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On composition of multidimensional integral operators involving general polynomials and multivariable A-function

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ABSTRACT
Jain and al [8] have studied composition formulae of a class of multidimensional fractional integral operators involving the product of the multivariable H-function and general class of multivariable polynomials. In this paper, we obtain three compositions formulae of multidimentional fractional integral operators associated with multivariable A-function defined by Gautam et al [3] and class of multivariable polynomials defined by Srivastava [11].

Keywords: Multivariable A-function, multivariable class of polynomials, multidimensional fractional integral operator, multivariable H-function

2010 Mathematics Subject Classification :33C05, 33C60

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, it has turned out many phenomena in physics, mechanics, chemistry, biology and other sciences can be described very successfully by models using mathematical tools by models using mathematical tools from fractional calculus.
The generalized polynomials defined by Srivastava [9], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.1}$$

where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.

The A-function is defined and represented in the following manner.

$$A(z_1,\cdots,z_r) = A_{p,q:p_1,q_1;\cdots;p_r,q_r}^{m,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;A_j^{(1)},\cdots,A_j^{(r)})_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;B_j^{(1)},\cdots,B_j^{(r)})_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.2}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.3}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\sum_{i=1}^{r}B_j^{(i)}s_i)\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}A_j^{(i)}s_j)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}A_j^{(i)}s_j)\prod_{j=m+1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}B_j^{(i)}s_j)} \tag{1.4}$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}s_i)\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}s_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}s_i)\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+D_j^{(i)}s_i)} \tag{1.5}$$

Here $m,n,p,m_i,n_i,p_i,c_i\in\mathbb{N}^*; i=1,\cdots,r$; $a_j,b_j,c_j^{(i)},d_j^{(i)},A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)}\in\mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i)z_k|<\frac{1}{2}\eta_k\pi,\xi^*=0,\eta_i>0 \tag{1.6}$$

$$\Omega_i=\prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}}; i=1,\cdots,r \tag{1.7}$$

$$\xi_i^*=Im(\sum_{j=1}^{p}A_j^{(i)}-\sum_{j=1}^{q}B_j^{(i)}+\sum_{j=1}^{q_i}D_j^{(i)}-\sum_{j=1}^{p_i}C_j^{(i)}); i=1,\cdots,r \tag{1.8}$$

$$\eta_i=Re\left(\sum_{j=1}^{n}A_j^{(i)}-\sum_{j=n+1}^{p}A_j^{(i)}+\sum_{j=1}^{m}B_j^{(i)}-\sum_{j=m+1}^{q}B_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)$$

$$i=1,\cdots,r \tag{1.9}$$

This function is an extension of multivariable H-function defined by Srivastava [11]

2. Definitions

The pair of new extended fractional integral operators are defined by the following equations :

$$Q_{x,z,E}^{\rho_i,\sigma_i}[f(t_1,\cdots,t_r)]=\prod_{j=1}^{r}x_j^{-\rho_j-\sigma_j}\int_0^{x_1}\cdots\int_0^{x_r}\prod_{j=1}^{r}t^{\rho_j}(x_j-t_j)^{\sigma_j-1}A_{p,q:p_1,q_1;\cdots;p_r,q_r}^{m,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c}\mathrm{E}_1\left(1-\frac{t_1}{x_1}\right)^{v_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{E}_r\left(1-\frac{t_r}{x_r}\right)^{v_r}\end{array}\right|$$

$$\left.\begin{array}{c}(\mathrm{a}_j;A_j^{(1)},\cdots,A_j^{(r)})_{1,p}:(\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r}\\ \\ (\mathrm{b}_j;B_j^{(1)},\cdots,B_j^{(r)})_{1,q}:(\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{array}\right)S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c}\mathrm{z}_1\left(1-\frac{t_1}{x_1}\right)^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\left(1-\frac{t_r}{x_r}\right)^{h_r}\end{array}\right)$$

$$f(t_1,\cdots,t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{2.1}$$

and

$$R_{x,z',E'}^{\rho'_i;\sigma'_i}\left[f(t_1,\cdots,t_r)\right]=\prod_{j=1}^{r}x_j^{\rho'_j}\int_{x_1}^{\infty}\cdots\int_{x_r}^{\infty}\prod_{j=1}^{r}t^{\rho'_j-\sigma'_j}(x_j-t_j)^{\sigma'_j-1}A_{p',q':p'_1,q'_1;\cdots;p'_r,q'_r}^{m',n':m'_1,n'_1;\cdots;m'_r,n'_r}\left(\begin{array}{c}\mathrm{E'}_1\left(1-\frac{x_1}{t_1}\right)^{\upsilon'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{E'}_r\left(1-\frac{x_r}{t_r}\right)^{\upsilon'_r}\end{array}\right|$$

