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On the covering radius of some classes of Block Repetition codes in R

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Abstract

In this paper, the covering radius of codes over $R = \mathbb{Z}_2\mathbb{Z}_8$ is determined. The block repetition codes over R are defined and also obtain the covering radius for block repetition codes over R.

Keywords: Finite Ring, Block Repetition code, Covering radius, Gray map, Different weight.

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1 Introduction

Recently, there has been substantial interest in the class of additive codes. In [10, 11], Delsarte contributes to the algebraic theory of association schemes where the main idea is to characterize the subgroups of the underlying abelian group in a given association scheme.

Additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ have been extensively studied in [1, 2, 3, 4]. In [6, 7], the authors, in particular, gave lower and upper bounds on the covering radius of codes over the ring $\mathbb{Z}_4$ and $\mathbb{Z}_6$ with respect to different distance and they explained the covering radius of various repetition codes.

2 Preliminaries

Let $R = \mathbb{Z}_2\mathbb{Z}_8$ be a finite ring, here $\mathbb{Z}_2 = \{0, 1\}$ and $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is an integer modulo 2 and 8 respectively.

In this section, some preliminary results are given based on [2] and [4]. A non empty set C is a $\mathbb{R}$-additive code if it is a subgroup of $\mathbb{Z}_2^\gamma \times {\mathbb{Z}_8}^\delta$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^\lambda \times {\mathbb{Z}_8}^\mu$, for some λ and μ. That C is of type $2^\lambda 8^\mu$ as a group. It follows that it has $\mid C \mid= 2^{\lambda+3\mu}$ codewords and the number of order two codewords in C is $2^{\lambda+\mu}$.

Consider the following extension of the Gray map

$$\psi : \mathbb{Z}_2^\gamma \times {\mathbb{Z}_8}^\delta \to \mathbb{Z}_2^n, \text{ with } n = \gamma + 3\delta,$$

given by

$$\psi(u, v) = (u, \phi(v_1), \cdots \phi(v_\delta)), \forall\ u \in\ \mathbb{Z}_2^\gamma, \forall\ (v_1, \cdots , v_\delta)\ \in {\mathbb{Z}_8}^\delta,$$

where

$$\phi : \mathbb{Z}_8 \to \mathbb{Z}_2^3,$$

is the Gray map given by

$$\phi(0) = (0, 0, 0),$$

$$\phi(1) = (0, 1, 0),$$

$$\phi(2) = (1, 0, 0),$$

$$\phi(3) = (1, 1, 0),$$

$$\phi(4) = (0, 0, 1),$$

$$\phi(5) = (0, 1, 1),$$

$$\phi(6) = (1, 0, 1),$$

$$\phi(7) = (1, 1, 1)$$

[13, 5]. Then the binary image of a R-additive code under the extended Gray map is called a *R-linear code* of length $n = \gamma + 3\delta$.

The Hamming weight of u, denoted by $wt_H(u)$ and $wt_L(v), wt_E(v), wt_{CE}(v), wt_{Hom}(v)$ the Lee(L), Euclidean(E), Chinese Euclidean(CE) and Homogeneous(Hom) weights of v respectively where $u \in \mathbb{Z}_2^\gamma$ and $v \in \mathbb{Z}_8^\delta$.

In [8, 12] are defined as for the vector $x = (x_1, x_2, \cdots, x_n) \in \mathbb{Z}_8$ with the following table

Code	$x \in \mathbb{Z}_8$
$w_L(x)$	0 if $x = 0, 3$ if $x = \{1, 3, 5, 7\}$, 2 if $x = \{2, 6\}$ and 4 otherwise
$w_E(x)$	0 if $x = 0, 1$ if $x = \{1, 7\}, 4$ if $x = \{2, 4, 6\}$ and 9 otherwise
$w_{CE}(x)$	0 if $x = 0, 1$ if $x = \{1, 7\}, 2$ if $x = \{2, 6\}, 3$ if $x = \{3, 5\}$ and 4 otherwise
$w_{Hom}(x)$	0 if $x = 0, 2$ if $x \neq 4$ and 4 otherwise

If $c_1, c_2 \in C$ or C any two distinct codewords of distance d_D are defined as

$$d_D(C) = \{d_D(c_1, c_2) | c_1 - c_2 \neq 0 \text{ and } c_1, c_2 \in C\}.$$

The minimum weight of C is

$$d_D(C) = \min\{d_D(c_1, c_2) | c_1 - c_2 \neq 0 \text{ and } c_1, c_2 \in C\}.$$

If C is a *linear code*, then $d_D(C) = \min\{w_D(c) | c \neq 0 \in C\}$. Therefore,

$$d_D(c_1, c_2) = w_D(c_1 - c_2).$$

Let $C \subseteq R^n$ be a linear code, where n is the length of the code, the number of codewords N and the minimum distance d_D is said to be an *(n, N, d_D)-code* in R.

