda+3\lambda p+\lambda p^2)} \left| \frac{p}{2[1]_q!} + Ap + \frac{bp^2}{4([1]_q!)^2(1+\lambda p+\lambda p^2)} \right|,$$

$$|a_{3+p}| \leq \frac{|b|p}{(3+6\lambda+5p\lambda+\lambda p^2)} \left| \frac{1}{2[1]_q!} + 2A + B + \frac{bp}{(2[1]_q!)^2(1+p\lambda+\lambda p^2)} + A \frac{bp}{2[1]_q!(1+p\lambda+\lambda p^2)} + \right. \\ \left. \frac{bp}{2[1]_q!(2+2\lambda+3\lambda p+\lambda p^2)} \left(\frac{1}{2[1]_q!} + A + \frac{bp}{4([1]_q!)^2(1+\lambda p+\lambda p^2)} \right) \right|$$

where

$$A = \left(\frac{[2]_q! - 2([1]_q!)^2}{4([1]_q!)^2[2]_q!} \right)$$

$$B = \left(\frac{4([1]_q!)^3[2]_q! - 4([1]_q!)^2[3]_q! + [2]_q![3]_q!}{8([1]_q!)^3[2]_q![3]_q!} \right)$$

Proof:

Suppose $f(z) \in G_{p,\lambda}(b, \gamma_{q,m,k}(z))$, then by definition and simple calculation, we have

$$\frac{1}{b} \left((p\lambda(p-1))z^p + (1+\lambda p+\lambda p^2)a_{1+p}z^{1+p} + (2+2\lambda+3p\lambda+\lambda p^2)a_{2+p}z^{2+p} + (3+6\lambda+5p\lambda+\lambda p^2)a_{3+p}z^{3+p} + \dots \right) = \\ \frac{pc_1}{2[1]_q!} z^{1+p} + \left(\frac{pc_2}{2[1]_q!} + Ap c_1^2 + \frac{pc_1 a_{1+p}}{2[1]_q!} \right) z^{2+p} + \left(\frac{pc_3}{2[1]_q!} + 2Ap c_1 c_2 + B p c_1^3 + \left(\frac{pc_2}{2[1]_q!} + Ap c_1^2 \right) a_{1+p} + \frac{pc_1}{2[1]_q!} a_{2+p} \right) z^{3+p} + \dots \quad (2.5)$$

Comparing coefficients of z^p , z^{p+1} , z^{p+2} and z^{p+3} in (2.5), we obtain a_{1+p} , a_{2+p} , a_{3+p} ,

$$\frac{(p\lambda(p-1))}{b} = 0 \quad (2.6)$$

$$a_{1+p} = \frac{bpc_1}{2[1]_q!(1+\lambda p+\lambda p^2)} \quad (2.7)$$

$$a_{2+p} = \frac{b}{(2+2\lambda+3p\lambda+\lambda p^2)} \left(\frac{pc_2}{2[1]_q!} + Ap c_1^2 + \frac{bp^2 c_1^2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} \right) \quad (2.8)$$

$$a_{3+p} = \frac{b}{(3+6\lambda+5p\lambda+\lambda p^2)} \left[\frac{pc_3}{2[1]_q!} + 2Ap c_1 c_2 + B p c_1^3 + \frac{bp^2 c_1 c_2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} + \frac{Ab p^2 c_1^3}{2[1]_q!(1+\lambda p+\lambda p^2)} + \right. \\ \left. \frac{bp^2}{2[1]_q!(2+2\lambda+3p\lambda+\lambda p^2)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} \right) \right] \quad (2.9)$$

Corollary 2.4. If $f(z) \in G_{1,\lambda}(b, \gamma_{q,m,k}(z))$, then

$$|a_2| \leq \left| \frac{b}{2[1]_q!(1+2\lambda)} \right|$$

$$|a_3| \leq \frac{|b|}{(2+6\lambda)} \left| \frac{1}{2[1]_q!} + A + \frac{b}{(2[1]_q!)^2(1+2\lambda)} \right|$$

$$|a_4| \leq \frac{|b|}{(3+12\lambda)} \left| \frac{1}{2[1]_q!} + 2A + B + \frac{b}{(2[1]_q!)^2(1+2\lambda)} + \frac{Ab}{2[1]_q!(1+2\lambda)} + \frac{b}{2[1]_q!(2+6\lambda)} \left(\frac{1}{2[1]_q!} + A + \frac{b}{(2[1]_q!)^2(1+2\lambda)} \right) \right|$$