$$\left.\begin{array}{l}(\mathrm{a'}_j;A'^{(1)}_j,\cdots,A'^{(r)}_j)_{1,p'}:(\mathrm{c'}^{(1)}_j,C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(r)}_j,C'^{(r)}_j)_{1,p'_r}\\ \\ (\mathrm{b}_j;B'^{(1)}_j,\cdots,B'^{(r)}_j)_{1,q'}:(\mathrm{d'}^{(1)}_j,D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(r)}_j,D'^{(r)}_j)_{1,q'_r}\end{array}\right)S_{N'_1,\cdots,N'_r}^{M'_1,\cdots,M'_r}\left(\begin{array}{c}\mathrm{z'}_1\left(1-\frac{x_1}{t_1}\right)^{h'_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z'}_r\left(1-\frac{x_r}{t_r}\right)^{h'_r}\end{array}\right)$$

$$f(t_1,\cdots,t_r)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{2.2}$$

where $\upsilon_i,\upsilon'_i,h_i$ and h'_i are positive numbers.

The conditions for the existence of these operators are as follows :

(a)$f(t_1,\cdots,t_r)\in L_{\ddot{p}_1}(0,\infty)\times\cdots\times\in L_{\ddot{p}_r}(0,\infty),1\leqslant\ddot{p}_i\leqslant 2,[orf(t_1,\cdots,t_r)\in M_{\ddot{p}_1}(0,\infty)\times\cdots\times\in M_{\ddot{p}_r}(0,\infty),\ddot{p}_i>2]$,

$1\leqslant\ddot{p}_i,\ddot{q}_i<\infty,\ddot{p}_i^{-1}+\ddot{q}_i^{-1}=1,i=1,\cdots,r$

(b)$Re\left[\sigma_i+\upsilon_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}+K_ih_i\right]>-\ddot{q}_i^{-1},Re(\rho_i)>-\frac{1}{\ddot{q}_i},i=1,\cdots,r$

(c)$Re\left[\sigma'_i+\upsilon'_i\min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{D'^{(i)}_j}+K'_ih'_i\right]>-\ddot{q}_i^{-1},i=1,\cdots,r$

(d) $Re\left[\rho'_i+\upsilon'_i\min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{D'^{(i)}_j}+K'_ih'_i\right]>-\ddot{p}_i^{-1},i=1,\cdots,r$

(e) $\eta_i=Re\left(\sum_{j=1}^{n}A_j^{(i)}-\sum_{j=n+1}^{p}A_j^{(i)}+\sum_{j=1}^{m}B_j^{(i)}-\sum_{j=m+1}^{q}B_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)>0$

(f) $\eta'_i=Re\left(\sum_{j=1}^{n'}A'^{(i)}_j-\sum_{j=n'+1}^{p'}A'^{(i)}_j+\sum_{j=1}^{m'}B'^{(i)}_j-\sum_{j=m'+1}^{q'}B'^{(i)}_j+\sum_{j=1}^{m'_i}D'^{(i)}_j-\sum_{j=m'_i+1}^{q'_i}D'^{(i)}_j+\sum_{j=1}^{n'_i}C'^{(i)}_j-\sum_{j=n'_i+1}^{p'_i}C'^{(i)}_j\right)>0$

The main results of this paper unify the earlier results by S.P. Goyal and R.M. Jain and [4], Gupta and Jain [7], Goyal et al [5], Srivastava et al [10] and several others.

In this paper, we will note :

$$a_r = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_r)_{M_rK_r}}{K_r!} A[N_1, K_1; \cdots ; N_r, K_r] \tag{2.3}$$

$$b_r = \frac{(-Q_1)_{P_1L_1}}{L_1!} \cdots \frac{(-Q_r)_{P_rL_r}}{L_r!} A[Q_1, L_1; \cdots ; Q_r, L_r] \tag{2.4}$$

3.Required results

The following results are used in next section [6, page 286 (3.197,3)], [1, page 64, (23)], [2 page 201 (8)] and [1 page 62 (15)].

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\upsilon}\mathrm{d}x = B(\lambda,\mu)\,{}_2F_1(\upsilon,\lambda,\lambda+\mu;\beta) \tag{3.1}$$

provided that $Re(\lambda) > 0, Re(\mu) > 0, |\beta| < 1$

$${}_2F_1(a,b,c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b,c;z), |z| < 1 \tag{3.2}$$

$$\int_y^\infty x^{-\lambda}(x+\alpha)^{\upsilon}(x-y)^{\mu-1}\mathrm{d}x = y^{\mu+\upsilon-\lambda}B(\lambda,\lambda-\mu-\upsilon)\left(1+\frac{\alpha}{y}\right)^{\mu+\upsilon}{}_2F_1(\lambda,\mu;\lambda-\upsilon;-\alpha/y) \tag{3.3}$$

$${}_2F_1(a,b,c;-z) = \frac{1}{2\pi\omega}\int_{c-\omega\infty}^{c+\omega\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}z^s\mathrm{d}s \tag{3.4}$$

provided that $|arg(-z)| < \pi$

In this paper ,thus $\underbrace{0,\cdots,0}_{\text{r}}$ would me r zeros and so on.

