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Certain integrals and series expansions involving the generalized multivariable Gimel-function

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ABSTRACT
In this paper, we evaluate two single finite integrals involving the product of Legendre functions, generalized hypergeometric functions and the generalized multivariable Gimel-function. These integrals are employed to evaluate two double finite integrals. We have also further utilized these integrals (single and double) to establish two Fourier-Legendre series and two Fourier-Legendre series expansions for he generalized multivariable Gimel-function.

KEYWORDS : Generalized multivariable Gimel-function, multiple integral contours, Legendre functions, generalized hypergeometric functions, double finite integrals, two Fourier-Legendre series expansions.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_j)_{n^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_j)_{m^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r)$, $\Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j} - \sum_{k=1}^{2} \beta^{(k)}_{2j} s_k\right) (j = 1, \cdots, m_2), \Gamma^{B_{3j}}\left(b_{3j} - \sum_{k=1}^{3} \beta^{(k)}_{3j} s_k\right) (j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{B_{rj}}\left(b_{rj} - \sum_{i=1}^{r} \beta^{(i)}_{rj}\right) (j = 1, \cdots, m_r)$, $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$\sum_{j=1}^{n_2} A_{2j} \alpha^{(k)}_{2j} + \sum_{j=1}^{m_2} B_{2j} \beta^{(k)}_{2j} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj} \alpha^{(k)}_{rj} + \sum_{j=1}^{m_r} B_{rj} \beta^{(k)}_{rj} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{\substack{1 \leqslant j \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta^{h'}_{hj}} + D^{(i)}_j \frac{d^{(i)}_j}{\delta^{(i)}_j} \right)$ and $\beta_i = \max_{\substack{1 \leqslant j \leqslant n_i \\ 1 \leqslant j \leqslant n^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} A_{hj} \frac{a_{hj} - 1}{\alpha^{h'}_{hj}} + C^{(i)}_j \frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right)$

Remark 1.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [5]).

Remark 3.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [4]).

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [7,8]).

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}= [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_j^{(r)})_{n^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}= [(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_j^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

$$F_1(z)={}_UF_V\left[\begin{array}{c|c}(\mathrm{A}_U)_s & \\ . & \mathrm{Z}\\ (\mathrm{B}_V)_s & \end{array}\right] \text{ and } F_2(z)={}_{U'}F_{V'}\left[\begin{array}{c|c}(\mathrm{A}_{U'})_t & \\ . & \mathrm{Z}\\ (\mathrm{B}_{V'})_t & \end{array}\right]$$

and $F(s)=\dfrac{[A_U]_sC^s}{[B_V]_ss!};G(t)=\dfrac{[A'_{U'}]_tD^t}{[B'_{V'}]_tt!}$

2. Single finite integrals.

In this section, we evaluate two single finite integrals.

Theorem 1.

$$\int_{-1}^{1}(1-x^2)^{\sigma-1}P_N^M(x)F_1\left[c(1-x^2)^h\right]F_2\left[d(1-x^2)^k\right]\gimel\left(z_1(1-x^2)^{-a_1},\cdots,z_r(1-x^2)^{-a_r}\right)\mathrm{d}x=$$

$$\frac{2^M\pi}{\Gamma\left(\frac{N-M}{2}+1\right)\Gamma\left(\frac{1-N-M}{2}\right)}\sum_{s,t=0}^{\infty}F(s)G(t)\,\gimel^{U;m_r+2,n_r:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\mathbf{A},\ \left(1+\sigma+\frac{N}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(\sigma-\frac{N}{2}+hs+kt;;a_1,\cdots,a_r;1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \left(\sigma+\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(\sigma-\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\mathbf{B}:B \end{array}\right) \tag{2.1}$$

provided

$$a_i>0(i=1,\cdots,r),\,Re(\sigma+hs+kt)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>|Re(M)|.$$

$\left|arg(z_i(1-x^2)^{-a_i})\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To establish (2.1), expressing the generalized hypergeometric functions in terms of series summation ([6], p. 73 Eq. 2), and the generalized multivariable Gimel-function as multiple integrals contour with the help of (1.1). Interchanging the order of integrations which is justified under the conditions mentioned above. We get

$$\sum_{s,t=0}^{\infty}F(s)G(t)\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_{-1}^{1}(1-x^2)^{\sigma+hs+kt-\sum_{i=1}^{r}a_is_i}P_N^M(x)\mathrm{d}x\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

Evaluating the inner integral with the help of ([3], p. 316 Eq. 16) and interpreting the resulting expression with the help of (1.1), we obtain the desired result.

