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New Ideas on Probability

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Abstract: Why is the Gambling Mathematics so distant from the Standard Probability Theory? Behind the use of rather different algebraic functions as predictive tools, there should be something more than their mere origins, respectively empirical and theoretical, to divide the two ways so sharply. If we approach the question through the Physical Probability, by analyzing on a semi-empirical basis a random binary experiment and by exploring the physical roots of the problem, we find out at least two original results. Epistemologically speaking, it emerges that the standard probability axioms need the *linear time* while the gambling strategies are founded on a manifold *non-linear* temporal conception, thus a recent topic like the search for the most effective probability patterns is reducible to the ancient issue of gleaning the intrinsic nature of Time. Heuristically speaking, it is possible a third-way model that supports a hypothetical *recursive time* and supplies new formulas for the geometric distribution.

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1. Introduction

We present a New Physical Approach to Probability (acronym $NPAP$) in the tradition of the Frequency Theory [36] developed by von Mises [59-61], with further elements recalling the Propensity Theory [36] introduced by Popper [46] and the Physicalist Theory [53] proposed by Szabó [54].

With such deeply inspiring works, this research has in common the will to investigate a random experiment from a semi-empirical perspective and the ambition to propose epistemological and heuristic innovations.

In particular, our efforts are focused on a rarely tried before [38-40] examination of the temporal conception behind each considered probability pattern and on the formulation of a new model.

The first results are about the Standard Probability Theory (SPT for brevity) whose axioms [36] imply the physical assumption of the *linear time*.

Other results concern the Gambling Mathematics (GM for short) whose strategies [56] seem independent of the SPT and imply a manifold *non-linear* temporal frame.

The major achievements regard the here developed $NPAP$ whose principles and formulas, independent of both the SPT and the GM, require a *non-linear time* less vague than the GM's. Some open questions about the $NPAP$ are eventually left to the keen reader.

2. A random binary game

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Let us flip an unbiased coin n times and let us place bets on each toss (also called "trial" in what follows).

Let us suppose to continue betting on the same outcome until the first success.

We want to solve this problem: how much should we bet on the next trial if the unfavorable outcomes have always occurred since we started the game?

It is related to the theoretical question of determining the probability distribution of the number of failures before a success, *i.e.*, the geometric distribution.

The answer is in the following two interrelated steps:

- we look for the most effective predictive formula;
- we bet coherently with our rational expectations.

Let E be the favorable event, arbitrarily "Heads" or "Tails", and let Ē be the unfavorable event.

Let n be the number of times the coin has already been flipped and let $n(\bar{\mathrm{E}})$ be the number of unfavorable events already occurred since the first trial; according to our premise:

$$n(\bar{\mathrm{E}}) = n \tag{1}$$

Let $p_{n+1}(\mathrm{E})$ be the probability of success in the next trial, after *n* consecutive failures.

Let $p_{n+1}(\bar{\mathrm{E}})$ be the probability of failure in the next trial, after *n* consecutive failures; obviously:

$$p_{n+1}(\bar{\mathrm{E}}) = 1 - p_{n+1}(\mathrm{E}) \tag{2}$$

Let $R_{n+1}(\mathrm{E})$ be the odds, *i.e.*, the ratio of the probability of the favorable event to the probability of the unfavorable event in the next trial:

$$R_{n+1}(\mathrm{E}) = \frac{p_{n+1}(\mathrm{E})}{p_{n+1}(\bar{\mathrm{E}})} \tag{3}$$

Let us infer the probability from the odds:

$$p_{n+1}(\mathrm{E}) = \frac{R_{n+1}(\mathrm{E})}{1+R_{n+1}(\mathrm{E})} \tag{4}$$

Let S_{n+1} be the bet size, *i.e.*, the sum of money we bet on the next trial; we assume that any player would place bets corresponding to the product between the minimum initial threshold S_1 and the odds $R_{n+1}(\mathrm{E})$:

$$S_{n+1} = S_1 R_{n+1}(\mathrm{E}) \tag{5}$$

Let L_n be the loss of money accumulated at the n^{th} unfavorable consecutive outcome, it depends on the predictive model the player has adopted.

