Int Jr. of Mathematical Sciences & Applications
Vol. 5, No. 2, (July-December, 2015)
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ISSN No: 2230-9888
www.journalshub.com

ON THE ZEROS OF POLYNOMIALS

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Abstract. *In this paper we concider the problem of finding the estimation of maximum number of zeros in a prescribed region and the results which we obtain generalizes and improves upon some well known results.*

Mathematics Subject Classification: 30C10, 30C15.

Keywords: Zeros of Polynomial,Enestrom Kakeya theorem, Prescribed Region.

1. Introduction

Eneström- Kakeya [1-2] states that if $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$, then all the zeros of P(z) lie in $|z| \leq 1$. Some results already proved in literature[6,7], concerning the number of zeros of the polynomial in the region $|z| \leq \frac{1}{2}$, the following result is due to Mohammad [3].

Theorem A_1**.** *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$ then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, does not exceed*

$$1 + \frac{1}{log2} log\frac{a_n}{a_0}.$$

Dewan [4] generalized Theorem A_1 to the polynomials with complex coefficients and obtained the following result.

Theorem A_2**.** *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n$$

for some real β and $|a_0| \leq |a_1| \leq ... \leq |a_{n-1}| \leq |a_n|$ then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{log2} log\frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n} |a_i|}{|a_0|}.$$

In this paper We want to prove the following results.

Theorem 1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n$$

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for some real β, $a_0 \neq 0$ *and* $|a_0| \geq |a_1| \geq ... \geq |a_{n-1}| \geq |a_n|$ *then the number of zeros of P(z) in* $|z| \leq r$, $0<r<1$, *does not exceed*

$$\frac{1}{log\frac{1}{r}} log\frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n}|a_i|}{|a_0|}.$$

Corollary 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n$$

for some real β, $a_0 \neq 0$ *and* $|a_0| \geq |a_1| \geq ... \geq |a_{n-1}| \geq |a_n|$ *then the number of zeros of P(z) in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{log2} log\frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n}|a_i|}{|a_0|}.$$

Remark 1. By taking $r = \frac{1}{2}$ in Theorem 1, then it reduces to Corollary 1.

Theorem 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n$$

for some real β, $a_0 \neq 0$ *and* $|a_0| \geq |a_1| \geq ... \geq |a_{m-1}| \geq |a_m| \leq |a_{m+1}| \leq ... \leq |a_{n-1}| \leq |a_n|$
then the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, *does not exceed*

$$\frac{1}{log\frac{1}{r}} log\frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1}|a_i| - 2|a_m|cos\alpha}{|a_0|}.$$

Corollary 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n$$

for some real β, $a_0 \neq 0$ *and* $|a_0| \geq |a_1| \geq ... \geq |a_{m-1}| \geq |a_m| \leq |a_{m+1}| \leq ... \leq |a_{n-1}| \leq |a_n|$
then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$\frac{1}{log2} log\frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1}|a_i| - 2|a_m|cos\alpha}{|a_0|}.$$

Corollary 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients such that* $|arg(a_i)| \leq \frac{\pi}{2}$, $a_0 \neq 0$ *and* $|a_0| \geq |a_1| \geq ... \geq |a_{m-1}| \geq |a_m| \leq |a_{m+1}| \leq ... \leq |a_{n-1}| \leq |a_n|$
then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$1 + \frac{1}{log\frac{1}{r}} log\frac{|a_n| + |a_0| - |a_m|}{|a_0|}.$$

Remark 2. (i) By taking $r = \frac{1}{2}$ in Theorem 2, then it reduces to Corollary 2.
(ii)By taking $\alpha = \beta = 0$ in Theorem 2, then it reduces to Corollary 3.