Theorem 2.

$$\int_{-1}^{1}(1-x^2)^{\sigma-1}P_N^M(x)F_1\left[c(1-x^2)^{-h}\right]F_2\left[d(1-x^2)^{-k}\right]\gimel\left(z_1(1-x^2)^{a_1},\cdots,z_r(1-x^2)^{a_r}\right)\mathrm{d}x=$$

$$\frac{2^M\pi}{\Gamma\left(\frac{N-M}{2}+1\right)\Gamma\left(\frac{1-N-M}{2}\right)}\sum_{s,t=0}^{\infty}F(s)G(t)\,\gimel^{U;m_r,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ \left(1-\sigma-\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(1-\sigma+\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ \left(-\sigma+\frac{N}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(-\sigma-\frac{N}{2}+hs+kt;;a_1,\cdots,a_r;1\right):B \end{array}\right) \tag{2.3}$$

provided

$$a_i > 0(i=1,\cdots,r),\ Re(\sigma - hs - kt) + \sum_{i=1}^{r} a_i \max_{\substack{1\leqslant j\leqslant n_i \\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_j^{(i)}\frac{c_j^{(i)}}{\gamma_j^{(i)}}\right) > |Re(M)|.$$

$\left|arg(z_i(1-x^2)^{a_i})\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Similarly, (2.3) can be established.

3. Double finite integrals.

In this section, we evaluate two double finite integrals.

Theorem 3.

$$\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c(1-x^2)^h\right]F_2\left[d(1-x^2)^k\right]F_1\left[c(1-y^2)^h\right]F_2\left[d(1-y^2)^k\right]$$

$$\gimel\left(z_1(1-x^2)^{-a_1}(1-y^2)^{-b_1},\cdots,z_r(1-x^2)^{-a_r}(1-y^2)^{-b_r}\right)\mathrm{d}x\mathrm{d}y =$$

$$\frac{2^{M_1+M_2}\pi^2}{\Gamma\left(\frac{N_1-M_1}{2}+1\right)\Gamma\left(\frac{1-N_1-M_1}{2}\right)\Gamma\left(\frac{N_2-M_2}{2}+1\right)\Gamma\left(\frac{1-N_2-M_2}{2}\right)}\sum_{s_1 t_1,s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)$$

$$\gimel_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;m_r+4,n_r:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\mathbf{A},\ \mathrm{A}_1(N_1,N_2):A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathrm{B}_1(M_1,M_2),\mathbf{B}:B \end{array}\right) \tag{3.1}$$

where

$$A_1(N_1,N_2) = \left(1+\sigma_1+\frac{N_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),\left(\sigma_1-\frac{N_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),$$

$$\left(1+\sigma_2+\frac{N_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right),\left(\sigma_2-\frac{N_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right) \tag{3.2}$$

$$B_1(M_1,M_2) = \left(\sigma_1+\frac{M_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),\left(\sigma_1-\frac{M_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),$$

$$\left(\sigma_2+\frac{M_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right),\left(\sigma_2-\frac{M_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right) \tag{3.3}$$

provided

$$a_i,b_i > 0(i=1,\cdots,r),\ Re(\sigma_1+hs_1+kt_1) - \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > |Re(M_1)|.$$

$$Re(\sigma_2+hs_2+kt_2) - \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant j\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > |Re(M_2)|.$$

$\left|arg(z_i(1-x^2)^{-a_i}(1-y^2)^{-b_i})\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To establish (3.1), evaluating the x-integral of (3.1) with the help of the theorem 1 and interchanging the order of integrations and summation which is justified under the conditions mentioned above, then evaluating the y-integral with the help of the theorem 1, we get the desired result (3.1).

Theorem 4.

$$\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c(1-x^2)^{-h}\right]F_2\left[d(1-x^2)^{-k}\right]F_1\left[c(1-y^2)^{-h}\right]$$

$$F_2[d(1-y^2)^{-k}]\gimel\left(z_1(1-x^2)^{-a_1}(1-y^2)^{-b_1},\cdots,z_r(1-x^2)^{-a_r}(1-y^2)^{-b_r}\right)\mathrm{d}x\mathrm{d}y=$$

$$\frac{2^{M_1+M_2}\pi^2}{\Gamma\left(\frac{N_1-M_1}{2}+1\right)\Gamma\left(\frac{1-N_1-M_1}{2}\right)\Gamma\left(\frac{N_2-M_2}{2}+1\right)\Gamma\left(\frac{1-N_2-M_2}{2}\right)}\sum_{s_1t_1,s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)$$