Let W_n be the sum a player should win on the successful n^{th} trial, *i.e.*, if the negative sequence of unfavorable outcomes had been interrupted just there, after $n-1$ consecutive failures; we assume that any bookmaker would pay $2:1$, *i.e.*, the double of the sum bet on the n^{th} outcome:

$$W_n = 2S_n \text{ , with } n > 0 \tag{6}$$

Let G_n be the actual gain on the n^{th} trial:

$$G_n = W_n - L_n \text{ , with } n > 0 \tag{7}$$

Let us now examine how the same binary event game is diversely tackled by the SPT, GM and $NPAP$.

3. The standard probability approach

3.1 How the SPT faces the problem

A repeated coin flipping is a Bernoulli process [2] where successive trials must be independent of each other. Past outcomes do not influence future events, *i.e.*, the probability of success or failure in the next trial is exactly the same of the first toss.

Since the two possible alternatives keep equally likely, the SPT's probabilities are:

$$p_{n+1}(\mathrm{E}) = \frac{1}{2} \tag{8}$$

$$p_{n+1}(\bar{\mathrm{E}}) = \frac{1}{2} \tag{9}$$

The consequent odds are:

$$R_{n+1}(\mathrm{E}) = 1 \tag{10}$$

According to the $Eq.(10)$ the bet size never changes:

$$S_{n+1} = S_1 \tag{11}$$

The loss of money grows linearly:

$$L_n = nS_1 \quad (12)$$

The potential winnings are stationary:

$$W_n = 2S_1 \quad (13)$$

The actual gain is:

$$G_n = (2 - n)S_1 \quad (14)$$

3.2 SPT and linear time

In the SPT each outcome is assigned a constant and equal probability by the "Principle of Indifference" [36] also known as the "Principle of Insufficient Reason".

In fact, behind the $Eq. (8)$ there is the conviction that, although the favorable event has not occurred for many consecutive times, we have *no reason* to overestimate it as next outcome or even to believe it will ever happen.

Similarly, behind the $Eq. (9)$ there is the conviction that, although the unfavorable event has occurred several consecutive times, we have *no reason* to underestimate it as next outcome or even to exclude it will happen forever.

Hence in the SPT there is *no reason* to exclude that one out of two incompatible events can occur infinite times in a row while the other event never.

This is an axiom involving the nature of Time that the SPT believes to be *linear*, meaning with a potentially infinite future not affected from past results.

As a consequence, the stochastic *independence* [33] nullifies the possibility to apply the equal distribution of probabilities at 50% between Heads and Tails outside a very large (potentially infinite) population, *i.e.*, in ordinary sequences of coin tosses.

The *incommensurability*, due to the linear time, between the infinite set of events (where the SPT's rules are properly formulated) and a finite sequence, hinders any intuitive compensation of the outcomes [57] in order to restore the population proportion in deviating sequences [31].

Time is therefore assumed *structureless*, without any scaling up allowing the Laws of Large Numbers [33] to be applicable also to short sequences of trials [44].

Such absence of structure requires a *scalar* and *homogeneous* time, two more features (in addition to its infinity) completing the definition of "linearity".

Beyond such implicit temporal hypothesis, there are also consequences from its *explicit* assumption: in an indefinitely extended linear time there is the often insurmountable problem to identify the first-ever event and to track a whole sequence; the only solution to this lack of data (that we could define as "Principle of Insufficient Information") is to assign a constantly equal probability to each possible outcome, regardless of the number of trials.

It means to rule out the temporal factor as irrelevant in the probability description and to abandon any *a posteriori* analysis of the phenomena, both consequences of the SPT's linearity of time.

Even interesting interpretations trying to infer the probability rules from symmetry only, *i.e.*, rejecting any ignorance-driven principle [52], need the scalar one-dimensional homogeneous time.

However described, the memoryless random processes [33, 55] require the classical linear time, but this physical conception is not free from paradoxical consequences when applied to chances, as acutely pointed out by Handfield [37], and it has recently been challenged by alternative *multidimensional time* theories such as the 2T [3] and the 3T [5-28, 43, 49, 50, 62-67].

3.3 SPT and Euclidean geometry

The graph described by the $Eq.(9)$ is a set of points belonging to the horizontal line $y = \frac{1}{2}$ which never intersects the null probability axis $y = 0$.

