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Composition of fractional integral operators involving the Prasad's multivariable I-function

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ABSTRACT
Saxena and Dave [6] have studied compositions of fractional integral operators involving the multivariable H-function and product of general class of polynomials. In this paper, the author derive compositions of fractional integral operators associated with multivariable I-function defined by Prasad [5] and class of multivariable polynomials defined by Srivastava [9]. We will quote the particular case concerning the multivariable H-function defined Srivastava et al [11].

Keywords: Multivariable I-function, multivariable class of polynomials, fractional integral operator, multivariable H-function

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1. Introduction and preliminaries.

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, it has turned out many phenomena in physics, mechanics, chemistry, biology and other sciences can be described very successfully by models using mathematical tools by models using mathematical tools from fractional calculus.
The generalized polynomials defined by Srivastava [9], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.1}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.
The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p',q';\cdots;p^{(r)},q^{(r)}}^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(r)},n^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha'_{rj},\cdots,\alpha_{rj}^{(r)})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta'_{rj},\cdots,\beta_{rj}^{(r)})_{1,q_r}:(b'_j,\beta'_j)_{1,q'};\cdots;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}} \end{array}\right) \tag{1.2}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.3}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [5].
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of

the corresponding conditions for multivariable H-function given by as : $|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) +$$

$$+ \left(\sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r} \beta_{rk}^{(i)}\right) \tag{1.4}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_r) = 0(\ |z_1|^{\alpha'_1}, \cdots, |z_r|^{\alpha'_r}\ ), max(\ |z_1|, \cdots, |z_r|\ ) \rightarrow 0$$

$$I(z_1, \cdots, z_r)\ = 0(\ |z_1|^{\beta'_1}, \cdots, |z_r|^{\beta'_r}\ ), min(\ |z_1|\ , \cdots, |z_r|\ ) \rightarrow \infty$$

where $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

2. Definitions

The pair of new extended fractional integral operators are defined by the following equations :

$$Q_{z_r}^{\eta,\alpha}[f(x)] = tx^{-\eta-t\alpha-1}\int_0^x y^\eta(x^t - y^t)^\alpha I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 v_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r v_r \end{array}\middle| \begin{array}{c} (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2} \\ \\ \\ \\ (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2} \end{array}\right.$$

$$\left.\begin{array}{c} ;\cdots;(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)})_{1,p_r} : (a_j^{(1)},\alpha_j^{(1)})_{1,p^{(1)}};\cdots;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ \\ ;\cdots;(b_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)})_{1,q_r} : (b_j^{(1)},\beta_j^{(1)})_{1,q^{(1)}};\cdots;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}} \end{array}\right)$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c} \mathrm{y}_1\left(1 - \frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1 - \frac{y^t}{x^t}\right)^{h_r} \end{array}\right) f(y)\mathrm{d}y \tag{2.1}$$

$$R_{\gamma_n}^{\delta,\beta}[f(x)] = tx^{\delta} \int_x^{\infty} y^{-\delta-t\beta-1}(y^t-x^t)^{\beta} \ I^{0,n_2';0,n_3';\cdots;0,n_r':m',n';\cdots;m'^{(r)},n'^{(r)}}_{p_2',q_2',p_3',q_3';\cdots;p_r'^{(1)},q_r'^{(1)}:p'^{(1)},q'^{(1)};\cdots;p'^{(r)},q'^{(r)}} \left(\begin{array}{c|c} \gamma_1\mu_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ \gamma_r\mu_r & \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{a'}_{2j};\alpha_{2j}'^{(1)},\alpha_{2j}'^{(2)})_{1,p_2'};\cdots;(\mathrm{a'}_{rj};\alpha_{rj}'^{(1)},\cdots,\alpha_{rj}'^{(r)})_{1,p_r'}:(a_j'^{(1)},\alpha_j'^{(1)})_{1,p'^{(1)}};\cdots;(a_j'^{(r)},\alpha_j'^{(r)})_{1,p'^{(r)}} \\ \\ (\mathrm{b'}_{2j};\beta_{2j}'^{(1)},\beta_{2j}'^{(2)})_{1,q_2'};\cdots;(\mathrm{b'}_{rj};\beta_{rj}'^{(1)},\cdots,\beta_{rj}'^{(r)})_{1,q_r'}:(b_j'^{(1)},\beta_j'^{(1)})_{1,q'^{(1)}};\cdots;(b_j'^{(r)},\beta_j'^{(r)})_{1,q'^{(r)}} \end{array} \right)$$

$$S^{M_1',\cdots,M_r'}_{N_1',\cdots,N_r'} \left(\begin{array}{c} \mathrm{y'}_1\left(1-\frac{x^t}{y^t}\right)^{h_1'} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y'}_r\left(1-\frac{x^t}{y^t}\right)^{h_r'} \end{array} \right) f(y)\mathrm{d}y \tag{2.2}$$

where $\upsilon_i = \left(1-\dfrac{y^t}{x^t}\right)^{\upsilon_i}$, $\mu_i = \left(1-\dfrac{x^t}{y^t}\right)^{\upsilon_i'}$; $t, \upsilon_i, \upsilon_i', h_i$ and h_i' are positive numbers.