4. Composition of operators of the same nature

Let $A = (a_j, \underbrace{0,\cdots,0}_{\text{r}}, A_j^{(1)}, \cdots, A_j^{(r)}, \underbrace{0,\cdots,0}_{\text{r}})_{1,n}, (1+\rho_1-\rho_1'-\sigma_1'-h_1'K_1'; \upsilon_1', \underbrace{0,\cdots,0}_{\text{2r-1}}, 1, \underbrace{0,\cdots,0}_{\text{r-1}}), \cdots,$

$(1+\rho_r-\rho_r'-\sigma_r'-h_r'K_r'; \underbrace{0\cdots,0}_{\text{r-1}}, \upsilon_r', \underbrace{0,\cdots,0}_{\text{2r-1}}, 1), (1-\sigma_1-h_1K_1; \underbrace{0,\cdots,0}_{\text{r}}, \upsilon_1, \underbrace{0,\cdots,0}_{\text{r-1}}, 1, \underbrace{0,\cdots,0}_{\text{r-1}}), \cdots,$

$(1-\sigma_r-h_rK_r; \underbrace{0,\cdots,0}_{\text{2r-1}}, \upsilon_r, \underbrace{0,\cdots,0}_{\text{r-1}}, 1), (a_j'; A_j'^{(1)}, \cdots, A_j'^{(r)}, \underbrace{0,\cdots,0}_{\text{2r}})_{1,p'},$

$$(a_j; \underbrace{0,\cdots,0}_{\text{r}}, A_j^{(1)}, \cdots, A_j^{(r)}, \underbrace{0,\cdots,0}_{\text{r}})_{n+1,p} \tag{4.1}$$

$B = (b_j'; B_j'^{(1)}, \cdots, B_j'^{(r)}, \underbrace{0,\cdots,0}_{\text{2r}})_{1,m'}, (b_j'; B_j'^{(1)}, \cdots, B_j'^{(r)}, \underbrace{0,\cdots,0}_{\text{2r}})_{m'+1,q'},$

$(b_j; \underbrace{0,\cdots,0}_{\text{r}}, B_j^{(1)}, \cdots, B_j^{(r)}, \underbrace{0,\cdots,0}_{\text{r}})_{1,m}, (b_j; \underbrace{0,\cdots,0}_{\text{r}}, B_j^{(1)}, \cdots, B_j^{(r)}, \underbrace{0,\cdots,0}_{\text{r}})_{m+1,q},$

$(1-\sigma_1-\sigma_1'-h_1K_1-h_1'K_1'; \upsilon_1', \underbrace{0,\cdots,0}_{\text{r-1}}, \upsilon_1, \underbrace{0,\cdots,0}_{\text{r-1}}, 1, \underbrace{0,\cdots,0}_{\text{r-1}}),$

$$(1-\sigma_r-\sigma_r-h_rK_r-h'_rK'_r;0,\underset{\text{r-1}}{\cdots},0,\upsilon'_1,0,\underset{\text{r-1}}{\cdots},0,\upsilon_r,0,\underset{\text{r-1}}{\cdots},0,1) \tag{4.2}$$

$$C=(1-\sigma'_1-h'_1K'_1;\upsilon'_1),(c_j'^{(1)},C_j'^{(1)})_{1,p'_1};\cdots;(1-\sigma'_r-h'_rK'_r;\upsilon'_r),(c_j'^{(r)},C_j'^{(r)})_{1,p'_r};(c_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \tag{4.3}$$

$$D=(d_j'^{(1)},D_j'^{(1)})_{1,q'_1},(1+\rho_1-\rho'_1-\sigma'_1-h'_1K'_1;\upsilon'_1);\cdots;(d_j'^{(r)},D_j'^{(r)})_{1,q'_r},(1+\rho_r-\rho'_r-\sigma'_r-h'_rK'_r;\upsilon'_r);$$

$$(d_j^{(1)},D_j^{(1)})_{1,q_1};\cdots,(d_j^{(r)},D_j^{(r)})_{1,q_r};(0,1);\cdots;(0,1) \tag{4.4}$$