It seems recalling Euclid's fifth postulate [32] in the Playfair's formulation [45] because there is a unique line parallel to the x-axis passing for the point $A(0, \frac{1}{2})$.

The possible geometrical interpretation of the SPT as a sort of *flat* probability is an open question.

4. The gambling mathematical approach

4.1 How the GM faces the problem

Contrarily to the SPT's interpretation of recurrences as unpredictable chance fluctuations, the GM is built on the expectation of regularities in the event sequences.

Although stigmatized as fallacious by several philosophers [51], this attitude is less disturbing than the ambiguity in its consequences. In fact, while the GM's approach to probability obtains here an epistemological absolution from the accusation to be cognitively biased, the variety of dissimilar gambling strategies denotes the GM's heuristic inconsistency.

Let us firstly notice that some elaborate betting systems (*e.g.*, Fibonacci, Labouchère, etc…) are inapplicable to the specific case we are examining.

Let us also notice that the betting systems based on augmented expectations of success after consecutive failures are prevailing on others which instead prefer the just occurred outcome.

With these premises, the GM's best representative for our purposes is known as "Martingale", an old-fashioned strategy where we double the bet after each losing toss; it means that in our situation (see the $Eq.(1)$) the bet size rises exponentially:

$$S_{n+1} = 2^n S_1 \tag{15}$$

The consequent odds are:

$$R_{n+1}(\mathrm{E}) = \frac{S_{n+1}}{S_1} = 2^n \tag{16}$$

We can infer the probabilities:

$$p_{n+1}(\mathrm{E}) = \frac{2^n}{1+2^n} \tag{17}$$

$$p_{n+1}(\bar{\mathrm{E}}) = \frac{1}{1+2^n} \tag{18}$$

According to the $Eq.(15)$ the loss of money is:

$$L_n = (2^n - 1)S_1 \tag{19}$$

The potential winnings grow in the same way:

$$W_n = 2^n S_1 \tag{20}$$

The actual gain is stationary:

$$G_n = S_1 \tag{21}$$

Other famous strategies consist of waiting for *two* or *three* consecutive failures before applying the Martingale betting system; it means to use the $Eqs.(17)$ and (18) after having shifted the starting point from $n(\bar{\mathrm{E}}) = 0$ to, respectively, $n(\bar{\mathrm{E}}) = 2$ or $n(\bar{\mathrm{E}}) = 3$.

Since the Martingale is one of the most extreme strategies and it is substantially inapplicable for hitting the table limits, we can establish the $Eqs.(17)$ and (18) as the upper bounds of intervals containing all the gambling behaviors from cautiousness to imprudence.

As lower bounds we obviously choose the SPT's probabilities expressed by the $Eqs.(8)$ and (9).

It is the maximum generalization of the GM's main tendency, *i.e.*, to place increasing bets after consecutive unfavorable events, whose formulas are:

$$\frac{1}{2} \leq p_{n+1}(\mathrm{E}) \leq \frac{2^n}{1+2^n} \tag{22}$$

$$\frac{1}{2} \leq p_{n+1}(\bar{\mathrm{E}}) \leq \frac{1}{1+2^n} \tag{23}$$

4.2 GM and non-linear time

The GM's different expectations on an upcoming trial on account of the previous sequence of outcomes, studied as "Gambler's fallacy" [48], seem motivated by a sort of "Principle of Sufficient Reason", *i.e.*, the opposite of the SPT's Principle of Insufficient Reason.

In fact, behind the $Eq.(17)$ there is the conviction that, since the favorable event has not occurred for many consecutive times, we have *a reason* to prefer it as next outcome, even if it could never happen.

Similarly, behind the $Eq.(18)$ there is the conviction that, since the unfavorable event has occurred several consecutive times, we have *a reason* to reject it as next outcome, even if it could happen forever.

Hence in the GM there is *a reason* (not the certainty) to exclude that one out of two incompatible events can occur infinite times in a row while the other event never.

This is an axiom involving the nature of Time that the GM considers *non-linear*, meaning with observable interactions among successive outcomes from any "present" into an indefinitely far away future.

The examined GM shows a belief in what Tversky and Kahneman [58] defined the "fairness" of the coin, *i.e.*, in the so called "Law of Small Numbers".

It is the propensity to regard a sample as highly representative of the whole, *e.g.*, the proportion of a whole random sequence manifested in each segment, with deviations inversely proportional to its length.