$$\gimel^{U;m_r,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\mathrm{A}_2(M_1,M_2),\mathbf{A}:\ \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ \mathrm{B}_2(N_1,N_2):B \end{array}\right) \tag{3.3}$$

where

$$A_1(M_1,M_2)=\left(1-\sigma_1-\frac{M_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),\left(1-\sigma_1+\frac{M_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),$$

$$\left(1-\sigma_2-\frac{M_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right),\left(1-\sigma_2+\frac{M_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right) \tag{3.4}$$

$$B_1(N_1,N_2)=\left(-\sigma_1+\frac{N_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),\left(-\sigma_1-\frac{N_1}{2}+hs_1+kt_1;a_1,\cdots,a_r;1\right),$$

$$\left(-\sigma_2+\frac{M_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right),\left(-\sigma_2-\frac{N_2}{2}+hs_2+kt_2;b_1,\cdots,b_r;1\right) \tag{3.5}$$

provided

$$a_i,b_i>0(i=1,\cdots,r),\,Re(\sigma_1-h_1s-k_1t)+\sum_{i=1}^{r}a_i\max_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}}{\gamma_j^{(i)}}\right)>|Re(M_1)|.$$

$$Re(\sigma_2-h_2s-k_2t)+\sum_{i=1}^{r}b_i\max_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}}{\gamma_j^{(i)}}\right)>|Re(M_2)|.$$

$\left|arg(z_i(1-x^2)^{a_i}(1-y^2)^{b_i})\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Similarly, (3.3) can be established.

4. Fourier-Legendre series.

The Fourier-Legendre series to be established are

Theorem 5.

$$(1-x^2)^{\sigma-1}P_N^M(x)F_1\left[c(1-x^2)^h\right]F_2\left[d(1-x^2)^k\right]\gimel\left(z_1(1-x^2)^{-a_1},\cdots,z_r(1-x^2)^{-a_r}\right)=$$

$$2^{M-1}\pi\sum_{U=0}^{\infty}\frac{(2U+1)(U-M)!}{(U+M)!\Gamma\left(\frac{U-M}{2}+1\right)\Gamma\left(\frac{1-U-M}{2}\right)}P_U^M(x)\sum_{s,t=0}^{\infty}F(s)G(t)\,\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r:V}$$

$$\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\mathbf{A},\ \left(1+\sigma+\frac{U}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(\sigma-\frac{U}{2}+hs+kt;;a_1,\cdots,a_r;1\right):A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\ \left(\sigma+\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(\sigma-\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\mathbf{B}:B\end{array}\right) \tag{4.1}$$

provided that : $M\leqslant U$ and the corresponding conditions stated in theorem 1 are satisfied.

Proof
To obtain (4.1), let

$$f(x)=(1-x^2)^{\sigma-1}F_1\left[c(1-x^2)^h\right]F_2\left[d(1-x^2)^k\right]\gimel\left(z_1(1-x^2)^{-a_1},\cdots,z_r(1-x^2)^{-a_r}\right)=\sum_{U=0}^{\infty}C_UP_U^M(x) \tag{4.2}$$

The above equation is valide, since $f(x)$ is continuous and bounded in $(-1,1)$. Multiplying both sides of (4.2) by $P_N^M(x)$ and integrating with respect to x from -1 to 1, using the theorem 1 and the orthogonality property of Legendre function ([3], p. 279), we obtain the value of C_U. Substituting the value of C_U in (4.2), we get the desired result.

Theorem 6.

$$(1-x^2)^{\sigma-1}P_N^M(x)F_1\left[c(1-x^2)^{-h}\right]F_2\left[d(1-x^2)^{-k}\right]\gimel\left(z_1(1-x^2)^{a_1},\cdots,z_r(1-x^2)^{a_r}\right)=$$

$$2^{M-1}\pi\sum_{U=0}^{\infty}\frac{(2U+1)(U-M)!}{(U+M)!\Gamma\left(\frac{U-M}{2}+1\right)\Gamma\left(\frac{1-U-M}{2}\right)}P_U^M(x)\sum_{s,t=0}^{\infty}F(s)G(t)\,\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r,n_r+2:V}$$

$$\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\ \left(1-\sigma-\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(1-\sigma+\frac{M}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ \left(-\sigma-\frac{U}{2}+hs+kt;;a_1,\cdots,a_r;1\right),\left(-\sigma+\frac{U}{2}+hs+kt;;a_1,\cdots,a_r;1\right):B\end{array}\right) \tag{4.3}$$

provided that : $M\leqslant U$ and the corresponding conditions stated in theorem 2 are satisfied.