3 The Fekete-Szegö Functional

Theorem 3.1. If $f(z)$ of the form (1.1) belongs to $M_{p,\lambda}(b, \gamma_{q,m,k}(z))$ and $\mu \in \mathbb{R}$ then

$$|a_{p+2} - \mu a_{p+1}^2| \leq |b|p(p\lambda+1-\lambda) \left| \frac{1}{(2+2\lambda+2p\lambda)} \left(\frac{1}{2[1]_q!} + A + \frac{bp}{4([1]_q!)^2} \right) - \mu \frac{pb}{(1+p\lambda)^2(2[1]_q!)^2} \right|$$

Proof: From (2.2) and (2.3), we have,

$$a_{p+2} - \mu a_{p+1}^2 = \frac{bp(p\lambda+1-\lambda)}{(2+2\lambda+2p\lambda)} \left[\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2} \right] - \mu \frac{(p\lambda+1-\lambda)^2 b^2 p^2 c_1^2}{(1+p\lambda)^2 (2[1]_q!)^2}$$

$$a_{p+2} - \mu a_{p+1}^2 = bp(p\lambda + 1 - \lambda) \left[\frac{1}{(2 + 2\lambda + 2p\lambda)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{4([1]_q!)^2} \right) - \mu \frac{pbpc_1^2}{(1 + p\lambda)^2(2[1]_q!)^2} \right]$$

Hence, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq |b|p(p\lambda + 1 - \lambda) \left| \frac{1}{(2 + 2\lambda + 2p\lambda)} \left(\frac{1}{2[1]_q!} + A + \frac{bp}{4([1]_q!)^2} \right) - \mu \frac{pb}{(1 + p\lambda)^2(2[1]_q!)^2} \right| \quad (3.1)$$

which completes the proof.

Corollary 3.2. If $f(z) \in M_{1,\lambda}(b, \gamma_{q,m,k}(z))$, then

$$|a_3 - \mu a_2^2| \leq |b| \left| \frac{1}{(2 + 4\lambda)} \left(\frac{1}{2[1]_q!} + A + \frac{b}{4([1]_q!)^2} \right) - \mu \frac{b}{(1 + \lambda)^2(2[1]_q!)^2} \right|$$

Theorem 3.3. If $f(z)$ of the form (1.1) belongs to $G_{p,\lambda}(b, \gamma_{q,m,k}(z))$ and $\mu \in \mathbb{R}$ then

$$|a_{p+2} - \mu a_{p+1}^2| \leq |b|p \left| \frac{1}{(2 + 2\lambda + 3p\lambda + \lambda p^2)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)} \right) - \mu \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)^2} \right|$$

Proof: From (2.7) and (2.8), we have,

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 &= \frac{bp}{(2 + 2\lambda + 3p\lambda + \lambda p^2)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)} \right) - \mu \frac{b^2 p^2 c_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)^2} \\ a_{p+2} - \mu a_{p+1}^2 &= bp \left[\frac{1}{(2 + 2\lambda + 3p\lambda + \lambda p^2)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)} \right) - \mu \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)^2} \right] \end{aligned}$$

Hence, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq |b|p \left| \frac{1}{(2 + 2\lambda + 3p\lambda + \lambda p^2)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)} \right) - \mu \frac{bpc_1^2}{(2[1]_q!)^2(1 + \lambda p + \lambda p^2)^2} \right| \quad (3.2)$$

which completes the proof.