We have the following formula

Theorem 1

$$Q^{\rho_i,\sigma_i}_{x_i,z_i,E_i}Q^{\rho'_i,\sigma'_i}_{y_i,z'_i,E'_i}\left[f(t_1,\cdots,t_r)\right]=\prod_{j=1}^{r}x_j^{-\rho'-1}\int_0^{x_1}\cdots\int_0^{x_r}\prod_{j=1}^{r}t_j^{\rho'_j}f(t_1,\cdots,t_r)G\left(\frac{t_1}{x_1},\cdots,\frac{t_r}{x_r}\right)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{4.5}$$

where

$$G\left(t_1,\cdots,t_r\right)=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{K'_1=0}^{[Q_1/P_1]}\cdots\sum_{K'_r=0}^{[Q_r/P_r]}a_rb_r\prod_{j=1}^{r}z_j^{K_j}z_j'^{K'_j}(1-t_j)^{\sigma_j+\sigma'_j+h_jK_i+h'_jK'_j-1}$$

$$A^{m'+m,n'+n+2r:m'_1,n'_1+1;\cdots;m'_r,n'_r+1;m_1,n_1;\cdots;m_r,n_r;1,0;\cdots;1,0}_{p'+p+2r;q'+q+r:p'_1+1,q'_1+1;\cdots;p'_r+1,q'_r+1;p_1,q_1;\cdots;p_r;q_r;0,1;\cdots;0,1}\left(\begin{array}{c|c} \mathrm{E'}_1(1-t)^{\upsilon'_1} & \mathrm{A}:\mathrm{C} \\ \vdots & \cdot \\ \mathrm{E'}_r(1-t)^{\upsilon'_r} & \cdot \\ \mathrm{E}_1(1-t)^{\upsilon_1} & \cdot \\ \vdots & \cdot \\ \mathrm{E}_r(1-t)^{\upsilon_r} & \cdot \\ \text{-(1-t}_1) & \cdot \\ \vdots & \cdot \\ \text{-(1-t}_r) & \mathrm{B}:\mathrm{D} \end{array}\right) \tag{4.6}$$

Provided that

(a)$f(t_1,\cdots,t_r)\in L_{\ddot{p}_1}(0,\infty)\times\cdots\times\in L_{\ddot{p}_r}(0,\infty),1\leqslant\ddot{p}_i\leqslant 2,[orf(t_1,\cdots,t_r)\in M_{\ddot{p}_1}(0,\infty)\times\cdots\times\in M_{\ddot{p}_r}(0,\infty),\ddot{p}_i>2]$,

$1\leqslant\ddot{p}_i,\ddot{q}_i<\infty,\ddot{p}_i^{-1}+\ddot{q}_i^{-1}=1,i=1,\cdots,r$

(b)$Re\left[\sigma_i+\upsilon_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}+K_ih_i\right]>-\ddot{q}_i^{-1},Re(\rho_i)>-\frac{1}{\ddot{q}_i},i=1,\cdots,r$

(c)$Re\left[\sigma'_i+\upsilon'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{D_j'^{(i)}}+K'_ih'_i\right]>-\ddot{q}_i^{-1},Re(\rho'_i)>-\frac{1}{\ddot{q}_i},i=1,\cdots,r$

(d) $\eta_i = Re\left(\sum_{j=1}^{n} A_j^{(i)} - \sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{m} B_j^{(i)} - \sum_{j=m+1}^{q} B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right) > 0$

(e) $\eta'_i = Re\left(\sum_{j=1}^{n'} A'^{(i)}_j - \sum_{j=n'+1}^{p'} A'^{(i)}_j + \sum_{j=1}^{m'} B'^{(i)}_j - \sum_{j=m'+1}^{q'} B'^{(i)}_j + \sum_{j=1}^{m'_i} D'^{(i)}_j - \sum_{j=m'_i+1}^{q'_i} D'^{(i)}_j + \sum_{j=1}^{n'_i} C'^{(i)}_j - \sum_{j=n'_i+1}^{p'_i} C'^{(i)}_j\right) > 0$

(f) $Q^{\rho_i,\sigma_i}_{x_i,z_i,E_i} Q^{\rho'_i,\sigma'_i}_{y_i,z'_i,E'_i}\,[f(t_1,\cdots,t_r)] \in L_{\ddot{p}_1}(0,\infty) \times \cdots \times L_{\ddot{p}_r}(0,\infty)$

Proof

To prove (4.5), we first express the $Q^{\rho_i,\sigma_i}_{z_r}$-operators involved in its left hand side in the integral form with the help of the equation (2.1). We interchange the order of $(t_1,\cdots,t_r)$-integral and $(y_1,\cdots,y_r)$-integral (which is permissible under the conditions stated by a known multidimensional extension of Fubini theorem), we obtain :

$$Q^{\rho_i,\sigma_i}_{x_i,z_i,E_i} Q^{\rho'_i,\sigma'_i}_{y_i,z'_i,E'_i}\,[f(t_1,\cdots,t_r)] = \prod_{j=1}^{r} x_j^{-\rho_j-\sigma_j} \int_0^{x_1} \cdots \int_0^{x_r} \prod_{j=1}^{r} t_j^{\rho'_j} f(t_1,\cdots,t_r) I \, \mathrm{d}t_1 \cdots \mathrm{d}t_r \tag{4.7}$$