If it was true, then the macroscopic probability rules should be somehow applicable also at microscopic level: it would mean to remove that *independence* between successive events postulated by the SPT and to abandon the SPT's structureless time.

Thus the GM implicitly assumes a *structured time* whose features are not univocally defined because of the multiplicity of the developed strategies, even opposite to each other.

Anyway a structured time lacks of a basic feature to be linear, so we get a further confirmation that the GM rejects the linear time hypothesis.

4.3 GM and hyperbolic geometry

The graph described by the solutions of the $Eq.(18)$ is a set of points belonging to the curves $y = \frac{1}{1+2^n}$ which never intersects the null probability axis $y = 0$.

It seems related to the negation of Euclid's fifth postulate, therefore to a non-Euclidean geometry. It recalls the hyperbolic geometry [1] by Lobachevsky [41, 42] because the x-axis is an unreachable asymptote for infinitely many curves and not for one only.

The possible geometrical interpretation of the GM as a sort of *hyperbolic* probability is an open question.

5. The new physical approach to probability

5.1 How the $NPAP$ faces the problem

The first information we need is empirical: what is the longest run (*i.e.*, sequence of identical events) ever registered in the history of binary event games?

As far as coin-flips are concerned it is impossible to answer because a coin has never been thrown in the air under reproducibility conditions, except the notable experiment conducted by Diaconis et al. [30] showing that coin tossing is "fair" to two decimals but not to three. Although not decisive, that surprising random binary test is enough to confirm the necessity to develop of a "third-way" model between the SPT and the GM.

A game whose features are similar to "Heads or Tails" is "Red or Black" at the roulette. It is not exactly the same, because the "zero" is an extra-choice beyond the binary event (even if a third outcome is possible also in our experiment, when the coin lands on its edge), but it constitutes a reliable basis for our research.

According to unofficial but trustworthy casino rumors the longest run ever occurred is a twenty-three times repetition of the same color.

We can formalize such information as $n_{max}(\bar{\mathrm{E}}) = 23$ in order to establish equations on the linear decrease from the maximum adverse probability before playing $p_1(\bar{\mathrm{E}}) = \frac{1}{2}$ until the null probability after the 23^{rd} consecutive unfavorable outcome $p_{24}(\bar{\mathrm{E}}) = 0$:

$$p_{n+1}(\mathrm{E}) = \frac{23+n}{46} \tag{24}$$

$$p_{n+1}(\bar{\mathrm{E}}) = \frac{23-n}{46} \tag{25}$$

The consequent odds are:

$$R_{n+1}(\mathrm{E}) = \frac{23+n}{23-n} \tag{26}$$

Given that we should not play games devoid of an execution protocol (*e.g.*, coin-tossing), according to the $Eq.(26)$ the bet size in a binary event game should grow as follows:

$$S_{n+1}(\mathrm{E}) = \frac{23+n}{23-n} S_1 \tag{27}$$

The loss of money would be:

$$L_n = S_1(1 + R_1 + \cdots + R_n) \tag{28}$$

The potential winnings would be:

$$W_n = 2R_n S_1 \tag{29}$$

The actual gain would be:

$$G_n = S_1(R_n - R_{n-1} + \cdots - R_2 - 1) \tag{30}$$

We may establish a semi-empirical extension of the empirical data collected above by choosing as turning point an integer $m \geq 23$. Such generalization constitutes the physical approach to probability we were looking for, whose equations are:

$$p_{n+1}(\mathrm{E}) = \frac{m+n}{2m} \text{ , with } m = 23,24,25 \ldots \tag{31}$$

$$p_{n+1}(\bar{\mathrm{E}}) = \frac{m-n}{2m} \text{ , with } m = 23,24,25 \ldots \tag{32}$$