The weights of x is defined as $wt_D(x) = wt_H(u) + wt_D(v)$ with $x = (u, v) \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_8^\delta$, and $u = (u_1, \cdots, u_\gamma) \in \mathbb{Z}_2^\gamma, v = (v_1, \cdots, v_\delta) \in {\mathbb{Z}_8}^\delta$, where $D = \{$ L, E, CE and Hom $\}$.

The Gray map defined above is an isometry which transforms the (weight) distance defined over $\mathbb{Z}_2^\gamma \times \mathbb{Z}_8^\delta$ to the Hamming distance defined over $\mathbb{Z}_2^n$, with $n = \gamma + 3\delta$.

3 The covering radius of codes and block repetition codes in R

Let C be a code of length n with minimum distance d over R. Then the spheres of radius $\lfloor \frac{d-1}{2} \rfloor$ around the codewords may not cover the whole space.

The least non-negative integer a such that sphere of radius r around the codewords cover the whole space R^n is called the *covering radius* of the code. That is, the covering radius of C is

$$r(C) = \max_{s \in R^n} \left\{ \min_{c \in C} \{d(s, c)\} \right\}.$$

For a binary code C, its covering radius $r(C)$ is defined as follows

$$r(C) = \max_{s \in \mathbb{F}_2}\{\min_{c \in C} d_H(s,\ c)\}$$

The extension of this definition to codes over R is that the covering radius of a code C is the smallest number r such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^\gamma \times \mathbb{Z}_8^\delta$. Hence, the covering radius of a code C over R, with respect to the distance(D), is given by

$$r_D(C) = \max_{s \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_8{}^\delta}\{\min_{c \in C} d_D(s, c)\}$$

respectively. It is easy to see that $r_D(C)$ is the minimum value of r_D such that

$$\mathbb{Z}_2^\gamma \times \mathbb{Z}_8^\delta = \bigcup_{c \in C} S_{r_D}(c),$$

respectively, where

$$S_{r_D}(u) = \{v \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_8^\delta : d_D(u, v) \le r_D\},$$

for $u \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_2^\delta$.

In order to determine the covering radii of some classes of block codes over $\mathbb{R}$ are defined. The approach in [10] is used to obtain the covering radius.

The block repetition code C^n over R is a R-additive code of length $n = \sum_{j=1}^{15} n_j$ with generator matrix

$$G = [\overbrace{01\cdots01}^{n_1}\overbrace{02\cdots02}^{n_2}\overbrace{03\cdots03}^{n_3}\overbrace{04\cdots04}^{n_4}\overbrace{05\cdots05}^{n_5}\overbrace{06\cdots06}^{n_6}\overbrace{07\cdots07}^{n_7}\overbrace{10\cdots10}^{n_8}\overbrace{11\cdots11}^{n_9}\overbrace{12\cdots12}^{n_{10}}$$
$$\overbrace{13\cdots13}^{n_{11}}\overbrace{14\cdots14}^{n_{12}}\overbrace{15\cdots15}^{n_{13}}\overbrace{16\cdots16}^{n_{14}}\overbrace{17\cdots17}^{n_{15}}].$$

If, for a fixed $1 \le i \le 15$. For all $1 \le j \ne i \le 15, n_j = 0$, then the code $C^n = C^{n_i}$ is denoted by C_i. Therefore, the fifteen basic codes from generator matrices are given,