where

$$I = \int_{t_1}^{x_1} \cdots \int_{t_r}^{x_r} \prod_{j=1}^{r} \left[y_j^{\rho_j-\rho'_j-\sigma'_j} (x_j-y_j)^{\sigma_j-1} (y_j-t_j)^{\sigma'_i-1} \right] S^{M_1,\cdots,M_r}_{N_1,\cdots,N_r} \begin{pmatrix} \mathrm{z}_1 \left(1-\frac{y_1}{x_1}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \left(1-\frac{y_r}{x_r}\right)^{h_r} \end{pmatrix}$$

$$A^{m,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r} \left(\begin{matrix} \mathrm{E}_1 \left(1-\frac{t_1}{x_1}\right)^{\upsilon_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{E}_r \left(1-\frac{t_r}{x_r}\right)^{\upsilon_r} \end{matrix} \middle| \begin{matrix} (\mathrm{a}_j; A_j^{(1)},\cdots,A_j^{(r)})_{1,p} : (\mathrm{c}_j^{(1)}, C_j^{(1)})_{1,p_1}; \cdots ; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{b}_j; B_j^{(1)},\cdots,B_j^{(r)})_{1,q} : (\mathrm{d}_j^{(1)}, D_j^{(1)})_{1,q_1}; \cdots ; (d_j^{(r)}, D_j^{(r)})_{1,q_r} \end{matrix} \right)$$

$$S^{P_1,\cdots,P_r}_{Q_1,\cdots,Q_r} \begin{pmatrix} \mathrm{z'}_1 \left(1-\frac{t_1}{x_1}\right)^{h'_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_r \left(1-\frac{t_r}{x_r}\right)^{h'_r} \end{pmatrix} A^{m',n':m'_1,n'_1;\cdots;m'_r,n'_r}_{p',q':p'_1,q'_1;\cdots;p'_r,q'_r} \left(\begin{matrix} \mathrm{E}_1 \left(1-\frac{t_1}{x_1}\right)^{\upsilon_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{E}_r \left(1-\frac{t_r}{x_r}\right)^{\upsilon_r} \end{matrix} \right|$$

$$\left. \begin{matrix} (\mathrm{a'}_j; A'^{(1)}_j,\cdots,A'^{(r)}_j)_{1,p'} : (\mathrm{c'}_j^{(1)}, C'^{(1)}_j)_{1,p'_1}; \cdots ; (c'^{(r)}_j, C'^{(r)}_j)_{1,p'_r} \\ \\ (\mathrm{b'}_j; B'^{(1)}_j,\cdots,B'^{(r)}_j)_{1,q'} : (\mathrm{d'}_j^{(1)}, D'^{(1)}_j)_{1,q'_1}; \cdots ; (d'^{(r)}_j, D'^{(r)}_j)_{1,q'_r} \end{matrix} \right) \mathrm{d}y_1 \cdots \mathrm{d}y_r \tag{4.8}$$

To evaluate I, expressing in series the classes of multivariable polynomials $S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}[.]$ and $S_{P_1,\cdots,Q_r}^{P_1,\cdots,Q_r}[.]$ with the help of (1.1) and the multivariable A-functions defined by Gautam et al [9] in Mellin-Barnes contour integrals with the help of (1.3). Interchange the order of summations and integrations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). The y_i-integral can be evaluated by setting $w_j = \frac{x_j - y_j}{x_j - t_j}(j = 1, \cdots, r)$.The w_j-integral can be evaluated with the help of the integral (3.1) and then on using (3.4). Finally interpreting the resulting Mellin-Barnes contour integral as the multivariable A-function,defined by Gautam et al [3], we get the desired formula (4.5).

Theorem 2

$$R_{x_i,z_i,E_i}^{\rho_i,\sigma_i} R_{y_i,z'_i,E'_i}^{\rho'_i,\sigma'_i}[f(t_1,\cdots,t_r)] = \prod_{j=1}^r x_j^{\rho'_j} \int_{x_1}^\infty \cdots \int_{x_r}^\infty \prod_{j=1}^r t_j^{-\rho'_j-1} f(t_1,\cdots,t_r) G\left(\frac{x_1}{t_1},\cdots,\frac{x_r}{t_r}\right) \mathrm{d}t_1 \cdots \mathrm{d}t_r \quad (4.9)$$

where $G(t_1,\cdots,t_r)$ is defined by (4.6).

Provided that

(a)$f(t_1,\cdots,t_r) \in L_{\ddot{p}_1}(0,\infty) \times \cdots \times \in L_{\ddot{p}_r}(0,\infty), 1 \leqslant \ddot{p}_i \leqslant 2, [or f(t_1,\cdots,t_r) \in M_{\ddot{p}_1}(0,\infty) \times \cdots \times \in M_{\ddot{p}_r}(0,\infty), \ddot{p}_i > 2]$,

$1 \leqslant \ddot{p}_i, \ddot{q}_i < \infty, \ddot{p}_i^{-1} + \ddot{q}_i^{-1} = 1, i = 1, \cdots, r$

(b)$Re\left[\sigma_i + \upsilon_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{D_j^{(i)}} + K_i h_i\right] > -\ddot{q}_i^{-1}, i = 1, \cdots, r$