The consequent odds are:

$$R_{n+1}(\mathrm{E}) = \frac{m+n}{m-n} \tag{33}$$

According to the $Eq.(33)$ the bet size in a binary event game should grow as follows:

$$S_{n+1}(\mathrm{E}) = \frac{m+n}{m-n} S_1, \tag{34}$$

The loss of money would be:

$$L_n = S_1(1 + R_1 + \cdots + R_n), \tag{35}$$

The potential winnings would be:

$$W_n = 2R_n S_1 \tag{36}$$

The actual gain would be:

$$G_n = S_1(R_n - R_{n-1} + \cdots - R_2 - 1) \tag{37}$$

Although the linear function of the $Eq.(32)$ seems the most rational choice to connect the extreme points $\mathrm{A}(0, \frac{1}{2})$ and $\mathrm{B}(m, 0)$, we can introduce the following higher degree curves with the integer parameters $k > 1$ and $m \geq 23$:

$$p_{n+1}(\mathrm{E}) = \frac{2m - \sqrt[k]{m^k - n^k}}{2m} \tag{38}$$

$$p_{n+1}(\bar{\mathrm{E}}) = \frac{\sqrt[k]{m^k - n^k}}{2m} \tag{39}$$

Let us notice that if $k = 2$ then the solutions of the $Eq.(39)$ are all contained in the elliptic function $\left(\frac{x}{m}\right)^2 + (2y)^2 = 1$.

We may also notice that if $k = 1$ then the $Eqs.(38)$ and (39) are respectively reducible to the $Eqs.(31)$ and (32); so the following equations, still characterized by the integer parameter $m \geq 23$, represent the most general functions applicable to our model:

$$p_{n+1}(\mathrm{E}) = \frac{2m - \sqrt[k]{m^k - n^k}}{2m} \text{ , with } k \in \mathbb{Z}^+ \tag{40}$$

$$p_{n+1}(\bar{\mathrm{E}}) = \frac{\sqrt[k]{m^k - n^k}}{2m} \text{ , with } k \in \mathbb{Z}^+ \tag{41}$$

The opportunity to use $k > 1$ in the $Eqs. (40)$ and (41), instead of the chosen $k = 1$, and the possible relation with the functions describing the risk-preference in coin-toss games [29] are open questions.

5.2 $NPAP$ and non-linear time

The $NPAP$'s different expectation on an upcoming trial on the basis of the previous sequence of outcomes is motivated by what we may call "Principle of Certainty", different from both the SPT's Principle of Insufficient Reason and the GM's Principle of Sufficient Reason.

In fact, behind the $Eq. (40)$ there is the conviction that, since the favorable event has not occurred for many consecutive times, we have the *certainty* it will happen and we have a propensity for it as next outcome.

Similarly, behind the $Eq. (41)$ there is the conviction that, since the unfavorable event has occurred several consecutive times, we have the *certainty* to exclude it will happen forever and we have a disinclination for it as next outcome.

Hence in the $NPAP$ there is the *certainty* to exclude that one out of two incompatible events can occur infinite times in a row while the other event never.

We could be sure that the number of trials necessary before the first success in a binary event game is *finite* in virtue of all the proofs showing that both the outcomes have always occurred in a finite time, but it would not be an irrefutable argument. In fact, it could be criticized by the SPT as an improper overestimation of the collected data: compared to the eternity, in a *linear time*, even the whole human experience about games is not sufficiently significant.

Therefore our "certainty" is nothing more than an axiom involving the nature of Time that the $NPAP$ considers *non-linear*, meaning with an indefinitely (but finitely) far away future linked with the past through cyclical (not necessarily identical) sequences similar, *e.g.*, to those tracked in skilled performances [35].

We must be aware that the $NPAP$ explicitly assumes a time with some *recursive structure* (*e.g.*, the canonical form supersystem-system-subsystem of the holographic paradigm [4]) by which the substantially same structure is scaled indefinitely up.

If it was true, some of the regularities invoked by the GM could find a physical justification because in a non-linear temporal frame the periodicity of the events is not as illusionary as postulated in the SPT's randomness.

The $NPAP$'s *geometric distribution* is innovative because the number of failures before a success is finite and not potentially infinite as the SPT's and the GM's.

Although the $NPAP$'s Future is supposed related to the Past like the GM's, differently from both the SPT's and the GM's Future it is not infinitely far from the Present.

Anyway, if the turning point is very high $(m \to \infty)$ then the intersection between the line containing the sets of points $p_{n+1}(\bar{E})$ and the x-axis is far enough to repute *quasi-linear* the $NPAP$'s time.

5.3 $NPAP$ and elliptic geometry

The graph described by the solutions of the $Eq. (32)$ is a set of points belonging to the straight lines $y = \frac{m-x}{2m}$ which intersect the null probability axis $y = 0$ in $x = m \geq 23$ integer root.