The conditions for the existence of these operators are as follows :

(a) $f(x) \in L_p(0,\infty)$, $1 \leqslant p,q < \infty, p^{-1}+q^{-1}=1$

(b) $Re\left[t\alpha + t\sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m^{(i)}} \dfrac{b_j^{(i)}}{\beta_j^{(i)}} + K_i h_i\right)\right] > -q^{-1}, Re(\eta) > -\dfrac{1}{q}$

(c) $Re\left[t\beta + t\sum_{i=1}^{r}\left(v_i' \min_{1\leqslant j\leqslant m'^{(i)}} \dfrac{b_j'^{(i)}}{\beta_j'^{(i)}} + K_i' h_i'\right)\right] > -q^{-1}$

(d) $Re\left[\delta + t\sum_{i=1}^{r}\left(v_i' \min_{1\leqslant j\leqslant m^{(i)}} \dfrac{b_j'^{(i)}}{\beta_j'^{(i)}} + K_i' h_i'\right)\right] > -p^{-1}$

(e) $$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r} \beta_{rk}^{(i)}\right) > 0$$

(f) $\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'^{(i)}_k - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'^{(i)}_k + \sum_{k=1}^{m'^{(i)}} \beta'^{(i)}_k - \sum_{k=m^{(i)}+1}^{q'^{(i)}} \beta'^{(i)}_k + \left(\sum_{k=1}^{n'_2} \alpha'^{(i)}_{2k} - \sum_{k=n_2+1}^{p'_2} \alpha'^{(i)}_{2k}\right)$

$$+\cdots+\left(\sum_{k=1}^{n'_r} \alpha'^{(i)}_{rk} - \sum_{k=n'_r+1}^{p'_r} \alpha'^{(i)}_{rk}\right) - \left(\sum_{k=1}^{q'_2} \beta'^{(i)}_{2k} + \sum_{k=1}^{q'_3} \beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q'_r} \beta'^{(i)}_{rk}\right) > 0$$

The main results of this paper unify the earlier results by S.P. Goyal, R.M. Jain and Neelima Gaur [3], R.K. Saxena and O.P. Dave [6], R.K. Saxena, Y. Singh and A. Ramawat [7], H.M. Srivastava, S.P. Goyal and R.M. Jain [10] and several others.

3. Required results

The following results are used in next section [4, page 286 (3.197,3)], [1, page 64, (23)], [2 page 201 (8)] and [1 page 62 (15)].

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\upsilon}\mathrm{d}x = B(\lambda,\mu)\,{}_2F_1(\upsilon,\lambda,\lambda+\mu;\beta) \tag{3.1}$$

provided that $Re(\lambda) > 0, Re(\mu) > 0, |\beta| < 1$

$${}_2F_1(a,b,c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b,c;z), |z| < 1 \tag{3.2}$$

$$\int_y^\infty x^{-\lambda}(x+\alpha)^{\upsilon}(x-y)^{\mu-1}\mathrm{d}x = y^{\mu+\upsilon-\lambda}B(\lambda,\lambda-\mu-\upsilon)\left(1+\frac{\alpha}{y}\right)^{\mu+\upsilon}{}_2F_1(\lambda,\mu;\lambda-\upsilon;-\alpha/y) \tag{3.3}$$

$${}_2F_1(a,b,c;-z) = \frac{1}{2\pi\omega}\int_{c-\omega\infty}^{c+\omega\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}z^s\mathrm{d}s \tag{3.4}$$

provided that $|arg(-z)| < \pi$

In this paper ,thus $0,\cdots,0$ would me r zeros and so on.

4. Composition of operators of the same nature

Let $U = p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1}$ (4.1)

$$V = 0,n_2;0,n_3;\cdots;0,n_{r-1};0,n_2;0,n_3;\cdots;0,n_{r-1} \tag{4.2}$$

$$X = m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)};m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)},0,1 \tag{4.3}$$

$$Y = p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)};p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)},0,1 \tag{4.4}$$

$$A = (a_{2k};\alpha^{(1)}_{2k},\alpha^{(2)}_{2k})_{1,p_2};\cdots;(a_{(r-1)k};\alpha^{(1)}_{(r-1)k},\alpha^{(2)}_{(r-1)k},\cdots,\alpha^{(r-1)}_{(r-1)k})_{1,p_{r-1}};$$

$$(a_{2k};\alpha^{(1)}_{2k},\alpha^{(2)}_{2k})_{1,p_2};\cdots;(a_{(r-1)k};\alpha^{(1)}_{(r-1)k},\alpha^{(2)}_{(r-1)k},\cdots,\alpha^{(r-1)}_{(r-1)k})_{1,p_{r-1}} \tag{4.5}$$

$$B = (b_{2k};\beta^{(1)}_{2k},\beta^{(2)}_{2k})_{1,q_2};\cdots;(b_{(r-1)k};\beta^{(1)}_{(r-1)k},\beta^{(2)}_{(r-1)k},\cdots,\beta^{(r-1)}_{(r-1)k})_{1,q_{r-1}};$$

$$(b_{2k};\beta_{2k}^{(1)},\beta_{2k}^{(2)})_{1,q_2};\cdots;(b_{(r-1)k};\beta_{(r-1)k}^{(1)},\beta_{(r-1)k}^{(2)},\cdots,\beta_{(r-1)k}^{(r-1)})_{1,q_{r-1}} \tag{4.6}$$