Corollary 3.4. If $f(z) \in G_{1,\lambda}(b, \gamma_{q,m,k}(z))$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq |b| \left| \frac{1}{(2 + 6\lambda)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bc_1^2}{(2[1]_q!)^2(1 + 2\lambda)} \right) - \mu \frac{bc_1^2}{(2[1]_q!)^2(1 + 2\lambda)^2} \right| \quad (3.3)$$

4 Hankel Determinant

Theorem 4.1. If $f(z)$ of the form (1.1) belongs to $M_{p,\lambda}(b, \gamma_{q,m,k}(z))$ and $\mu \in \mathbb{R}$ then

$$\begin{aligned} |a_{p+1}a_{p+3} - \mu a_{p+2}^2| &\leq |b|^2 p^2 (1 + p\lambda - \lambda)^2 \left| \frac{1}{(3 + 6\lambda + 3p\lambda)(1 + p\lambda)2[1]_q!} \left(\frac{1}{2[1]_q!} + 2A + B + \frac{bp}{(2[1]_q!)^2} + \right. \right. \\ &\quad A \frac{bp}{2[1]_q!} + \frac{bp}{8([1]_q!)^2} + A \frac{bp}{4[1]_q!} + \frac{b^2 p^2}{16([1]_q!)^2} \Big) - \frac{\mu}{(2 + 2\lambda + 2p\lambda)^2} \left(\frac{1}{4[1]_q!^2} + \frac{A}{[1]_q!} + \frac{bp}{4([1]_q!)^3} + A^2 + \right. \\ &\quad \left. \left. A \frac{bp}{2([1]_q!)^2} + \frac{b^2 p^2}{16([1]_q!)^4} \right) \right| \end{aligned}$$

$$\begin{aligned} \text{Proof:} &\text{ From (2.2), (2.3) and (2.4)), we have, } a_{1+p}a_{3+p} - \mu a_{2+p}^2 = \frac{(p\lambda + 1 - \lambda)bpc_1}{(1 + p\lambda)2[1]_q!} \left[\frac{bp(p\lambda + 1 - \lambda)}{(3 + 6\lambda + 3p\lambda)} \left(\frac{c_3}{2[1]_q!} + \right. \right. \\ &2Ac_1c_2 + Bc_3^3 + \frac{bpc_1c_2}{(2[1]_q!)^2} + A \frac{bpc_1^3}{2[1]_q!} + \frac{bpc_1}{4[1]_q!} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2} \right) \Big) \Big] - \mu \left[\frac{bp(p\lambda + 1 - \lambda)}{(2 + 2\lambda + 2p\lambda)} \left(\frac{c_2}{2[1]_q!} + \right. \right. \\ &\quad \left. \left. Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2} \right) \right]^2 \\ a_{p+1}a_{p+3} - \mu a_{p+2}^2 &= b^2 p^2 (1 + p\lambda - \lambda)^2 \left[\frac{c_1}{(3 + 6\lambda + 3p\lambda)(1 + p\lambda)2[1]_q!} \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_3^3 + \frac{bpc_1c_2}{(2[1]_q!)^2} + \right. \right. \\ &A \frac{bpc_1^3}{2[1]_q!} + \frac{bpc_1c_2}{8([1]_q!)^2} + A \frac{bpc_1^3}{4[1]_q!} + \frac{b^2 p^2 c_1^3}{16([1]_q!)^2} \Big) - \frac{\mu}{(2 + 2\lambda + 2p\lambda)^2} \left(\frac{c_2^2}{4[1]_q!^2} + \frac{Ac_1^2 c_2}{[1]_q!} + \frac{bpc_1^2 c_2}{4([1]_q!)^3} + A^2 c_1^4 + \right. \\ &\quad \left. \left. A^2 c_1^4 \right) \right] \end{aligned}$$

$$A \frac{bp c_1^4}{2([1]_q!)^2} + \frac{b^2 p^2 c_1^4}{16([1]_q!)^4} \Big]$$

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leq |b^2| p^2 (1+p\lambda-\lambda)^2 \left| \frac{1}{(3+6\lambda+3p\lambda)(1+p\lambda)2[1]_q!} \left(\frac{1}{2[1]_q!} + 2A+B + \frac{bp}{(2[1]_q!)^2} + A \frac{bp}{2[1]_q!} + \frac{bp}{8([1]_q!)^2} \right. \right. \\ \left. \left. + A \frac{bp}{4[1]_q!} + \frac{b^2 p^2}{16([1]_q!)^2} \right) - \frac{\mu}{(2+2\lambda+2p\lambda)^2} \left(\frac{1}{4[1]_q!^2} + \frac{A}{[1]_q!} + \frac{bp}{4([1]_q!)^3} + A^2 + A \frac{bp}{2([1]_q!)^2} + \frac{b^2 p^2}{16([1]_q!)^4} \right) \right| \quad (4.1)$$

