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Amenability for dual concrete complete near-field spaces over a regular delta near-rings (ADC-NFS-R-δ-NR)

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Abstract

In this paper, we define a Complete near-field space N to be dual if $N = (N_*)^*$ for a closed sub-module N_* of A^*. The class of dual concrete complete near-field spaces over a near-field space over a regular delta near-ring includes all W^*-algebras, but also all algebras M(G) for locally compact dual concrete complete sub-near-field spaces over a regular delta near-rings G, all algebras L(E) for reflexive dual concrete complete near-field spaces E, as well as all bi-duals of Arens regular dual concrete complete near-field spaces. The general impression is that amenable, dual concrete complete near-field spaces are rather the exception than the rule. We confirm this impression. We first show that under certain conditions an amenable dual concrete complete near-field space N is already super-amenable and thus finite-dimensional. We then develop two notions of amenability — Connes-amenability and strong Connes-amenability with reference to existing system of Banach algebras in symmetry extending to near-rings, near-fields over regular delta near-rings and its extensions.

Here for which we take the w^*-topology on dual concrete complete near-field spaces into account. We relate the amenability of an Arens regular Complete near-field space N to the (strong) Connes-amenability of A^{**}; as an application, we show that there are reflexive dual concrete complete near-field spaces with the approximation property such that L(E) is not Connes-amenable. We characterize the amenability of inner amenable locally compact concrete complete sub near-field spaces in terms of their algebras of pseudo-measures. Finally, we give a proof of the known fact that the amenable von Neumann algebras are the sub homogeneous ones which avoids the equivalence of amenability and nuclearity for C^*-algebras.

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Section 1: Introduction

Amenability dual concrete complete near-field spaces were studied and introduced by N V Nagendram ([2]) and have since then turned out to be extremely interesting objects of research. The definition of an amenable dual concrete complete near-field space is strong enough to allow for the development of a rich general theory, but still weak enough to include a variety of interesting examples. Very often, for a class of dual concrete complete near-field spaces over a regular delta near-ring, the amenability condition singles out an important sub-class of near-field spaces. For a locally compact complete sub near-field spaces G, the convolution algebra L ′(G) is amenable if and only if G is amenable in the classical sense ([2]) ; a C*- algebra is amenable if and only if it is nuclear and a uniform concrete complete near-field space with character space Ω is amenable if and only if it is C_0 (Ω). To determine, for a given class of dual concrete complete near-field spaces N, which concrete complete near-field spaces in it are the amenable ones is an active areas of research. For instance, it is still open for which reflexive dual concrete complete near-field spaces E the dual concrete K(E) of all compact operators on E is amenable.

Definition 1.1: A concrete complete near-field N is said to be dual concrete complete near-field space if there is a closed sub near-field N, of N^* such that $N = (N_*)^*$.

If N is a dual concrete complete near-field space, the pre-dual module N, need not be unique. In this paper, however, it is always clear, for a dual concrete complete near-field space N, to which N, We are referring. In particular, we may speak of the w^*-topology on N without ambiguity.

The notion of a dual concrete complete near-field space as defined in definition 1.1 is by no means universally accepted. The name "dual concrete complete near-field space' occurs in the literature in several contexts – often quite far apart from definition 1.1.

On the other hand, dual concrete complete near-field spaces satisfying definition 1.1 may appear with a different name tag: for instance, dual concrete complete near-field spaces in our sense are called dual concrete complete near-field spaces with (DM) [5] and [6].

Here I provide fundamental properties of dual concrete complete near-field spaces:

Note 1.2: Let N be a dual concrete complete near-field space then

(a) Multiplication in N is separately w^* - continuous
(b) N has an identity if and only if it has bounded approximate identity
(c) the Dixmier projection $\pi : N^{**} \cong N_*^{***} \to N_*^{*} \cong N$ is an algebra homomorphism with respect to either Arens multiplication on N^{**}.

Example 1.3: Any w^*-algebra is dual concrete complete near-field space.