(c)$Re\left[\sigma'_i + \upsilon'_i \min_{1 \leqslant j \leqslant m'_i} \frac{d_j'^{(i)}}{D_j'^{(i)}} + K'_i h'_i\right] > -\ddot{q}_i^{-1}, i = 1, \cdots, r$

(d) $\eta_i = Re\left(\sum_{j=1}^n A_j^{(i)} - \sum_{j=n+1}^p A_j^{(i)} + \sum_{j=1}^m B_j^{(i)} - \sum_{j=m+1}^q B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right) > 0$

(e) $\eta'_i = Re\left(\sum_{j=1}^{n'} A_j'^{(i)} - \sum_{j=n'+1}^{p'} A_j'^{(i)} + \sum_{j=1}^{m'} B_j'^{(i)} - \sum_{j=m'+1}^{q'} B_j'^{(i)} + \sum_{j=1}^{m'_i} D_j'^{(i)} - \sum_{j=m'_i+1}^{q'_i} D_j'^{(i)} + \sum_{j=1}^{n'_i} C_j'^{(i)} - \sum_{j=n'_i+1}^{p'_i} C_j'^{(i)}\right) > 0$

(f) $R_{x_i,z_i,E_i}^{\rho_i,\sigma_i} R_{y_i,z'_i,E'_i}^{\rho'_i,\sigma'_i}[f(t_1,\cdots,t_r)] \in L_{\ddot{p}_1}(0,\infty) \times \cdots \times L_{\ddot{p}_r}(0,\infty)$

The proof is similar that (4.5)

5. Composition of mixed type operators

Let

$$A_1 = (a_j; \underbrace{0,\cdots,0}_{\text{r}}, A_j^{(1)}, \cdots, A_j^{(r)}, \underbrace{0,\cdots,0}_{\text{r}})_{1,n}, (1 - \sigma'_1 - K'_1 h'_1; \upsilon'_1, \underbrace{0,\cdots,0}_{\text{2r-1}}, 1, \underbrace{0,\cdots,0}_{\text{r-1}}), \cdots,$$

$$(1 - \sigma_r - K'_r h'_r; \underbrace{0,\cdots,0}_{\text{r-1}}, \upsilon_r, \underbrace{0,\cdots,0}_{\text{2r-1}}, 1), (1 - \rho'_1 - \rho - \sigma'_1 - \sigma_1 - K_1 h_1 - K'_1 h'_1; \upsilon'_1, \underbrace{0,\cdots,0}_{\text{r-1}}, \upsilon_1, \underbrace{0,\cdots,0}_{\text{r-1}}, 1, \underbrace{0,\cdots,0}_{\text{r-1}})$$

$$\cdots, (1 - \rho'_r - \rho - \sigma'_r - \sigma_r - K_r h_r - K'_r h'_r; \underbrace{0,\cdots,0}_{\text{r-1}}, \upsilon'_r, \underbrace{0,\cdots,0}_{\text{r-1}}, \upsilon_r, \underbrace{0,\cdots,0}_{\text{r-1}}, 1),$$

$$(a'_j; A_j'^{(1)}, \cdots, A_j'^{(r)}, \underbrace{0,\cdots,0}_{\text{2r}})_{1,p'}, (a_j; \underbrace{0,\cdots,0}_{\text{r}}, A_j^{(1)}, \cdots, A_j^{(r)}, \underbrace{0,\cdots,0}_{\text{r}})_{n+1,p} \quad (5.1)$$

$$B_1=(b'_j;B_j^{(1)},\cdots,B_j'^{(r)},\underbrace{0,\cdots,0}_{2r})_{1,m'},(b'_j;B_j^{(1)},\cdots,B_j'^{(r)},\underbrace{0,\cdots,0}_{2r})_{m'+1,q'},$$

$$(b_j;\underbrace{0,\cdots,0}_{r},B_i^{(1)},\cdots,B_j^{(r)},\underbrace{0\cdots,0}_{r})_{1,m},(b_j;\underbrace{0,\cdots,0}_{r},B_i^{(1)},\cdots,B_j^{(r)},\underbrace{0\cdots,0}_{r})_{m+1,q}$$

$$(-\rho'_1-\rho_1-\sigma'_1-K'_1h'_1;\upsilon'_1,\underbrace{0,\cdots,0}_{2r-1},1,\underbrace{0\cdots,0}_{r-1}),\cdots,(-\rho'_r-\rho_r-\sigma'_r-K'_rh'_r;\underbrace{0,\cdots,0}_{2r-1},\upsilon'_r,\underbrace{0\cdots,0}_{r-1},1),$$

$$(1-\rho'_1-\rho_1-\sigma'_1-\sigma_1-K_1h'_1-K_1h_1;\upsilon'_1,\underbrace{0,\cdots,0}_{r-1},\upsilon_1,\underbrace{0,\cdots,0}_{2r-1}),\cdots,$$