Corollary 4.2. If $f(z) \in M_{1,\lambda}(b, \gamma_{q,m,k}(z))$, then

$$|a_2 a_4 - \mu a_3^2| \leq |b^2| \left| \frac{1}{(3+9\lambda)(1+\lambda)2[1]_q!} \left(\frac{1}{2[1]_q!} + 2A+B + \frac{b}{(2[1]_q!)^2} + A \frac{b}{2[1]_q!} + \frac{b}{8([1]_q!)^2} + A \frac{b}{4[1]_q!} \right. \right. \\ \left. \left. + \frac{b^2}{16([1]_q!)^2} \right) - \frac{\mu}{(2+4\lambda)^2} \left(\frac{1}{4[1]_q!^2} + \frac{A}{[1]_q!} + \frac{b}{4([1]_q!)^3} + A^2 + A \frac{b}{2([1]_q!)^2} + \frac{b^2}{16([1]_q!)^4} \right) \right| \quad (4.2)$$

Theorem 4.3. If $f(z) \in A_p$ of the form (1.1) is belongs to $G_{p,\lambda}(b, \gamma_{q,m,k}(z))$ and $\mu \in \mathbb{R}$ then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leq |b^2| p^2 \left| \frac{c_1}{2[1]_q!(1+\lambda p+\lambda p^2)(3+6\lambda+5p\lambda+\lambda p^2)} \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3 + \frac{bpc_1c_2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} + \right. \right. \\ \left. \left. \frac{Abpc_1^3}{2[1]_q!(1+\lambda p+\lambda p^2)} + \frac{bpc_2}{(2[1]_q!)^2(2+2\lambda+3p\lambda+\lambda p^2)} + \frac{Abpc_1^2}{2[1]_q!(2+2\lambda+3p\lambda+\lambda p^2)} + \frac{b^2 p^2 c_1^3}{(2[1]_q!)^3(1+\lambda p+\lambda p^2)(2+2\lambda+3p\lambda+\lambda p^2)} \right. \right. \\ \left. \left. + \frac{\mu}{(2+2\lambda+3p\lambda+\lambda p^2)^2} \left(\frac{c_2}{(2[1]_q!)^2} + \frac{Ac_1^2 c_2}{[1]_q!} + \frac{bpc_1^2 c_2}{4([1]_q!)^3(1+\lambda p+\lambda p^2)} + A^2 c_1^4 + \frac{Abpc_1^4}{2([1]_q!)^2(1+\lambda p+\lambda p^2)} + \right. \right. \right. \\ \left. \left. \left. \frac{b^2 p^2 c_1^4}{(2[1]_q!)^4(1+\lambda p+\lambda p^2)^2} \right) \right) \right|$$

Proof:

From (2.7), (2.8) and (2.9), we have,

$$a_{p+1}a_{p+3}-\mu a_{p+2}^2 = \frac{bpc_1}{2[1]_q!(1+\lambda p+\lambda p^2)} \left(\frac{b}{(3+6\lambda+5p\lambda+\lambda p^2)} \left(\frac{pc_3}{2[1]_q!} + 2Ap c_1 c_2 + Bpc_1^3 + \frac{bp^2 c_1 c_2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} + \right. \right. \\ \left. \left. \frac{Abp^2 c_1^3}{2[1]_q!(1+\lambda p+\lambda p^2)} + \frac{bp^2}{2[1]_q!(2+2\lambda+3p\lambda+\lambda p^2)} \left(\frac{c_2}{2[1]_q!} + Ac_1^2 + \frac{bpc_1^2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} \right) \right) \right) - \mu \left(\frac{b}{(2+2\lambda+3p\lambda+\lambda p^2)} \left(\frac{pc_2}{2[1]_q!} + \right. \right. \\ \left. \left. \frac{Ap c_1^2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} \right) \right)^2 \\ a_{p+1}a_{p+3}-\mu a_{p+2}^2 = b^2 p^2 \left(\frac{c_1}{2[1]_q!(1+\lambda p+\lambda p^2)(3+6\lambda+5p\lambda+\lambda p^2)} \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3 + \frac{bpc_1c_2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} + \right. \right. \\ \left. \left. \frac{Abpc_1^3}{2[1]_q!(1+\lambda p+\lambda p^2)} + \frac{bpc_2}{(2[1]_q!)^2(2+2\lambda+3p\lambda+\lambda p^2)} + \frac{Abpc_1^2}{2[1]_q!(2+2\lambda+3p\lambda+\lambda p^2)} + \frac{b^2 p^2 c_1^3}{(2[1]_q!)^3(1+\lambda p+\lambda p^2)(2+2\lambda+3p\lambda+\lambda p^2)} \right. \right. \\ \left. \left. + \frac{\mu}{(2+2\lambda+3p\lambda+\lambda p^2)^2} \left(\frac{c_2}{(2[1]_q!)^2} + \frac{Ac_1^2 c_2}{[1]_q!} + \frac{bpc_1^2 c_2}{4([1]_q!)^3(1+\lambda p+\lambda p^2)} + A^2 c_1^4 + \frac{Abpc_1^4}{2([1]_q!)^2(1+\lambda p+\lambda p^2)} + \right. \right. \right. \\ \left. \left. \left. \frac{b^2 p^2 c_1^4}{(2[1]_q!)^4(1+\lambda p+\lambda p^2)^2} \right) \right) \right)$$

Hence,

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leq |b^2| p^2 \left| \frac{c_1}{2[1]_q!(1+\lambda p+\lambda p^2)(3+6\lambda+5p\lambda+\lambda p^2)} \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3 + \right. \right. \\ \left. \left. \frac{bpc_1c_2}{(2[1]_q!)^2(1+\lambda p+\lambda p^2)} + \frac{Abpc_1^3}{2[1]_q!(1+\lambda p+\lambda p^2)} + \frac{bpc_2}{(2[1]_q!)^2(2+2\lambda+3p\lambda+\lambda p^2)} + \frac{Abpc_1^2}{2[1]_q!(2+2\lambda+3p\lambda+\lambda p^2)} + \right. \right. \\ \left. \left. \frac{b^2 p^2 c_1^3}{(2[1]_q!)^3(1+\lambda p+\lambda p^2)(2+2\lambda+3p\lambda+\lambda p^2)} \right) - \frac{\mu}{(2+2\lambda+3p\lambda+\lambda p^2)^2} \left(\frac{c_2}{(2[1]_q!)^2} + \frac{Ac_1^2 c_2}{[1]_q!} + \frac{bpc_1^2 c_2}{4([1]_q!)^3(1+\lambda p+\lambda p^2)} + \right. \right. \\ \left. \left. + A^2 c_1^4 + \frac{Abpc_1^4}{2([1]_q!)^2(1+\lambda p+\lambda p^2)} + \frac{b^2 p^2 c_1^4}{(2[1]_q!)^4(1+\lambda p+\lambda p^2)^2} \right) \right| \quad (4.3)$$

Corollary 4.4. If $f(z) \in G_{1,\lambda}(b, \gamma_{q,m,k}(z))$, then

$$|a_2a_4 - \mu a_3^2| \leq |b^2| \left| \frac{c_1}{2[1]_q!(1+2\lambda)(3+12\lambda)} \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3 + \frac{bc_1c_2}{(2[1]_q!)^2(1+2\lambda)} + \frac{Abc_1^3}{2[1]_q!(1+2\lambda)} + \frac{bc_2}{(2[1]_q!)^2(2+6\lambda)} + \frac{Abc_1^2}{2[1]_q!(2+6\lambda)} + \frac{b^2c_1^3}{(2[1]_q!)^3(1+2\lambda)(2+6\lambda)} \right) - \frac{\mu}{(2+6\lambda)^2} \left(\frac{c_2^2}{(2[1]_q!)^2} + \frac{Ac_1^2c_2}{[1]_q!} + \frac{bc_1^2c_2}{4([1]_q!)^3(1+2\lambda)} + A^2c_1^4 + \frac{Abc_1^4}{2([1]_q!)^2(1+2\lambda)} + \frac{b^2c_1^4}{(2[1]_q!)^4(1+2\lambda)^2} \right) \right| \quad (4.4)$$

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