The same intersection point is shared by the curves $y = \frac{\sqrt[k]{m^k - n^k}}{2m}$ with $k \in \mathbb{Z}^+$, containing the solutions of the $Eq. (41)$.

They both seem related to the negation of Euclid's fifth postulate, therefore to a non-Euclidean geometry. It recalls the elliptic geometry [34] by Riemann [47] because there is no line parallel to the x-axis.

The possible geometrical interpretation of the $NPAP$ as a sort of *elliptic* probability is an open question.

6. Comparisons among the results

Let us compare the numerical results obtained in this paper.

Table 1 Probability of the favorable event

		SPT	GM	$NPAP$
$n(\bar{E})$	$p_{n+1}(E)$	$Eq.(8)$	$Eq.(17)$	$Eq.(24)$
0	$p_1(E)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	$p_2(E)$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{12}{23}$
2	$p_3(E)$	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{25}{46}$
3	$p_4(E)$	$\frac{1}{2}$	$\frac{8}{9}$	$\frac{13}{23}$
4	$p_5(E)$	$\frac{1}{2}$	$\frac{16}{17}$	$\frac{27}{46}$
...	...	...	...	...

The $Table$ 1 evidences that the $NPAP$'s probability of success grows in dependence of the number of previous consecutive failures, but slower than the GM's.

Table 2 Probability of the unfavorable event

		SPT	GM	$NPAP$
$n(\bar{E})$	$p_{n+1}(\bar{E})$	$Eq.(8)$	$Eq.(18)$	$Eq.(25)$
0	$p_1(\bar{E})$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	$p_2(\bar{E})$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{11}{23}$
2	$p_3(\bar{E})$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{21}{46}$
3	$p_4(\bar{E})$	$\frac{1}{2}$	$\frac{1}{9}$	$\frac{10}{23}$
4	$p_5(\bar{E})$	$\frac{1}{2}$	$\frac{1}{17}$	$\frac{19}{46}$
...	...	...	...	...

The $Table$ 2 evidences that the $NPAP$'s probability of failure decreases in dependence of the number of previous consecutive failures, but slower than the GM's.

Table 3 Odds of the favorable event

		SPT	GM	$NPAP$
$n(\bar{E})$	$R_{n+1}(E)$	$Eq.(10)$	$Eq.(16)$	$Eq.(26)$
0	$R_1(E)$	1	1	1
1	$R_2(E)$	1	2	$\frac{12}{11}$
2	$R_3(E)$	1	4	$\frac{25}{21}$
3	$R_4(E)$	1	8	$\frac{13}{10}$
4	$R_5(E)$	1	16	$\frac{27}{19}$
...	...	...	...	...

The $Table$ 3 evidences that the $NPAP$'s odds of success grow in dependence of the number of previous consecutive failures, but slower than the GM's.

Table 4 Bet on the favorable odds

		SPT	GM	$NPAP$
$n(\bar{E})$	$S_{n+1}(E)$	$Eq.(11)$	$Eq.(15)$	$Eq.(27)$
0	$S_1(E)$	S_1	S_1	S_1
1	$S_2(E)$	S_1	$2S_1$	$\frac{12S_1}{11}$
2	$S_3(E)$	S_1	$4S_1$	$\frac{25S_1}{21}$
3	$S_4(E)$	S_1	$8S_1$	$\frac{13S_1}{10}$
4	$S_5(E)$	S_1	$16S_1$	$\frac{27S_1}{19}$
...	...	...	...	...

The $Table$ 4 evidences that the $NPAP$'s bet size grows in dependence of the number of previous consecutive failures, but slower than the GM's.

Table 5 Loss in case of unfavorable outcome

		SPT	GM	$NPAP$
$n(\bar{\text{E}})$	$L_n(\text{E})$	$Eq.(12)$	$Eq.(19)$	$Eq.(28)$
1	$L_1(\text{E})$	S_1	S_1	S_1
2	$L_2(\text{E})$	$2S_1$	$3S_1$	$\frac{23S_1}{11}$
3	$L_3(\text{E})$	$3S_1$	$7S_1$	$\frac{758S_1}{231}$
4	$L_4(\text{E})$	$4S_1$	$15S_1$	$\frac{10583S_1}{2310}$
...	...	...	...	...