$$\mathbb{A}=(a_{rk};\alpha_{rk}^{(1)},\alpha_{rk}^{(2)},\cdots,\alpha_{rk}^{(r)},0,\cdots,0,0)_{1,p_r};(a_{rk};0,\cdots,0,\alpha_{rk}^{(1)},\alpha_{rk}^{(2)},\cdots,\alpha_{rk}^{(r)},0)_{1,p_r} \tag{4.7}$$

$$\mathbb{B}=(b_{rk};\beta_{rk}^{(1)},\beta_{rk}^{(2)},\cdots,\beta_{rk}^{(r)},0,\cdots,0,0)_{1,q_r};(b_{rk};0,\cdots,0,\beta_{rk}^{(1)},\beta_{rk}^{(2)},\cdots,\beta_{rk}^{(r)},0)_{1,q_r} \tag{4.8}$$

$$A_1=(-\alpha-\sum_{i=1}^{r}K_ih_i;v_1,\cdots,v_r,0,\cdots,0,0),(-\beta-\sum_{i=1}^{r}L_ih_i;0,\cdots,0,v_1,\cdots,v_r,0),$$

$$\left(-\beta-\frac{(\delta-\eta)}{t}-\sum_{i=1}^{r}L_ih_i;0,\cdots,0,0,v_1,\cdots,v_r\right);(-\alpha-\sum_{i=1}^{r}K_ih_i;v_1,\cdots,v_r,0,\cdots,0,1) \tag{4.9}$$

$$B_1=(-1-\alpha-\beta-\sum_{i=1}^{r}(K_i+L_i)h_i;v_1,\cdots,v_r;v_1,\cdots,v_r,0);$$

$$(-1-\alpha-\beta-\sum_{i=1}^{r}(K_i+L_i)h_i;v_1,\cdots,v_r;v_1,\cdots,v_r,1) \tag{4.10}$$

$$A'=(a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}};(a_k^{(1)},\alpha_k^{(1)})_{1,p^{(1)}};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}} \tag{4.11}$$

$$B'=(b_k^{(1)},\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}};(b_k^{(1)},\beta_k^{(1)})_{1,q^{(1)}};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}},(0,1) \tag{4.12}$$

$$a_r=\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_r)_{M_rK_r}}{K_r!}A[N_1,K_1;\cdots;N_r,K_r] \tag{4.13}$$

$$b_r=\frac{(-Q_1)_{P_1L_1}}{L_1!}\cdots\frac{(-Q_r)_{P_rL_r}}{L_r!}A[Q_1,L_1;\cdots;Q_r,L_r] \tag{4.14}$$

Theorem 1

We have

$$Q_{z_r}^{\eta,\alpha}Q_{z_r}^{\delta,\beta}\left[f(x)\right]=tx^{-t(\alpha+\beta+1+\frac{\delta+1}{t})}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}a_rb_rx^{-t\sum_{i=1}^{r}(K_i+L_i)}\prod_{i=1}^{r}y_i^{K_i+L_i}$$

$$\int_0^x u^{\delta}(x^t-u^t)^{\alpha+\beta+1+\sum_{i=1}^{r}(K_i+L_i)h_i}I_{U:2p_r+4,2q_r+2;Y}^{V;0,2n_r+4;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathrm{A}\ ;\mathbb{A}\ ,\mathrm{A}_1;A' \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \left(\frac{u^t}{x^t}-1\right) & \mathrm{B};\mathbb{B}\ ,\mathrm{B}_1;B' \end{array}\right)f(u)\mathrm{d}u \tag{4.15}$$

Provided that

(a) $f(x) \in L_p(0,\infty)$, $1 \leqslant p \leqslant 2$ [or $f(x) \in M_p(0,\infty), p > 2]$, $1 \leqslant p,q < \infty, p^{-1}+q^{-1}=1$

(b)$Re\left[t\alpha + t\sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}} + K_i h_i\right)\right] > -q^{-1}$

(c) $Re\left[t\beta + t\sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}} + L_i h_i\right)\right] > -q^{-1}, Re(\eta) > -\frac{1}{q}$

(d) $Re\left[\alpha + \beta + \sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}} + (K_i+L_i) h_i\right)\right] > -2\,, Re(\delta) > -\frac{1}{p}$

(e) $\Omega_i = \sum_{k=1}^{n^{(i)}}\alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}}\beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)} + \left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right) + \cdots +$

$\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right) - \left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)} + \sum_{k=1}^{q_3}\beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) > 0$

(f) $Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] \in L_p(0,\infty)$

Proof

$$Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] = tx^{-\eta-t\alpha-1}\int_0^x y^{\eta}(x^t-y^t)^{\alpha} I\begin{pmatrix} \mathrm{z}_1 v_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r v_r \end{pmatrix} S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r} \end{pmatrix} \Bigg\{ \; ty^{-\delta-t\beta-1}$$

$$\int_0^y u^{\delta}(y^t-u^t)^{\beta} f(u) I\begin{pmatrix} \mathrm{z}_1\left(1-\frac{u^t}{y^t}\right)^{v_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{y^t}\right)^{v_r} \end{pmatrix} S_{P_1,\cdots,Q_r}^{P_1,\cdots,Q_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{u^t}{y^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{u^t}{y^t}\right)^{h_r} \end{pmatrix}\Bigg\}\mathrm{d}u\mathrm{d}y \qquad (4.16)$$