Generator Matrix	Codes
$G_{01(03)(05)(07)} = [01\cdots01]$	$C_{01(03)(05)(07)} = \{c_{i, i=0 \text{ to } 7}\}$
$G_{02(06)} = [02\cdots02]$	$C_{02(06)} = \{c_0, c_2, c_6\}$
$G_{04} = [04\cdots04]$	$C_{04} = \{c_0, c_4\}$
$G_{10} = [10\cdots10]$	$C_{10} = \{c_0, c_8\}$
$G_{11(13)(15)(17)} = [11\cdots11]$	$C_{11(13)(15)(17)} = \{c_{i, i=0 \text{ to } 15}\}$
$G_{12(16)} = [12\cdots12]$	$C_{12(16)} = \{c_{i, i=0,2,4,6,8,10,12,14,16}\}$
$G_{14} = [14\cdots14]$	$C_{14} = \{c_0, c_{12}\}$

where $\{c_0 = 00\cdots00, c_1 = 01\cdots01, c_2 = 02\cdots02, c_3 = 03\cdots03, c_4 = 04\cdots04,$
$c_5 = 05\cdots05,\ c_6 = 06\cdots06, c_7 = 07\cdots07, c_8 = 10\cdots10, c_9 = 11\cdots11, c_{10} = 12\cdots12,$
$c_{11} = 13\cdots13, c_{12} = 14\cdots14, c_{13} = 15\cdots15, c_{14} = 16\cdots16, c_{15} = 17\cdots17\}.$

The following theorems provide the covering radius of C_j, for $1 \le j \le 15$.

Theorem 3.1. *The covering radius of $C_{j,\ 1 \le j \le 15}$ with respect to the Lee weight is given by*

1. $\frac{3n}{4} \leq r_L(C_{01}) = r_L(C_{03}) = r_L(C_{05}) = r_L(C_{07}) \leq \frac{5n}{2}$,
2. $n \leq r_L(C_{02}) = r_L(C_{06}) \leq 3n$,
3. $n \leq r_L(C_{04}) \leq 3n$,
4. $2n \leq r_L(C_{10}) \leq 4n$,
5. $r_L(C_{11}) = r_L(C_{13}) = r_L(C_{15}) = r_L(C_{17}) = 2n$,
6. $\frac{5n}{4} \leq r_L(C_{12}) = r_L(C_{16}) \leq \frac{3n}{2}$ *and*
7. $\frac{5n}{4} \leq r_L(C_{14}) \leq 3n$.

Proof. (1) For $c \in C_j, 1 \leq j \leq 15$, let $t_i(c), 0 \leq i \leq 15$ denote the number of occurrences of symbol i in the codeword c.

By the definition of covering radii with Lee distance(weight) is

$$r_L(C_j) = \max_{x \in R^n} \{d_L(x, C_j); 1 \leq j \leq 15\}.$$

Let $x \in R^n$. If x is given $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15})$, where $\sum_{j=0}^{15} t_j = n$, then