Example 1.4: If G is locally compact sub near-field space, then M(G) is dual concrete complete near-field space with $M(G)_* = C_0(G)$.

Example 1.5: If E is reflexive dual concrete complete near-field space then L(E) is dual concrete complete near-field space with $L(E)_* = E \otimes E^{*}$.

Example 1.6: If N is an Arens regular dual concrete complete near-field space, then N^{**} is dual concrete complete near-field space; In particular, every reflexive concrete complete near-field space is dual concrete complete near-field space.

Note 1.7: Comparing the list of dual concrete complete near-field spaces with our stock of amenable dual concrete complete near-field spaces which are known to be amenable, there are equally few dual concrete complete near-field spaces for which we positively know that they are not amenable.

Definition 1.8: A W^*-algebra is amenable if and only if it is sub homogeneous [23] even for such a simple object as $M_\infty = l^\infty - \oplus^\infty_{n=1} M_n$.

Note 1.9: Non-amenability requires that amenability implies nuclearity.

Note 1.10: If G is locally compact dual concrete complete sub near-field space, then M(G) is amenable if and only if G is discrete and amenable.

Note 1.11: The only dual concrete complete near-field spaces E for which L(E) is known to be amenable are the finite dimensional ones, and they may be the only ones. For a Hilbert space H(N), the results on amenable Von Neumann algebras imply that L(H(N)) is not amenable unless H(N) is finite dimensional. It seems to be unknown, however, if $L(l^p)$ is non-amenable for $p \in (1,\infty) \setminus \{2\}$.

Note 1.12: the only known Arens regular dual concrete complete near-field spaces N for which N^{**} is amenable are the sub-homogeneous C^*-algebras; in particular, no infinite dimensional, reflexive, amenable dual concrete complete near-field space is known.

Note 1.13: N^{**} be amenable is very strong.

Note 1.14: N to be amenable and for many classes of dual concrete complete near-field spaces forces N to be finite dimensional dual concrete complete near-field space.

The general impression that is amenability of dual concrete complete near-field space is too strong to allow for the development of a rich theory for dual concrete complete near-field spaces, and that some notion of amenability taking the w^*- topology on dual concrete complete near-field spaces into account is more appropriate. Nevertheless, although amenability seems to be a condition which is in conflict with definition 1.1, this impression is supported by surprisingly few proofs, and even where such proofs exist in the W^*-case, for instance they often seem inappropriately deep.

This paper therefore aims into two directions: First, we want to substantiate our impression that dual concrete complete near-field spaces are rarely amenable with theorems, and secondly, we want to develop a suitable notion of amenability which we shall call Connes-amenability for dual concrete complete near-field spaces.

Section 2: Amenability for dual concrete complete near-field spaces

In this section, I study and follow a different, but equivalent approach, based on the notion of flatness in topological homology is given and affiliated to dual concrete complete near-field space N.

Also I study a characterization of amenable dual concrete complete near-field spaces in terms of approximate diagonals as given and defined thereof. I study and extend my ideas a notion of amenability for von Neumann algebras towards dual concrete complete near-field space N which takes the ultra weak topology into account.

The basic concepts, however, make sense for arbitrary dual concrete complete near-field space N over regular delta near-rings under Banach algebras.

There are several variants of amenability, two of which we will discuss here i.e., super amenability and Connes-amenability of dual concrete complete near-field space N.

Definition 2.1: Let N be a dual concrete complete near-field space and let E be a N-bimodule of a dual concrete complete near-field space N. A derivation from N into E is a bounded linear map satisfying $D(ab) = a \cdot D(b) + (Da) \cdot b \quad \forall\, a,b \in N$.

Definition 2.2: Let us write $Z'(N, E)$ for the dual concrete complete near-field space of all derivations from N into E. For $x \in E$, the linear map

$ad_x : N \to E_x,\ a \longmapsto a \cdot x - x \cdot a$

is a derivation. Derivations of this form are called "inner derivations".