If we interchange the order of integration, which is permissible under the conditions stated, we have

$$Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] = t^2 x^{-\eta-t\alpha-1}\int_0^x I_1 u^{\delta} f(u)\mathrm{d}u \qquad (4.17)$$

where $I_1 = \int_u^x y^{\eta-\delta-t\beta-1}(x^t-y^t)^{\alpha}(y^t-u^t)^{\beta}\, I\begin{pmatrix} \mathrm{z}_1 \upsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \upsilon_r \end{pmatrix} S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r} \end{pmatrix}$

$$I\begin{pmatrix} \mathrm{z}_1\left(1-\frac{u^t}{y^t}\right)^{v_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{y^t}\right)^{v_r} \end{pmatrix} S_{P_1,\cdots,Q_r}^{P_1,\cdots,Q_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{u^t}{y^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{u^t}{y^t}\right)^{h_r} \end{pmatrix}\mathrm{d}y \tag{4.18}$$

Now, expressing in serie the classes of multivariable polynomials $S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}[.]$ and $S_{P_1,\cdots,Q_r}^{P_1,\cdots,Q_r}[.]$ with help of (1.1), expressing the multivariable I-functions defined by Prasad [5] in Mellin-Barnes contour integrals with the help of (1.3). Interchange the order of summations and integrations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process), then

$$I_1 = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]} a_r b_r \prod_{i=1}^{r} y_i^{K_i+L_i}\frac{1}{(2\pi\omega)^{2r}}\int_{L_1}\cdots\int_{L_{2r}}\xi(s_1,\cdots,s_r)\xi(s_1',\cdots,s_r')$$

$$\prod_{i=1}^{r}\phi_i(s_i)\phi_i(s_i')z_i^{s_i+s_i'}\left\{\int_u^x y^{\eta-\delta-t\beta-t\sum_{i=1}^{r}(v_is_i'+h_iL_i)-1}(x^t-u^t)^{\alpha+\sum_{i=1}^{r}(v_is_i+h_iK_i)}x^{-t(\sum_{i=1}^{r}(v_is_i+h_iK_i)}\right.$$

$$\left.(y^t-u^t)^{\beta+\sum_{i=1}^{r}(v_is_i'+h_iL_i)}\mathrm{d}y\right)\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s_1'\cdots\mathrm{d}s_r' \tag{4.19}$$

The y-integral can be evaluated fairly easily by the substitution $w = \dfrac{x^t-y^t}{x^t-u^t}$

The w-integral can be evaluated with the help of the integral (3.1) and then on using (3.4), the equation (4.19) transforms into the desired form (4.15), when we apply the definition (1.3).

Let $U = p_2', q_2'; p_3', q_3'; \cdots; p_{r-1}', q_{r-1}'; p_2', q_2'; p_3', q_3'; \cdots; p_{r-1}', q_{r-1}'$ (4.20)

$$V = 0, n_2'; 0, n_3'; \cdots; 0, n_{r-1}'; 0, n_2'; 0, n_3'; \cdots; 0, n_{r-1}' \tag{4.21}$$

$$X = m'^{(1)}, n'^{(1)}; \cdots; m'^{(r)}, n'^{(r)}; m'^{(1)}, n'^{(1)}; \cdots; m'^{(r)}, n'^{(r)}, 1, 0 \tag{4.22}$$

$$Y = p'^{(1)}, q'^{(1)}; \cdots; p'^{(r)}, q'^{(r)}; p'^{(1)}, q'^{(1)}; \cdots; p'^{(r)}, q'^{(r)}, 0, 1 \tag{4.23}$$

$$A = (a_{2k}'; \alpha_{2k}'^{(1)}, \alpha_{2k}'^{(2)})_{1,p_2'}; \cdots; (a_{(r-1)k}'; \alpha_{(r-1)k}'^{(1)}, \alpha_{(r-1)k}'^{(2)}, \cdots, \alpha_{(r-1)k}'^{(r-1)})_{1,p_{r-1}'};$$

$$(a_{2k}'; \alpha_{2k}'^{(1)}, \alpha_{2k}'^{(2)})_{1,p_2'}; \cdots; (a_{(r-1)k}'; \alpha_{(r-1)k}'^{(1)}, \alpha_{(r-1)k}'^{(2)}, \cdots, \alpha_{(r-1)k}'^{(r-1)})_{1,p_{r-1}'} \tag{4.24}$$

$$B = (b_{2k}'; \beta_{2k}'^{(1)}, \beta_{2k}'^{(2)})_{1,q_2'}; \cdots; (b_{(r-1)k}'; \beta_{(r-1)k}'^{(1)}, \beta_{(r-1)k}'^{(2)}, \cdots, \beta_{(r-1)k}'^{(r-1)})_{1,q_{r-1}'};$$

$$(b_{2k}'; \beta_{2k}'^{(1)}, \beta_{2k}'^{(2)})_{1,q_2'}; \cdots; (b_{(r-1)k}'; \beta_{(r-1)k}'^{(1)}, \beta_{(r-1)k}'^{(2)}, \cdots, \beta_{(r-1)k}'^{(r-1)})_{1,q_{r-1}'} \tag{4.25}$$

$$\mathbb{A} = (a'_{rk};\alpha'^{(1)}_{rk},\alpha'^{(2)}_{rk},\cdots,\alpha'^{(r)}_{rk},0,\cdots,0,0)_{1,p'_r};(a'_{rk};0,\cdots,0,\alpha'^{(1)}_{rk},\alpha'^{(2)}_{rk},\cdots,\alpha'^{(r)}_{rk},0)_{1,p'_r} \quad (4.26)$$