Similarly, (4.3) can be established.

5. Double Fourier-Legendre series.

The double Fourier-Legendre series to be established are

Theorem 7.

$$(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c(1-x^2)^h\right]F_2\left[d(1-x^2)^k\right]F_1\left[c(1-y^2)^h\right]F_2\left[d(1-y^2)^k\right]$$

$$\gimel\left(z_1(1-x^2)^{-a_1}(1-y^2)^{-b_1},\cdots,z_r(1-x^2)^{-a_r}(1-y^2)^{-b_r}\right)=2^{M_1+M_2-2}\pi^2$$

$$\sum_{U_1,U_2=0}^{\infty}\frac{(2U_1+1)(2U_2+1)(U_1-M_1)!(U_2-M_2)!}{\Gamma\left(\frac{U_1-M_1}{2}+1\right)\Gamma\left(\frac{1-U_1-M_1}{2}\right)(U_1+M_1)!\Gamma\left(\frac{U_2-M_2}{2}+1\right)\Gamma\left(\frac{1-U_2-M_2}{2}\right)(U_2+M_2)!}$$

$$\sum_{s_1t_1,s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)\,\gimel^{U;m_r+4,n_r:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\mathbf{A},\ \mathrm{A}_1(U_1,U_2):A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\ \mathrm{B}_1(M_1,M_2),\mathbf{B}:B\end{array}\right)P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)\qquad(5.1)$$

where $A_1(.,.), B_1(.,.)$are defined respectively by (3.2) and (3.3).

provided that : $M_1\leqslant U_1, M_2\leqslant U_2$ and the corresponding conditions stated in theorem 3 are satisfied.

Proof
To obtain (5.1), let

$$f(x,y)=(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}F_1\left[c(1-x^2)^h\right]F_2\left[d(1-x^2)^k\right]F_1\left[c(1-y^2)^h\right]F_2\left[d(1-y^2)^k\right]$$

$$\gimel\left(z_1(1-x^2)^{-a_1}(1-y^2)^{-b_1},\cdots,z_r(1-x^2)^{-a_r}(1-y^2)^{-b_r}\right)=\sum_{U_1,U_2=0}^{\infty}C_{U_1,U_2}P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)\qquad(5.4)$$

The above equation is valide, since $f(x,y)$ is continuous and bounded in $(-1,1)\times(-1,1)$. Multiplying both sides of (5.4) by $P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)$ and integrating with respect to x and y respectively from -1 to 1, using the theorem 3 and the orthogonality property of Legendre function ([3], p. 279), we obtain the value of C_{U_1,U_2}. Substituting the value of C_{U_1,U_2} in (5.4), we get the desired result.

Theorem 8.

$$(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c(1-x^2)^{-h}\right]F_2\left[d(1-x^2)^{-k}\right]F_1\left[c(1-y^2)^{-h}\right]F_2\left[d(1-y^2)^{-k}\right]$$

$$\gimel\left(z_1(1-x^2)^{a_1}(1-y^2)^{b_1},\cdots,z_r(1-x^2)^{a_r}(1-y^2)^{b_r}\right)=2^{M_1+M_2-2}\pi^2$$

$$\sum_{U_1,U_2=0}^{\infty}\frac{(2U_1+1)(2U_2+1)(U_1-M_1)!(U_2-M_2)!}{\Gamma\left(\frac{U_1-M_1}{2}+1\right)\Gamma\left(\frac{1-U_1-M_1}{2}\right)(U_1+M_1)!\Gamma\left(\frac{U_2-M_2}{2}+1\right)\Gamma\left(\frac{1-U_2-M_2}{2}\right)(U_2+M_2)!}$$

$$\sum_{s_1t_1,s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)\,\gimel^{U;m_r,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\mathrm{A}_2(M_1,M_2),\mathbf{A}:\ \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ \mathrm{B}_2(U_1,U_2):B\end{array}\right)P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)\qquad(5.5)$$

where $A_2(.,.), B_2(.,.)$are defined respectively by (3.4) and (3.5).

provided that : $M_1\leqslant U_1, M_2\leqslant U_2$ and the corresponding conditions stated in theorem 4 are satisfied.

On applying the above method, (5.5) can be established.

6. Conclusion.

The generalized Gimel-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various single and double Fourier-Legendre series expansions concerning a large variety of special functions of one variable and several variables.

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