$$(1-\rho'_r-\rho_r-\sigma'_r-\sigma_r-K_rh'_r-K_rh_r;\underbrace{0,\cdots,0}_{r-1},\upsilon'_r,\underbrace{0,\cdots,0}_{r-1},\upsilon_r,\underbrace{0,\cdots,0}_{r-1}) \tag{5.2}$$

$$A_2=(a_j;\underbrace{0,\cdots,0}_{r},A_j^{(1)},\cdots,A_j^{(r)},\underbrace{0,\cdots,0}_{r})_{1,n},(1-\sigma_1-K_1h_1;\underbrace{0,\cdots,0}_{r-1},\upsilon_1,\underbrace{0,\cdots,0}_{r-1},1,\underbrace{0\cdots,0}_{r}),\cdots,$$

$$(1-\sigma_r-K_rh_r;\underbrace{0,\cdots,0}_{2r-1},\upsilon_r,\underbrace{0,\cdots,0}_{r-1},1),(1-\rho'_1-\rho_1-\sigma'_1-\sigma_1-K'_1h'_1-K_1h_1;\upsilon'_1,\underbrace{0,\cdots,0}_{r-1},\upsilon_1,\underbrace{0,\cdots,0}_{r-1},1,\underbrace{0,\cdots,0}_{r-1})$$

$$\cdots,(1-\rho'_r-\rho-\sigma'_r-\sigma_r-K_rh_r-K'_rh'_r;\underbrace{0,\cdots,0}_{r-1},\upsilon'_r,\underbrace{0,\cdots,0}_{r-1},\upsilon_r,\underbrace{0,\cdots,0}_{r-1},1),$$

$$(a'_j;A_j'^{(1)},\cdots,A_j'^{(r)},\underbrace{0,\cdots,0}_{2r})_{1,p'},(a_j;\underbrace{0,\cdots,0}_{r},A_j^{(1)},\cdots,A_j^{(r)},\underbrace{0,\cdots,0}_{r})_{n+1,p} \tag{5.3}$$

$$B_2=(b'_j;B_j^{(1)},\cdots,B_j'^{(r)},\underbrace{0,\cdots,0}_{2r})_{1,m'},(b'_j;B_j^{(1)},\cdots,B_j'^{(r)},\underbrace{0,\cdots,0}_{2r})_{m'+1,q'},$$

$$(b_j;\underbrace{0,\cdots,0}_{r},B_i^{(1)},\cdots,B_j^{(r)},\underbrace{0\cdots,0}_{r})_{1,m},(b_j;\underbrace{0,\cdots,0}_{r},B_i^{(1)},\cdots,B_j^{(r)},\underbrace{0\cdots,0}_{r})_{m+1,q}$$

$$(-\rho'_1-\rho_1-\sigma_1-K_1h_1;\underbrace{0,\cdots,0}_{r-1},\upsilon_1,\underbrace{0\cdots,0}_{r-1},1,\underbrace{0,\cdots,0}_{r-1}),\cdots,(-\rho'_r-\rho_r-\sigma_r-K_rh_r;\underbrace{0,\cdots,0}_{2r-1},\upsilon_r,\underbrace{0\cdots,0}_{r-1},1),$$

$$(1-\rho'_1-\rho_1-\sigma'_1-\sigma_1-K_1h'_1-K_1h_1;\upsilon'_1,\underbrace{0,\cdots,0}_{r-1},\upsilon_1,\underbrace{0,\cdots,0}_{2r-1}),\cdots,$$

$$(1-\rho'_r-\rho_r-\sigma'_r-\sigma_r-K_rh'_r-K_rh_r;\underbrace{0,\cdots,0}_{r-1},\upsilon'_r,\underbrace{0,\cdots,0}_{r-1},\upsilon_r,\underbrace{0,\cdots,0}_{r}) \tag{5.4}$$

$$C'=(c_j'^{(1)},C_j'^{(1)})_{1,p'_1};\cdots;(c_j'^{(r)},C_j'^{(r)})_{1,p'_r};(c_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \tag{5.5}$$

$$D'=(d_j'^{(1)},D_j'^{(1)})_{1,q'_1};\cdots;(d_j'^{(r)},D_j'^{(r)})_{1,q'_r};(d_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r};(0,1),\cdots,(0,1) \tag{5.6}$$