$$\begin{aligned}
d_L(x, \overline{00}) &= n - t_0 + t_2 + 3t_4 + t_6 + t_9 + 2t_{10} + t_{11} + 4t_{12} + t_{13} + 2t_{14} + t_{15},\\
d_L(x, \overline{01}) &= n - t_1 + t_3 + 3t_5 + t_7 + t_8 + t_{10} + 2t_{11} + t_{12} + 4t_{13} + t_{14} + 2t_{15},\\
d_L(x, \overline{02}) &= n - t_2 + t_0 + t_4 + 3t_6 + t_9 + t_{11} + 2t_{12} + t_{13} + t_{15} + 4t_{14} + 2t_8,\\
d_L(x, \overline{03}) &= n - t_3 + t_1 + t_5 + 3t_7 + t_8 + 2t_9 + t_{10} + t_{12} + 2t_{13} + t_{14} + 4t_{15},\\
d_L(x, \overline{04}) &= n - t_4 + 3t_0 + t_2 + t_6 + 4t_8 + t_9 + 2t_{10} + t_{11} + t_{13} + 2t_{14} + t_{15},\\
d_L(x, \overline{05}) &= n - t_5 + 3t_1 + t_3 + t_7 + t_8 + 4t_9 + t_{10} + 2t_{11} + t_{12} + t_{14} + 2t_{15},\\
d_L(x, \overline{06}) &= n - t_6 + t_0 + 3t_2 + t_4 + 2t_8 + t_9 + 4t_{10} + t_{11} + 2t_{12} + t_{13} + t_{15},\\
d_L(x, \overline{07}) &= n - t_7 + t_1 + 3t_3 + t_5 + t_8 + 2t_9 + t_{10} + 4t_{11} + t_{12} + 2t_{13} + t_{14},\\
d_L(x, \overline{10}) &= n - t_8 + t_1 + 2t_2 + t_3 + 4t_4 + t_5 + 2t_6 + t_7 + t_{10} + 3t_{12} + t_{14},\\
d_L(x, \overline{11}) &= n - t_9 + t_0 + t_2 + 2t_3 + t_4 + 4t_5 + t_6 + 2t_7 + t_{11} + 3t_{13} + t_{15},\\
d_L(x, \overline{12}) &= n - t_{10} + 2t_0 + t_1 + t_3 + 2t_4 + t_5 + 4t_6 + t_7 + t_8 + t_{12} + 3t_{14},\\
d_L(x, \overline{13}) &= n - t_{11} + t_0 + 2t_1 + t_2 + t_4 + 2t_5 + t_6 + 4t_7 + t_9 + t_{13} + 3t_{15},\\
d_L(x, \overline{14}) &= n - t_{12} + 4t_0 + t_1 + 2t_2 + t_3 + t_5 + 2t_6 + t_7 + 3t_8 + t_9 + 3t_{14},\\
d_L(x, \overline{15}) &= n - t_{13} + t_0 + 4t_1 + t_2 + 2t_3 + t_4 + t_6 + 2t_7 + 3t_9 + t_{11} + t_{15},\\
d_L(x, \overline{16}) &= n - t_{14} + 2t_0 + t_1 + 4t_2 + t_3 + 2t_4 + t_5 + t_7 + t_8 + 3t_{10} + t_{12},\\
d_L(x, \overline{17}) &= n - t_{15} + t_0 + 2t_1 + t_2 + 4t_3 + t_4 + 2t_5 + t_6 + t_9 + 3t_{11} + t_{13},
\end{aligned}$$

Therefore, $d_L(x, C_{01}) = d_L(x, C_{03}) = d_L(x, C_{05}) = d_L(x, C_{07}) = \min\{d_L(x, \overline{00}), d_L(x, \overline{01}), d_L(x, \overline{02}), d_L(x, \overline{03}), d_L(x, \overline{04}), d_L(x, \overline{05}), d_L(x, \overline{06}), d_L(x, \overline{07})\} \leq \frac{5n}{2}$ and hence

$$r_L(C_{01}) = r_L(C_{03}) = r_L(C_{05}) = r_L(C_{07}) \leq \frac{5n}{2}.$$

If $x = (\overbrace{00\cdots00}^{\frac{n}{8}}\overbrace{01\cdots01}^{\frac{n}{8}}\overbrace{02\cdots02}^{\frac{n}{8}}\overbrace{03\cdots03}^{\frac{n}{8}}\overbrace{04\cdots04}^{\frac{n}{8}}\overbrace{05\cdots05}^{\frac{n}{8}}\overbrace{06\cdots06}^{\frac{n}{8}}\overbrace{07\cdots07}^{\frac{n}{8}}) \in R^n$,
then $d_L(x, C_{01}) = d_L(x, C_{03}) = d_L(x, C_{05}) = d_L(x, C_{07}) = \frac{n}{16}+2(\frac{n}{16})+\frac{n}{16}+4(\frac{n}{16})+\frac{n}{16}+2(\frac{n}{16})+\frac{n}{16} = \frac{3n}{4}$. Thus $r_L(C_{01}) = r_L(C_{03}) = r_L(C_{05}) = r_L(C_{07}) \geq \frac{3n}{4}$ and hence,

$\frac{3n}{4} \leq r_L(C_{01}) = r_L(C_{03}) = r_L(C_{05}) = r_L(C_{07}) \leq \frac{5n}{2}$.

(2) In $C_{02(06)}$, $d_L(x, C_{02}) = d_L(x, C_{06}) = \min\{d_L(x, \overline{00}), d_L(x, \overline{02}), d_L(x, \overline{04}), d_L(x, \overline{06})\} \leq 3n$. Thus,

$$r_L(C_{02}) = r_L(C_{06}) \leq 3n.$$

If $x = (\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{02\cdots02}^{\frac{n}{4}}\overbrace{04\cdots04}^{\frac{n}{4}}\overbrace{06\cdots06}^{\frac{n}{4}}) \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{02}) = d_L(x, \overline{04}) = d_L(x, \overline{06}) = 2(\frac{n}{8}) + 4(\frac{n}{8}) + 2(\frac{n}{8}) = n$. Thus $r_L(C_{02}) = r_L(C_{06}) \geq n$ and so, $n \leq r_L(C_2) = r_L(C_6) \leq 3n$.