The $Table\,5$ evidences that the $NPAP$ is less dangerous than the GM in case of consecutive failures, but more than the SPT.

Table 6 Winnings in case of favorable outcome

		SPT	GM	$NPAP$
$n(\bar{\text{E}})$	$W_n(\text{E})$	$Eq.(13)$	$Eq.(20)$	$Eq.(29)$
1	$W_1(\text{E})$	$2S_1$	$2S_1$	$2S_1$
2	$W_2(\text{E})$	$2S_1$	$4S_1$	$\frac{24S_1}{11}$
3	$W_3(\text{E})$	$2S_1$	$8S_1$	$\frac{50S_1}{21}$
4	$W_4(\text{E})$	$2S_1$	$16S_1$	$\frac{13S_1}{5}$
...	...	...	...	...

The $Table\,6$ evidences that the $NPAP$ is less rewarding than the GM in case of success, but more than the SPT.

Table 7 Actual gain in case of favorable outcome

		SPT	GM	$NPAP$
$n(\bar{\text{E}})$	$G_n(\text{E})$	$Eq.(14)$	$Eq.(21)$	$Eq.(30)$
1	$G_1(\text{E})$	S_1	S_1	S_1
2	$G_2(\text{E})$	0	S_1	$\frac{S_1}{11}$
3	$G_3(\text{E})$	$-S_1$	S_1	$-\frac{208S_1}{231}$
4	$G_4(\text{E})$	$-2S_1$	S_1	$-\frac{4577S_1}{2310}$
...	...	...	...	...

The $Table\,7$ evidences that according to the SPT it is convenient to play only once whatever is the result, without placing any further bets, while the GM entices to continue playing until the winnings whatever is the sum of money, even huge, to spend in the meantime (in reality, the earning limits imposed by casinos have largely reduced the possibility to apply the Martingale strategy).

The $NPAP$ seems less discouraging towards the binary event games than the SPT, because a player is enticed to bet two consecutive times, also when using the $Eqs.(31)$ and (32) or the $Eqs.(40)$ and (41), because G_1 and G_2 are positive while G_3 is negative independently of the chosen m or k.

According to the $NPAP$ if we do not win at the second trial then we progressively lose money, but not so quickly as in the GM.

7. Conclusions

We have analyzed a binary event game from different perspectives in order to investigate the SPT's and GM's well-consolidated predictive patterns and for proposing a new semi-empirical model.

We have found that, however motivated (by the Principle of Indifference or by symmetry reasons), the SPT has the physical implication of a *linear time* and its probability of success after n consecutive failures is $p_{n+1}(\text{E}) = \frac{1}{2}$.

The GM's examined strategies seem founded on what we have defined "Principle of Sufficient Reason" which implies a not precisely identifiable *non-linear* temporal frame. The GM's mainstream probability of success after n consecutive failures is $\frac{1}{2} \leq p_{n+1}(\mathrm{E}) \leq \frac{2^n}{1+2^n}$.

The here proposed $NPAP$ is based on the "Principle of Certainty" (*i.e.*, the axiom that the success must occur after a finite number of trials) which has the explicit assumption of a recursively structured *non-linear time*. The $NPAP$'s probability of success after n consecutive failures is $p_{n+1}(\mathrm{E}) = \frac{m+n}{2m}$ with $m \geq 23$ integer and its improvement $p_{n+1}(\mathrm{E}) = \frac{2m-\sqrt[k]{m^k-n^k}}{2m}$ (with $k \in \mathbb{Z}^+$) is an open question.

To the enthusiastic and skilled readers are also left the following intriguing questions:

- Is there any relation between our results and the functions describing the risk-preference in coin-toss games studied by mathematical psychology?

- Is it possible to extend the algebraic achievements here supplied beyond the "binary event" limited context where they were attained?

- Is the observed behavior of the natural events, apparently closer to both the GM and the $NPAP$ than to the SPT, a significant clue in favor of a non-linear time model yet to be exhaustively formulated in Physics?

- Would it be plausible a three-fold geometrical classification of probability in *flat*, *hyperbolic* and *elliptic* respectively referred to the SPT, GM and $NPAP$?

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