Theorem 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients.*
If $Re(a_i) = \alpha_i,\ Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, ..., n$ *and* $a_0 \neq 0$, $\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_{n-1} \geq \alpha_n$
then the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, *does not exceed*

$$\frac{1}{log\frac{1}{r}} log\frac{|\alpha_0| + \alpha_0 + |\alpha_n| - \alpha_n + 2\sum_{i=0}^{n}|\beta_i|}{|a_0|}.$$

Corollary 4. *Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients. If $Re(a_i)=\alpha_i,\ Im(a_i)=\beta_i$ for $i=0,1,2,...,n$ and $\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_{n-1} \geq \alpha_n > 0$ then the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, does not exceed*

$$1+\frac{1}{log\frac{1}{r}} log\frac{\alpha_0+\sum_{i=0}^{n}|\beta_i|}{|a_0|}.$$

Remark 3. By taking $\alpha_n > 0$ in Theorem 3, then it reduces to Corollary 4.

Theorem 4. *Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients. If $Re(a_i)=\alpha_i,\ Im(a_i)=\beta_i$ for $i=0,1,2,...,n$ and $a_0 \neq 0$,*

$$\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_{m-1} \geq \alpha_m \leq \alpha_{m+1} \leq ... \leq \alpha_{n-1} \leq \alpha_n$$

then the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, does not exceed

$$\frac{1}{log\frac{1}{r}} log\frac{|\alpha_0|+\alpha_0-2\alpha_m+|\alpha_n|+\alpha_n+2\sum_{i=0}^{n}|\beta_i|}{|a_0|}.$$

Corollary 5. *Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients. If $Re(a_i)=\alpha_i>0,\ Im(a_i)=\beta_i$ for $i=0,1,2,...,n$ and*

$$\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_{m-1} \geq \alpha_m \leq \alpha_{m+1} \leq ... \leq \alpha_{n-1} \leq \alpha_n$$

then the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, does not exceed

$$1+\frac{1}{log\frac{1}{r}} log\frac{\alpha_0-\alpha_m+\alpha_n+\sum_{i=0}^{n}|\beta_i|}{|a_0|}.$$

Corollary 6. *Let $P(z)=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with real coefficients. $\alpha_0 \neq 0$,*

$$\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_{m-1} \geq \alpha_m \leq \alpha_{m+1} \leq ... \leq \alpha_{n-1} \leq \alpha_n$$

then the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, does not exceed

$$\frac{1}{log\frac{1}{r}} log\frac{|\alpha_0|+\alpha_0-2\alpha_m+|\alpha_n|+\alpha_n|}{|a_0}.$$

Remark 4. (i) By taking $\alpha_i>0$ for $i=0,1,2,...,n$ in Theorem 4, then it reduces to Corollary 5.

(ii) By taking $\beta_i=0$ for $i=0,1,2,...,n$ in Theorem 4, then it reduces to Corollary 6.

We need the following lemmas for proof of the theorems.

2. Lemmas

Lemma 1. Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients such that

$$|arg(a_i)-\beta| \leq \alpha \leq \frac{\pi}{2};\ |a_{i-1}| \leq |a_i|\ for\ some\ i=0,1,2,...,n$$

then $|a_i-a_{i-1}| \leq (|a_i|-|a_{i-1}|)cos\alpha+(|a_i|+|a_{i-1}|)sin\alpha$.
The above lemma is due to Govil [5].

Lemma 2. [4]: If $f(z)$ is regular $f(0) \neq 0$ and $f(z) \leq M$ in $|z| \leq 1$ then the number of zeros of $f(z)$ in $|z| \leq r$, $0<r<1$ does not exceed $\frac{1}{log\frac{1}{r}} log\frac{M}{|f(0)|}$.

3. Proof of the Theorems

Proof of Theorem 1.

Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a polynomial of degree n

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -a_nz^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$. Then for $|z|>1$, we have

$Q(z) \leq |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0|$

$\leq |a_n| + \sum_{i=1}^{n}(|a_{i-1}| - |a_i|)cos\alpha + \sum_{i=1}^{n}(|a_{i-1}| + |a_i|)sin\alpha + |a_0|$ (by using lemma 1)

$= |a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| - |a_n|(cos\alpha + sin\alpha - 1)$

$\leq |a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i|$

Apply Lemma 2 to Q(z), we get that the number of zeros of Q(z) in $|z| \leq r$ does not exceed

$$\frac{1}{log\frac{1}{r}} log\frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i|}{|a_0|}.$$

Since all the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, is also equal to the number of zeros of Q(z) in $|z| \leq r$, we get the required result.