(3) In C_{04}, $d_L(x, C_{04}) = \min\{d_L(x, \overline{00}), d_L(x, \overline{04})\} \leq 3n$ and hence

$$r_L(C_{04}) \leq 3n.$$

If $x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{04\cdots04}^{\frac{n}{2}}) \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{04}) = n$. Therefore, $r_L(C_{04}) \geq n$ and thus $n \leq r_L(C_{04}) \leq 3n$.

(4) In C_{10}, $d_L(x, C_{10}) = \min\{d_L(x, \overline{00}), d_L(x, \overline{01})\} \leq 4n$, then

$$r_L(C_{10}) \leq 4n.$$

If $x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{01\cdots01}^{\frac{n}{2}}) \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{01}) = 2n$. Thus $r_L(C_{10}) \geq 2n$ and hence $2n \leq r_L(C_{10}) \leq 4n$.

(5) In $C_{11(13)(15)(17)}$, $d_L(x, C_{11}) = d_L(x, C_{13}) = d_L(x, C_{15}) = d_L(x, C_{17}) = \min\{d_L(x, \overline{00}),$
$d_L(x, \overline{01}), d_L(x, \overline{02}), d_L(x, \overline{03}), d_L(x, \overline{04}), d_L(x, \overline{05}), d_L(x, \overline{06}), d_L(x, \overline{07}), d_L(x, \overline{10}), d_L(x, \overline{11}),$
$d_L(x, \overline{12}), d_L(x, \overline{13}), d_L(x, \overline{14}), d_L(x, \overline{15}), d_L(x, \overline{16}), d_L(x, \overline{17})\} \leq 2n$

$$r_L(C_{11}) = r_L(C_{13}) = r_L(C_{15}) = r_L(C_{17}) \leq 2n.$$

Let $x = (\overbrace{00\cdots00}^{\frac{n}{16}}\overbrace{01\cdots01}^{\frac{n}{16}}\overbrace{02\cdots02}^{\frac{n}{16}}\overbrace{03\cdots03}^{\frac{n}{16}}\overbrace{04\cdots04}^{\frac{n}{16}}\overbrace{05\cdots05}^{\frac{n}{16}}\overbrace{06\cdots06}^{\frac{n}{16}}\overbrace{07\cdots07}^{\frac{n}{16}}$
$\overbrace{10\cdots10}^{\frac{n}{16}}\overbrace{11\cdots11}^{\frac{n}{16}}\overbrace{12\cdots12}^{\frac{n}{16}}\overbrace{13\cdots13}^{\frac{n}{16}}\overbrace{14\cdots14}^{\frac{n}{16}}\overbrace{15\cdots15}^{\frac{n}{16}}\overbrace{16\cdots16}^{\frac{n}{16}}\overbrace{17\cdots17}^{\frac{n}{16}}) \in R^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{01}) = d_L(x, \overline{02}) = d_L(x, \overline{03}) = d_L(x, \overline{04}) = d_L(x, \overline{05}) = d_L(x, \overline{06}) = d_L(x, \overline{07}) = d_L(x, \overline{10}) = d_L(x, \overline{11}) = d_L(x, \overline{12}) = d_L(x, \overline{13}) = d_L(x, \overline{14}) = d_L(x, \overline{15}) = d_L(x, \overline{16}) = d_L(x, \overline{17}) = \frac{n}{16} + 2(\frac{n}{16}) + \frac{n}{16} + 4(\frac{n}{16}) + \frac{n}{16}+2(\frac{n}{16})+\frac{n}{16}+\frac{n}{16}+2(\frac{n}{16})+3(\frac{n}{16})+2(\frac{n}{16})+5(\frac{n}{16})+2(\frac{n}{16})+3(\frac{n}{16})+2(\frac{n}{16}) = \frac{32n}{16} = 2n$. Thus $r_L(C_{11}) = r_L(C_{13}) = r_L(C_{15}) = r_L(C_{17}) \geq 2n$ and hence, $r_L(C_{11}) = r_L(C_{13}) = r_L(C_{15}) = r_L(C_{17}) = 2n$.