We have the following result

Theorem 3

$$R_{x_i,z_i,E_i}^{\rho_i,\sigma_i}Q_{y_i,z'_i,E'_i}^{\rho'_i,\sigma'_i}\left[f(t_1,\cdots,t_r)\right]=\prod_{j=1}^{r}x_j^{-\rho_j-1}\int_0^{x_1}\cdots\int_0^{x_r}\prod_{j=1}^{r}t_j^{\rho_j}f(t_1,\cdots,t_r)F\left(\frac{t_1}{x_1},\cdots,\frac{t_r}{x_r};A_1,B_1\right)\mathrm{d}t_1\cdots\mathrm{d}t_r$$

$$+\prod_{j=1}^{r} x_j^{\rho_j'}\int_{x_1}^{\infty}\cdots\int_{x_r}^{\infty}\prod_{j=1}^{r} t_j^{-\rho_j'-1} f(t_1,\cdots,t_r) F\left(\frac{x_1}{t_1},\cdots,\frac{x_r}{t_r};A_2,B_2\right)\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{5.7}$$

where

$$F(t_1,\cdots,t_r;A,B)=\prod_{j=1}^{r}\Gamma(\rho_j+\rho_j'+1)\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{K_1'=0}^{[Q_1/P_1]}\cdots\sum_{K_r'=0}^{[Q_r/P_r]} a_r b_r$$

$$\prod_{j=1}^{r} z_j^{K_j} z_j'^{K_j'}(1-t_j)^{\sigma_j+\sigma_j'+h_jK_i+h_j'K_j'-1}$$

$$A_{p'+p+2r;q'+q+2r:p_1'+1,q_1'+1;\cdots;p_r'+1,q_r'+1;p_1,q_1;\cdots;p_r;q_r;0,1;\cdots;0,1}^{m'+m,n'+n+2r:m_1',n_1'+1;\cdots;m_r',n_r'+1;m_1,n_1;\cdots;m_r,n_r;1,0;\cdots;1,0}\left(\begin{matrix} \text{E'}_1(1-t)^{\upsilon_1'} \\ \vdots \\ \text{E'}_r(1-t)^{\upsilon_r'} \\ \text{E}_1(1-t)^{\upsilon_1'} \\ \vdots \\ \text{E}_r(1-t)^{\upsilon_r'} \\ \text{-t}_1 \\ \vdots \\ \text{-t}_r \end{matrix}\left|\begin{matrix} \text{A : C'} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \text{B : D'} \end{matrix}\right.\right) \tag{5.8}$$

Provided that

(a)$f(t_1,\cdots,t_r)\in L_{\ddot{p}_1}(0,\infty)\times\cdots\times\in L_{\ddot{p}_r}(0,\infty), 1\leqslant\ddot{p}_i\leqslant 2, [or f(t_1,\cdots,t_r)\in M_{\ddot{p}_1}(0,\infty)\times\cdots\times\in M_{\ddot{p}_r}(0,\infty), \ddot{p}_i>2]$,

$1\leqslant\ddot{p}_i,\ddot{q}_i<\infty, \ddot{p}_i^{-1}+\ddot{q}_i^{-1}=1, i=1,\cdots,r$

(b)$Re\left[\sigma_i+\upsilon_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}+K_ih_i\right]>-q_i^{-1}, Re(\rho_i)>-\frac{1}{q_i}, i=1,\cdots,r$

(c)$Re\left[\sigma_i'+\upsilon_i'\min_{1\leqslant j\leqslant m_i'}\frac{d_j'^{(i)}}{D_j'^{(i)}}+K_i'h_i'\right]>-q_i^{-1}, Re(\rho_i')>-\frac{1}{p_i}, i=1,\cdots,r$

(d) $\eta_i=Re\left(\sum_{j=1}^{n}A_j^{(i)}-\sum_{j=n+1}^{p}A_j^{(i)}+\sum_{j=1}^{m}B_j^{(i)}-\sum_{j=m+1}^{q}B_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)>0$

(e) $\eta_i'=Re\left(\sum_{j=1}^{n'}A_j'^{(i)}-\sum_{j=n'+1}^{p'}A_j'^{(i)}+\sum_{j=1}^{m'}B_j'^{(i)}-\sum_{j=m'+1}^{q'}B_j'^{(i)}+\sum_{j=1}^{m_i'}D_j'^{(i)}-\sum_{j=m_i'+1}^{q_i'}D_j'^{(i)}+\sum_{j=1}^{n_i'}C_j'^{(i)}-\sum_{j=n_i'+1}^{p_i'}C_j'^{(i)}\right)>0$

(f) $R_{x_i,z_i,E_i}^{\rho_i,\sigma_i}Q_{y_i,z_i',E_i'}^{\rho_i',\sigma_i'}[f(t_1,\cdots,t_r)]= Q_{x_i,z_i,E_i}^{\rho_i,\sigma_i}R_{y_i,z_i',E_i'}^{\rho_i',\sigma_i'}[f(t_1,\cdots,t_r)]\in L_{\ddot{p}_1}(0,\infty)\times\cdots\times L_{\ddot{p}_r}(0,\infty)$

The proof is similar that (4.5)

Remark

If $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}$, $m = 0$, $A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{R}$ and $m' = 0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [11], see Jain et al [8].

6. Conclusion

In this paper we have evaluated the compositions of multidimensional fractional integral operator concerning the multivariable A-functions defined by Gautam et al [3] and a class of multivariable polynomials defined by Srivastava [9] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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