This completes the proof of Theorem 1.

Proof of Theorem 2.

Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a polynomial of degree n

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -a_nz^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$. Then for $|z|>1$, we have

$Q(z) \leq |a_0| + \sum_{i=1}^{m} |a_i - a_{i-1}| + \sum_{i=m+1}^{n} |a_i - a_{i-1}| + |a_n|$

$\leq |a_0| + \sum_{i=1}^{m}(|a_{i-1}| - |a_i|)cos\alpha + \sum_{i=1}^{m}(|a_{i-1}| + |a_i|)sin\alpha +$

$\sum_{i=m+1}^{n}(|a_i| - |a_{i-1}|)cos\alpha + \sum_{i=m+1}^{n}(|a_i| + |a_{i-1}|)sin\alpha + |a_n|$ (by using lemma 1)

$= (|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| - 2|a_m|cos\alpha$

Apply Lemma 2 to Q(z), we get that the number of zeros of Q(z) in $|z| \leq r$ does not exceed

$$\frac{1}{log\frac{1}{r}} log\frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| - 2|a_m|cos\alpha}{|a_0|}.$$

Since all the number of zeros of P(z) in $|z| \leq r$, $0<r<1$, is also equal to the number of zeros of Q(z) in $|z| \leq r$, we get the required result.

This completes the proof of Theorem 2.

Proof of Theorem 3.

Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a polynomial of degree n

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -a_nz^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$. Then for $|z|>1$, we have

$Q(z) \leq |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0|$

$\leq |\alpha_n| + |\beta_n| + \sum_{i=1}^{n} \left[(|\alpha_i - \alpha_{i-1}|) + (|\beta_i - \beta_{i-1}|)\right] + |\alpha_0| + |\beta_0|$

$\leq |\alpha_n| + |\beta_n| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) + \sum_{i=1}^{n}(|\beta_i| + |\beta_{i-1}|) + |\alpha_0| + |\beta_0|$

$\leq |\alpha_0| + \alpha_0 + |\alpha_n| - \alpha_n + 2\sum_{i=0}^{n} |\beta_i|$

Apply Lemma 2 to Q(z), we get that the number of zeros of Q(z) in $|z| \leq r$ does not exceed

$$\frac{1}{log\frac{1}{r}} log \frac{|\alpha_0| + \alpha_0 + |\alpha_n| - \alpha_n + 2\sum_{i=0}^{n} |\beta_i|}{|a_0|}.$$

Since all the number of zeros of P(z) in $|z| \leq r$, 0<r<1, is also equal to the number of zeros of Q(z) in $|z| \leq r$, we get the required result.

This completes the proof of Theorem 3.

Proof of Theorem 4.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degree n

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$. Then for $|z|$>1 , we have

$Q(z) \leq |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0|$

$\leq |\alpha_n| + |\beta_n| + \sum_{i=1}^{n} \left[(|\alpha_i - \alpha_{i-1}|) + (|\beta_i - \beta_{i-1}|)\right] + |\alpha_0| + |\beta_0|$

$\leq |\alpha_n| + |\beta_n| + \sum_{i=1}^{m}(\alpha_{i-1} - \alpha_i) + \sum_{i=m+1}^{n}(\alpha_i - \alpha_{i-1}) + \sum_{i=1}^{n}(|\beta_i| + |\beta_{i-1}|) + |\alpha_0| + |\beta_0|$

$\leq |\alpha_0| + \alpha_0 - 2\alpha_m + |\alpha_n| + \alpha_n + 2\sum_{i=0}^{n} |\beta_i|$

Apply Lemma 2 to Q(z), we get that the number of zeros of Q(z) in $|z| \leq r$ does not exceed

$$\frac{1}{log\frac{1}{r}} log \frac{|\alpha_0| + \alpha_0 - 2\alpha_m + |\alpha_n| + \alpha_n + 2\sum_{i=0}^{n} |\beta_i|}{|a_0|}.$$

Since all the number of zeros of P(z) in $|z| \leq r$, 0<r<1, is also equal to the number of zeros of Q(z) in $|z| \leq r$, we get the required result.

This completes the proof of Theorem 4.

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