(6) In C_{10}, $d_L(x, C_{12}) = d_L(x, C_{16}) = \min\{d_L(x, \overline{00}), d_L(x, \overline{04}), d_L(x, \overline{12}), d_L(x, \overline{16})\} \leq \frac{3n}{2}$, then

$$r_L(C_{12}) = r_L(C_{16}) \leq \frac{3n}{2}.$$

If $x = (\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{04\cdots04}^{\frac{n}{4}}\overbrace{12\cdots12}^{\frac{n}{4}}\overbrace{16\cdots16}^{\frac{n}{4}}) \in R^n$, then $d_L(x,\overline{00}) = d_L(x,\overline{04}) = d_L(x,\overline{12}) = d_L(x,\overline{16}) = 4(\frac{n}{8}) + 3(\frac{n}{8}) + 3(\frac{n}{8}) = \frac{5n}{4}$. Thus $r_L(C_{12}) = r_L(C_{16}) \geq \frac{5n}{4}$ and so, $\frac{5n}{4} \leq r_L(C_{12}) = r_L(C_{16}) \leq \frac{3n}{2}$.

(7) In C_{14}, $d_L(x,C_{14}) = \min\{d_L(x,\overline{00}), d_L(x,\overline{14})\} \leq 3n$, then

$$r_L(C_{14}) \leq 3n.$$

If

$x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{14\cdots14}^{\frac{n}{2}}) \in R^n$. Therefore $d_L(x,\overline{00}) = d_L(x,\overline{14}) = \frac{5n}{4}$. Thus $r_L(C_{14}) \geq \frac{5n}{4}$ and hence $\frac{5n}{4} \leq r_L(C_{14}) \leq 3n$.

□

Theorem 3.2. *These are bounds on the covering radii of $C_j, 1 \leq j \leq 15$, with respect to the Euclidean, Chinese Euclidean and Homogeneous weights are given by*

Codes	Euclidean Weight	Chinese Euclidean Weight	Homogeneous Weight
$(C_{01(03)(05)(07)}) = C_1$	$2n \leq r_E(C_1) \leq 5n$	$n \leq r_{CE}(C_1) \leq 3n$	$n \leq r_{Hom}(C_1) \leq 4n$
$(C_{02(06)}) = C_2$	$\frac{3n}{2} \leq r_E(C_2) \leq 6n$	$n \leq r_{CE}(C_2) \leq 3n$	$n \leq r_{Hom}(C_2) \leq 5n$
$(C_{04}) = C_3$	$n \leq r_E(C_3) \leq 6n$	$n \leq r_{CE}(C_3) \leq 3n$	$n \leq r_{Hom}(C_3) \leq 4n$
$(C_{10}) = C_4$	$\frac{n}{4} \leq r_E(C_4) \leq 7n$	$\frac{n}{4} \leq r_{CE}(C_4) \leq 3n$	$\frac{n}{4} \leq r_{Hom}(C_4) \leq 5n$
$(C_{11(13)(15)(17)}) = C_5$	$\frac{9n}{4} \leq r_E(C_5) \leq \frac{9n}{2}$	$\frac{5n}{4} \leq r_{CE}(C_5) \leq \frac{5n}{2}$	$\frac{5n}{4} \leq r_{Hom}(C_5) \leq 3n$
$(C_{12(16)}) = C_6$	$\frac{7n}{4} \leq r_E(C_6) \leq \frac{11n}{2}$	$\frac{5n}{4} \leq r_{CE}(C_6) \leq 4n$	$\frac{5n}{4} \leq r_{Hom}(C_6) \leq 5n$
$(C_{14}) = C_7$	$\frac{5n}{4} \leq r_E(C_7) \leq \frac{11n}{2}$	$\frac{5n}{4} \leq r_{CE}(C_7) \leq \frac{5n}{2}$	$\frac{5n}{4} \leq r_{Hom}(C_7) \leq 3n$

Proof. The proof is follows from Theorem 3.1 with different weights such as Euclidean, Chinese Euclidean and Homogeneous. □

Block repetition code in R

The block repetition code $C^n : BRep^{n_1+n_2+\cdots+n_{15}}$ over R is a R-additive code.