$$\mathbb{B} = (b'_{sk};\beta'^{(1)}_{sk},\beta'^{(2)}_{sk},\cdots,\beta'^{(s)}_{sk},0,\cdots,0,0)_{1,q'_r};(b'_{rk};0,\cdots,0,\beta'^{(1)}_{rk},\beta'^{(2)}_{rk},\cdots,\beta'^{(r)}_{rk},0)_{1,q'_r} \quad (4.27)$$

$$A_2 =(-\alpha-\sum_{i=1}^{r}K_i h'_i;v'_1,\cdots,v'_r,0,\cdots,0,0),(-\beta-\sum_{i=1}^{r}L_i h'_i;0,\cdots,0,v'_1,\cdots,v'_r,0),$$

$$(\frac{(\delta-\eta)}{t}-\alpha-\sum_{i=1}^{r}L_i h'_i;v'_1,\cdots,v'_r;0,\cdots,0,1);(-\beta-\sum_{i=1}^{r}K_i h'_i;0,\cdots,0;v'_1,\cdots,v'_r,1) \quad (4.28)$$

$$B_2 = (-1-\alpha-\beta-\sum_{i=1}^{r}(K_i+L_i)h'_i;v'_1,\cdots,v'_r;v'_1,\cdots,v'_r,0);$$

$$(-1-\alpha-\beta-\sum_{i=1}^{r}(K_i+L_i)h'_i;v'_1,\cdots,v'_r;v'_1,\cdots,v'_r,1) \quad (4.29)$$

$$A' = (a'^{(1)}_k,\alpha'^{(1)}_k)_{1,p'^{(1)}};\cdots;(a'^{(r)}_k,\alpha'^{(r)}_k)_{1,p'^{(r)}};(a'^{(1)}_k,\alpha'^{(1)}_k)_{1,p'^{(1)}};\cdots;(a'^{(r)}_k,\alpha'^{(r)}_k)_{1,p'^{(r)}} \quad (4.30)$$

$$B' =(b'^{(1)}_k,\beta'^{(1)}_k)_{1,q'^{(1)}};\cdots;(b'^{(r)}_k,\beta'^{(r)}_k)_{1,q'^{(r)}};(b'^{(1)}_k,\beta'^{(1)}_k)_{1,q'^{(1)}};\cdots;(b'^{(r)}_k,\beta'^{(r)}_k)_{1,q'^{(r)}},(0,1) \quad (4.31)$$

a_r and b_r are defined by (4.13) and (4.14) respectively

Theorem 2

We have

$$R^{\eta,\alpha}_{z_r}R^{\delta,\beta}_{z_r}[f(x)] =tx^{\eta}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]} a_r b_r x^{-t\sum_{i=1}^{r}(K_i+L_i)}\prod_{i=1}^{r}y'^{K_i+L_i}_i$$

$$\int_x^{\infty}u^{-(\eta+1+t(\alpha+\beta+1+\sum_{i=1}^{r}h'_i(K_i+L_i)))}(u^t-x^t)^{\alpha+\beta+1+\sum_{i=1}^{r}(K_i+L_i)h'_i}$$

$$I^{V;0,2n'_r+4;X}_{U:2p'_r+4,2q'_r+2;Y}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v'_1} & \mathrm{A};\mathbb{A},\mathrm{A}_2;A' \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v'_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v'_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v'_r} & \mathrm{B};\mathbb{B},\mathrm{B}_2;B' \\ \left(\frac{x^t}{u^t}-1\right) & \end{array}\right)f(u)\mathrm{d}u \quad (4.32)$$

Provided that

(a) $f(x)\in L_p(0,\infty), 1\leqslant p\leqslant 2$ [or $f(x)\in M_p(0,\infty), p>2$], $1\leqslant p,q<\infty, p^{-1}+q^{-1}=1$

(b) $Re\left[t\alpha + t\sum_{i=1}^{r}\left(v'_i \min_{1\leqslant j\leqslant m'^{(i)}} \frac{b'^{(i)}_j}{\beta'^{(i)}_j} + K_i h'_i\right)\right] > -q^{-1}$

(c) $Re\left[t\beta + t\sum_{i=1}^{r}\left(v'_i \min_{1\leqslant j\leqslant m'^{(i)}} \frac{b'^{(i)}_j}{\beta'^{(i)}_j} + L_i h'_i\right)\right] > -p^{-1}$

(d) $\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'_k{}^{(i)} - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'_k{}^{(i)} + \sum_{k=1}^{m'^{(i)}} \beta'_k{}^{(i)} - \sum_{k=m^{(i)}+1}^{q'^{(i)}} \beta'_k{}^{(i)} + \left(\sum_{k=1}^{n'_2} \alpha'_{2k}{}^{(i)} - \sum_{k=n_2+1}^{p'_2} \alpha'_{2k}{}^{(i)}\right)$

$+\cdots+\left(\sum_{k=1}^{n'_r} \alpha'_{rk}{}^{(i)} - \sum_{k=n'_r+1}^{p'_r} \alpha'_{rk}{}^{(i)}\right) - \left(\sum_{k=1}^{q'_2} \beta'_{2k}{}^{(i)} + \sum_{k=1}^{q'_3} \beta'_{3k}{}^{(i)} + \cdots + \sum_{k=1}^{q'_r} \beta'_{rk}{}^{(i)}\right) > 0$