Let $G = [\overbrace{0101\cdots01}^{n_1}\overbrace{0202\cdots02}^{n_2}\overbrace{0303\cdots03}^{n_3}\overbrace{0404\cdots04}^{n_4}\overbrace{0505\cdots05}^{n_5}\overbrace{0606\cdots06}^{n_6}\overbrace{0707\cdots07}^{n_7}$ $\overbrace{1010\cdots10}^{n_8}\overbrace{1111\cdots11}^{n_9}\overbrace{1212\cdots12}^{n_{10}}\overbrace{1313\cdots13}^{n_{11}}\overbrace{1414\cdots14}^{n_{12}}\overbrace{1515\cdots15}^{n_{13}}\overbrace{1616\cdots16}^{n_{14}}\overbrace{1717\cdots17}^{n_{15}}]$ be a generator matrix with the parameters of C^n :

$n = \sum_{j=1}^{15} n_j$,
$N = 16$,

$d_L = \min\{(32n_1 + 32n_2 + 24n_3 + 32n_4 + 24n_5 + 32n_6 + 24n_7 + 8n_8 + 32n_9 + 32n_{10} + 32n_{11} + 40n_{12} + 32n_{13} + 40n_{14} + 32n_{15}), 32(n_1 + n_2 + n_4 + n_6 + n_9 + n_{10} + n_{11} + n_{13} + n_{15}) + 24(n_3 + n_5 + n_7) + 8n_8 + 40(n_{12} + n_{14})\}$

$d_E = \min\{(72n_1+48n_2+64n_3+32n_4+64n_5+48n_6+64n_7+8n_8+72n_9+56n_{10}+72n_{11}+56n_{12}+64n_{13}+56n_{14}+72n_{15}), 72(n_1+n_9+n_{11}+n_{15})+48(n_2+n_6)+64(n_3+n_5+n_7)+32n_4+8n_8+56(n_{10}+n_{12}+n_{14})+64n_{13}\}$

$d_{CE} = \min\{(40n_1+32n_2+32n_3+32n_4+32n_5+32n_6+32n_7+8n_8+40n_9+40n_{10}+40n_{11}+40n_{12}+40n_{13}+40n_{14}+40n_{15}), 40(n_1+n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})+32(n_2+n_3+n_4+n_5+n_6+n_7)+8n_8\}$

$d_{Hom} = \min\{(40n_1+32n_2+32n_3+32n_4+32n_5+32n_6+32n_7+8n_8+40n_9+40n_{10}+40n_{11}+40n_{12}+40n_{13}+40n_{14}+40n_{15}), 40(n_1+n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})+32(n_2+n_3+n_4+n_5+n_6+n_7)+8n_8\}$

Theorem 3.3. *Let C^n be the block repetition code in R with length is n. Then the covering radius of block repetition code is*

1. $\frac{3(n_1+n_3+n_5+n_7)+4(n_2+n_4+n_6)+8(n_8+n_9+n_{11}+n_{13}+n_{15})+5(n_{10}+n_{12}+n_{14})}{4} \leq r_L(C^n) \leq \frac{40(n_1+n_3+n_5+n_7+n_{10}+n_{14})+48(n_2+n_4+n_6+n_8+n_{12})+32(n_9+n_{11}+n_{13}+n_{15})}{16}$,
2. $\frac{3(n_1+n_3+n_5+n_7)+6(n_2+n_4)+4n_4+n_8+9(n_9+n_{11}+n_{13}+n_{15})+7(n_{10}+n_{14})+5n_{12}}{4} \leq r_E(C^n) \leq \frac{80(n_1+n_3+n_5+n_7+n_9+n_{11}+n_{13}+n_{14}+n_{15})+62(n_2+n_{12})+96(n_4+n_6+n_{10})+144n_8}{16}$,
3. $\frac{4(n_1+n_2+n_3+n_4+n_5+n_6+n_7)+n_8+5(n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})}{4} \leq r_{CE}(C^n) \leq 3(n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})$,
4. $\frac{4(n_1+n_2+n_3+n_4+n_5+n_6+n_7)+n_8+5(n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})}{4} \leq r_{Hom}(C^n) \leq 3(n_1+n_8+n_{11}+n_{12}+n_{13}+n_{15})+4(n_2+n_3+n_4+n_5+n_6+n_7+n_9+n_{10}+n_{14})$.