(e) $R^{\eta,\alpha}_{z_r} R^{\delta,\beta}_{z_r}\left[f(x)\right] \in L_p(0,\infty)$

The proof is similar that (4.15)

5. Composition of mixed type operators

Let $U = p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}; p'_2, q'_2; p'_3, q'_3; \cdots ; p'_{r-1}, q'_{r-1}$ (5.1)

$V = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}; 0, n'_2; 0, n'_3; \cdots ; 0, n'_{r-1}$ (5.2)

$X = m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)}; m'^{(1)}, n'^{(1)}; \cdots ; m'^{(r)}, n'^{(r)}, 1, 0$ (5.3)

$Y = p^{(1)}, q^{(1)}; \cdots ; p^{(r)}, q^{(r)}; p'^{(1)}, q'^{(1)}; \cdots ; p'^{(r)}, q'^{(r)}, 0, 1$ (5.4)

$A = (a_{2k}; \alpha^{(1)}_{2k}, \alpha^{(2)}_{2k})_{1,p_2}; \cdots ; (a_{(r-1)k}; \alpha^{(1)}_{(r-1)k}, \alpha^{(2)}_{(r-1)k}, \cdots , \alpha^{(r-1)}_{(r-1)k})_{1,p_{r-1}};$

$(a'_{2k}; \alpha'^{(1)}_{2k}, \alpha'^{(2)}_{2k})_{1,p'_2}; \cdots ; (a'_{(r-1)k}; \alpha'^{(1)}_{(r-1)k}, \alpha'^{(2)}_{(r-1)k}, \cdots , \alpha'^{(r-1)}_{(r-1)k})_{1,p'_{r-1}}$ (5.5)

$B = (b_{2k}; \beta^{(1)}_{2k}, \beta^{(2)}_{2k})_{1,q_2}; \cdots ; (b_{(r-1)k}; \beta^{(1)}_{(r-1)k}, \beta^{(2)}_{(r-1)k}, \cdots , \beta^{(r-1)}_{(r-1)k})_{1,q_{r-1}};$

$(b'_{2k}; \beta'^{(1)}_{2k}, \beta'^{(2)}_{2k})_{1,q'_2}; \cdots ; (b'_{(r-1)k}; \beta'^{(1)}_{(r-1)k}, \beta'^{(2)}_{(r-1)k}, \cdots , \beta'^{(r-1)}_{(r-1)k})_{1,q'_{r-1}}$ (5.6)

$\mathbb{A} = (a_{rk}; \alpha^{(1)}_{rk}, \alpha^{(2)}_{rk}, \cdots , \alpha^{(r)}_{rk}, 0, \cdots , 0, 0)_{1,p_r}; (a'_{rk}; 0, \cdots , 0, \alpha'^{(1)}_{rk}, \alpha'^{(2)}_{rk}, \cdots , \alpha'^{(r)}_{rk}, 0)_{1,p'_r}$ (5.7)

$\mathbb{B} = (b_{rk}; \beta^{(1)}_{rk}, \beta^{(2)}_{rk}, \cdots , \beta^{(r)}_{rk}, 0, \cdots , 0, 0)_{1,q_r}; (b'_{rk}; 0, \cdots , 0, \beta'^{(1)}_{rk}, \beta'^{(2)}_{rk}, \cdots , \beta'^{(r)}_{rk}, 0)_{1,q'_r}$ (5.8)

$A' = (a^{(1)}_k, \alpha^{(1)}_k)_{1,p^{(1)}}; \cdots ; (a^{(r)}_k, \alpha^{(r)}_k)_{1,p^{(r)}};\ (a'^{(1)}_k, \alpha'^{(1)}_k)_{1,p'^{(1)}}; \cdots ; (a'^{(r)}_k, \alpha'^{(r)}_k)_{1,p'^{(r)}}$ (5.9)

$B' = (b^{(1)}_k, \beta^{(1)}_k)_{1,q^{(1)}}; \cdots ; (b^{(r)}_k, \beta^{(r)}_k)_{1,q^{(r)}}; (b'^{(1)}_k, \beta'^{(1)}_k)_{1,q'^{(1)}}; \cdots ; (b'^{(r)}_k, \beta'^{(r)}_k)_{1,q'^{(r)}}, (0, 1)$ (5.10)

$A_3 = \Big(-\alpha - \beta - \frac{\eta+\delta+1}{t} - \sum_{i=1}^{r}(K_i h_i + L_i h'_i); v_1, \cdots , v_r; v'_1, \cdots , v'_r, 0\Big);$

$\Big(-\alpha - \beta - \frac{\eta+\delta+1}{t} - \sum_{i=1}^{r}(K_i h_i + L_i h'_i); v_1, \cdots , v_r; v'_1, \cdots , v'_r, 1\Big);$

$$\big(-\beta-\sum_{i=1}^{r}L_ih'_i;0,\cdots,0;v'_1,\cdots,v'_r,1\big) \tag{5.11}$$

r

$$B_3=\big(-\alpha-\beta-\frac{2(\eta+\delta+1)}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0\big);$$

$$\big(\beta-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}L_ih'_i;0,\cdots,0;v'_1,\cdots,v'_r,1\big) \tag{5.12}$$

$$A_4=\big(-\alpha-\beta-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0\big);$$

$$\big(-\alpha-\beta-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,1\big);$$

$$\big(-\alpha-\sum_{i=1}^{r}K_ih_i;v_1,\cdots,v_r;0,\cdots,0,0\big) \tag{5.13}$$

$$B_4=\ \big(-\alpha-\beta-\frac{2(\eta+\delta+1)}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0\big);$$

$$\big(-\alpha-\sum_{i=1}^{r}K_ih_i;v_1,\cdots,v_r;0,\cdots,0,0\big) \tag{5.14}$$

a_r and b_r are defined by (4.13) and (4.14) respectively

We have the following result

Theorem 3

$$Q^{\eta,\alpha}_{z_r}R^{\delta,\beta}_{z_r}\left[f(x)\right]=R^{\eta,\alpha}_{z_r}Q^{\delta,\beta}_{z_r}\left[f(x)\right]=tx^{-\eta-1}\Gamma\left(\frac{\eta+\delta+1}{t}\right)\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}a_rb_r\prod_{i=1}^{r}y_i^{K_i}y_i'^{L_i}$$

$$\int_0^x u^{\eta}\left(1-\frac{u^t}{x^t}\right)^{\alpha+\beta+1+\sum_{i=1}^{r}(K_ih_i+L_ih'_i)}I^{V;0,n_r+n'_r+3;X}_{U:p_r+p'_r+3,q_r+q'_r+2;Y}\left(\begin{array}{c|c}\mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathrm{A}\ ;\mathbb{A}\ ,\mathrm{A}_3;A'\\ \vdots & \cdot\\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot\\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot\\ \vdots & \cdot\\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot\\ \left(\frac{u^t}{x^t}-1\right) & \mathrm{B};\mathbb{B}\ ,\mathrm{B}_3;B'\end{array}\right)$$

$$f(u)\mathrm{d}u\ \ +tx^{\delta}\Gamma\left(\frac{\eta+\delta+1}{t}\right)\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}a_rb_r\prod_{i=1}^{r}y_i^{K_i}y_i'^{L_i}$$

$$\int_x^\infty u^{-\delta-1}\left(1-\frac{x^t}{u^t}\right)^{\alpha+\beta+1+\sum_{i=1}^r(K_ih_i+L_ih_i')} I^{V;0,n_r+n_r'+3;X}_{U:p_r+p_r'+3,q_r+q_r'+2;Y}\left(\begin{array}{c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} \\ \vdots \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} \\ \vdots \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} \\ \left(\frac{x^t}{u^t}-1\right) \end{array}\right|$$

$$\left.\begin{array}{c} \mathrm{A}\ ;\mathbb{A}\ ,\mathrm{A}_4;A' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{B};\mathbb{B}\ ,\mathrm{B}_4;B' \end{array}\right) f(u)\mathrm{d}u \tag{5.15}$$

Provided that

(a) $f(x)\in L_p(0,\infty)$, $1\leqslant p\leqslant 2$ [or $f(x)\in M_p(0,\infty), p>2]$, $1\leqslant p,q<\infty, p^{-1}+q^{-1}=1$

(b)$Re\left[t\alpha+t\sum_{i=1}^r\left(v_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+K_ih_i\right)\right]>-q^{-1}$

(c)$Re\left[t\beta+t\sum_{i=1}^r\left(v_i'\min_{1\leqslant j\leqslant m'^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+L_ih_i'\right)\right]>-q^{-1}$

(d) $Re(\delta)>-\frac{1}{p}, Re(\eta)>-\frac{1}{q}$

(e) $Q^{\eta,\alpha}_{z_r}R^{\delta,\beta}_{z_r}[f(x)]=R^{\eta,\alpha}_{z_r}Q^{\delta,\beta}_{z_r}[f(x)]\in L_p(0,\infty)$

(f) $\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$

$$\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right)>0$$

$$(g)\ \Omega_i' = \sum_{k=1}^{n'^{(i)}} \alpha_k'^{(i)} - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha_k'^{(i)} + \sum_{k=1}^{m'^{(i)}} \beta_k'^{(i)} - \sum_{k=m^{(i)}+1}^{q'^{(i)}} \beta_k'^{(i)} + \left(\sum_{k=1}^{n_2'} \alpha_{2k}'^{(i)} - \sum_{k=n_2+1}^{p_2'} \alpha_{2k}'^{(i)}\right)$$

$$+\cdots+\left(\sum_{k=1}^{n_r'} \alpha_{rk}'^{(i)} - \sum_{k=n_r'+1}^{p_r'} \alpha_{rk}'^{(i)}\right) - \left(\sum_{k=1}^{q_2'} \beta_{2k}'^{(i)} + \sum_{k=1}^{q_3'} \beta_{3k}'^{(i)} + \cdots + \sum_{k=1}^{q_r'} \beta_{rk}'^{(i)}\right) > 0$$

The proof is similar that (4.15)

6. Multivariable H-function

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad [5] reduces to multivariable H-function defined by Srivastava et al [11]. We obtain the following fractional integral operators.

$$Q_{z_r}^{\eta,\alpha}\left[f(x)\right] = tx^{-\eta-t\alpha-1}\int_0^x y^{\eta}(x^t-y^t)^{\alpha} H\begin{pmatrix} \mathrm{z}_1 \upsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \upsilon_r \end{pmatrix} S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r} \end{pmatrix} f(y)\mathrm{d}y \qquad (6.1)$$

and

$$R_{\gamma_n}^{\delta,\beta}\left[f(x)\right] = tx^{\delta} \int_x^{\infty} y^{-\delta-t\beta-1}(y^t-x^t)^{\beta} H\begin{pmatrix} \gamma_1 \mu_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_r \mu_r \end{pmatrix} S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix} \mathrm{y'}_1\left(1-\frac{x^t}{y^t}\right)^{h_1'} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y'}_r\left(1-\frac{x^t}{y^t}\right)^{h_r'} \end{pmatrix} f(y)\mathrm{d}y \qquad (6.2)$$

We have the following results.