Proof. Use to ref. [9] and Theorem 3.1, 3.2, thus
$r_L(C^n) \geq \frac{3(n_1+n_3+n_5+n_7)+4(n_2+n_4+n_6)+8(n_8+n_9+n_{11}+n_{13}+n_{15})+5(n_{10}+n_{12}+n_{14})}{4}$.

Let $x = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}x_{15} \in R^n$ with $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}$ is $(a_i), (b_i), (c_i), (d_i), (e_i), (f_i), (g_i), (h_i), (k_i), (l_i), (m_i), (n_i), (o_i), (p_i), (q_i)$, $_{i=0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}$, respectively such that $n_1 = \sum_{j=0}^{15} a_j$, $n_2 = \sum_{j=0}^{15} b_j$, $n_3 = \sum_{j=0}^{15} c_j$, $n_4 = \sum_{j=0}^{15} d_j$, $n_5 = \sum_{j=0}^{15} e_j$, $n_6 = \sum_{j=0}^{15} f_j$, $n_7 = \sum_{j=0}^{11} g_j$, $n_8 = \sum_{j=0}^{15} h_j$, $n_9 = \sum_{j=0}^{15} k_j$, $n_{10} = \sum_{j=0}^{15} l_j$, $n_{11} = \sum_{j=0}^{15} m_j$, $n_{12} = \sum_{j=0}^{15} n_j$, $n_{13} = \sum_{j=0}^{15} o_j$, $n_{14} = \sum_{j=0}^{15} p_j$, $n_{15} = \sum_{j=0}^{15} q_j$.

Thus, $r_L(C^n) \leq \frac{40(n_1+n_3+n_5+n_7+n_{10}+n_{14})+48(n_2+n_4+n_6+n_8+n_{12})+32(n_9+n_{11}+n_{13}+n_{15})}{16}$.

Similarly, $r_E(C^n), r_{CE}(C^n), r_{Hom}(C^n)$. □

References

[1] Abualrub T., Siap I. and Aydin N., *$\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes,* IEEE Trans. Inform. Theory, 60(3) (2014), 115–121.

[2] Bilal M., Borges J., Dougherty S. T. and Fernandez-Cordoba C., *Maximum distance separable codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_4$,* Des. Codes Cryptogr., 61(1),(2011), 31–40.

[3] Borges, Dougherty S. T. and Fernandez-Cordoba C., *Characterization and constructions of self-dual codes over* $\mathbb{Z}_2\mathbb{Z}_4$, Adv. Math. Commun., 6(3), (2012), 287-303.

[4] Borges, Fernandez-Cordoba C., Pujol J., Rifa J. and Villanueva M., $\mathbb{Z}_2\mathbb{Z}_4$*-linear codes: generator matrices and duality,* Des. Codes Cryptogr., 54(2),(2010), 167–179.

[5] Carlet C., $\mathbb{Z}_{2^k}$*-linear codes,* IEEE Trans. Inform. Theory 44 no. 4 (1998), 1543-1547.

[6] Chella Pandian P., *On the Covering Radius of Codes Over* $\mathbb{Z}_4$ *with Chinese Euclidean Weight,* Journal on Information Theory, 4(4), (2015), pp.1-8.

[7] Chella Pandian P., *On the Covering Radius of Codes Over* $\mathbb{Z}_6$,Journal on Information Theory, 5(2), (2016), pp.1-9.

[8] Constantinescu I., Heise W. and Honold T., *Monomial extensions of isometries between codes over* $\mathbb{Z}_m$, Proc. workshop ACCT'96, Sozopol, Bulgaria, 1996, 98-104.

[9] Cohen G. D., Karpovsky M. G., Mattson H. F. and Schatz J. R., *Covering radius-survey and recent results,* IEEE Trans. Inform. Theory, 31(3), (1985), 328–343.

[10] Delsarte P., *An algebraic approach to the association schemes of coding theory,* Philips Research Rep. Suppl., 10, (1973).

[11] Delsarte P. and Levenshtein V., *Association schemes and coding theory,* IEEE Trans. Inform. Theory, 44(6), (1998), 2477–2504.

[12] Gupta M. K., Glynn D. G. and Aaron Gulliver T., *On senary simplex codes,* in International Symposium, on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes(Springer, Berlin Heidelberg, 2001), pp. 112-121.

[13] Raut M. K. and Gupta M. K., *On octonary codes and their covering radii,* arxiv:1411.1822v3 [cs.IT], (2014).