Corollary 1

$$Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] = tx^{-t(\alpha+\beta+1+\frac{\delta+1}{t})} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \sum_{L_1=0}^{[Q_1/P_1]} \cdots \sum_{L_r=0}^{[Q_r/P_r]} a_r b_r x^{-t\sum_{i=1}^r (K_i+L_i)} \prod_{i=1}^r y_i^{K_i+L_i}$$

$$\int_0^x u^{\delta}(x^t-u^t)^{\alpha+\beta+1+\sum_{i=1}^r (K_i+L_i)h_i} H_{2p_r+4,2q_r+2;Y}^{0,2n_r+4;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathbb{A}\ ,\mathrm{A}_1; A' \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \left(\frac{u^t}{x^t}-1\right) & \mathbb{B}\ ,\mathrm{B}_1; B' \end{array}\right) f(u)\mathrm{d}u \qquad (6.3)$$

under the same notations and conditions that (4.15) with $U = V = A = B = 0$.

Corollary 2

We have

$$R_{z_r}^{\eta,\alpha} R_{z_r}^{\delta,\beta} [f(x)] = tx^{\eta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \sum_{L_1=0}^{[Q_1/P_1]} \cdots \sum_{L_r=0}^{[Q_r/P_r]} a_r b_r x^{-t\sum_{i=1}^{r}(K_i+L_i)} \prod_{i=1}^{r} y_i'^{K_i+L_i}$$

$$\int_x^{\infty} u^{-(\eta+1+t(\alpha+\beta+1+\sum_{i=1}^{r} h_i'(K_i+L_i)))} (u^t - x^t)^{\alpha+\beta+1+\sum_{i=1}^{r}(K_i+L_i)h_i'}$$

$$H_{2p_r'+4,2q_r'+2;Y}^{0,2n_r'+4;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \mathbb{A}\ ,\mathrm{A}_2; A' \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \cdot \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \cdot \\ \left(\frac{x^t}{u^t}-1\right) & \mathbb{B}\ ,\mathrm{B}_2; B' \end{array}\right) f(u)\mathrm{d}u \tag{6.4}$$

under the same notations and conditions that (4.34) with $U = V = A = B = 0$.

Corollary 3

$$Q_{z_r}^{\eta,\alpha} R_{z_r}^{\delta,\beta} [f(x)] = R_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta} [f(x)] = tx^{-\eta-1}\Gamma\left(\frac{\eta+\delta+1}{t}\right) \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \sum_{L_1=0}^{[Q_1/P_1]} \cdots \sum_{L_r=0}^{[Q_r/P_r]} a_r b_r \prod_{i=1}^{r} y_i^{K_i} y_i'^{L_i}$$

$$\int_0^x u^{\eta}\left(1-\frac{u^t}{x^t}\right)^{\alpha+\beta+1+\sum_{i=1}^{r}(K_i h_i + L_i h_i')} H_{p_r+p_r'+3,q_r+q_r'+2;Y}^{0,n_r+n_r'+3;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathbb{A}\ ,\mathrm{A}_3; A' \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \left(\frac{u^t}{x^t}-1\right) & \mathbb{B}\ ,\mathrm{B}_3; B' \end{array}\right)$$

$$f(u)\mathrm{d}u \quad +tx^{\delta}\Gamma\left(\frac{\eta+\delta+1}{t}\right)\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}a_r b_r\prod_{i=1}^{r}y_i^{K_i}{y'_i}^{L_i}$$

$$\int_x^{\infty}u^{-\delta-1}\left(1-\frac{x^t}{u^t}\right)^{\alpha+\beta+1+\sum_{i=1}^{r}(K_ih_i+L_ih'_i)}H^{0,n_r+n'_r+3;X}_{p_r+p'_r+3,q_r+q'_r+2;Y}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v'_1} & \\ \vdots & \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v'_r} & \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v'_1} & \\ \vdots & \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v'_r} & \\ \left(\frac{x^t}{u^t}-1\right) & \end{array}\right.$$

$$\left.\begin{array}{c} \mathbb{A}\ ,\mathrm{A}_4;A' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{B}\ ,\mathrm{B}_4;B' \end{array}\right)f(u)\mathrm{d}u \tag{6.5}$$

under the same notations and conditions that (5.15) with $U=V=A=B=0$.

Remark : If $A[N_1,K_1;\cdots;N_r,K_r]=B(N_1,K_1)\cdots B(N_r,K_r)$, the class of multivariable polynomials $S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[.]$ reduces to product of classes of polynomials of one variable defined by Srivastava [8], see the paper of Saxena et al [6].

7. Conclusion

In this paper we have evaluated the compositions of fractional integral operator concerning the multivariable I-functions defined by Prasad [1] and a class of multivariable polynomials defined by Srivastava [9] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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