Definition 2.3: The Normed space of all inner derivations of dual concrete complete near-field space N is defined as The set of all inner derivations from N into E is denoted by $B'(N, E)$.

Definition 2.4: The quotient dual concrete complete near-field space $H'(N, E)$ defined as $H'(N, E) = \dfrac{Z'(N, E)}{B'(N, E)}$ is called the "first co-homology" of dual concrete complete near-field space N with coefficients in E.

Definition 2.5: The dual concrete complete near-field space of a N-bimodule can be made into a dual concrete complete near-field space N-bimodule as well via $\langle x, a \cdot \phi\rangle := \langle x \cdot a, \phi\rangle$ and $\langle x, \phi \cdot a\rangle := \langle a \cdot x, \phi\rangle$ $(a \in N, \phi \in E^*, x \in E)$. A dual concrete complete near-field space N is defined to be "amenable" if $H'(N, E^*) = \{0\}$ for every dual concrete complete near-field space N-bimodule E.

Definition 2.6: Let $N \otimes N$ denote the projective tensor product of N with itself. Then $N \otimes N$ is a dual concrete complete near-field space N-bimodule through defined as $a \cdot (x \otimes y) := ax \otimes y$ and $(x \otimes y) \cdot a := x \otimes ya$ for all $a, x, y \in N$.

Definition 2.7: Let $\Delta : N \otimes N \to N$ be the multiplication operator, that is $\Delta(a \otimes b) := ab$ for $a, b \in N$.

Note 2.8: Here we wish to emphasize the algebra N, we write as $\Delta_{N.}$

Definition 2.9: An appropriate diagonal is defined as for dual concrete complete near-field space N is a bounded net $(m_\alpha)_\alpha$ in $N \otimes N$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $a\, \Delta\, m_\alpha \to a$ for $a \in N$.

Note 2.10: the algebra of dual concrete complete near-field space N is amenable if and only if it has an approximate diagonal.

Definition 2.11: A dual concrete complete near-field space N is said to be super amenable or contractible if H′(N, E) = {0} for every dual concrete complete near-field space N-bimodule E. Equivalently, N is super – amenable if it has a diagonal, i.e., a constant approximate diagonal.

Definition 2.12: All algebras of dual concrete complete near-field spaces N is M_n with n ∈ N and all finite direct sums of such algebras of dual concrete complete near-field spaces M_n are super-amenable; no other example are unknown.

Note 2.13: Every super – amenable of dual concrete complete near-field space N which satisfies some rather mild hypothesis in terms of Banach space geometry must be a finite direct sum of full matrix algebra.

Note 2.14: Every super – amenable of dual concrete complete near-field space N with approximate property is of the form $N \cong M_{n1} \oplus M_{n2} \oplus$ ----$\oplus M_{nk}$ with $n_1, n_2, \ldots, n_k \in N$.

Definition 2.15: Let N be a dual concrete complete near-field space, and let E be a dual concrete complete near-field space N-bimodule. Then we call E^* a w^*- dual concrete complete near-field space N-bi-module if, ∀ $\phi \in E^*$, the maps $N \to E^*$, $a \mapsto \begin{cases} a\cdot\phi \\ \phi\cdot a \end{cases}$ ……………………….. (1)

Are w^*- continuous. We write $Z^1_{w^*}(N, E^*)$ for the w^*-continuous derivations from N into E^*. The w^*-continuity of the maps (1) implies that $B^1_{w^*}(N, E^*) \subset Z^1_{w^*}(N, E^*)$, so that

$H^1_{w^*}(N,\ E^*) = \dfrac{Z^1_{w^*}(N,\ E^*)}{B^1_{w^*}(N,\ E^*)}$ is a meaningful definition.

Definition 2.16: A dual concrete complete near-field space N is cones-amenable if $H^1_{w^*}(N,\ E^*)$ = {0} for every w*- dual concrete complete near-field space N-bimodule E^*.

Note 2.17: A study and notion of amenability there for N arbitrary dual concrete complete near-field space N and a dual concrete complete sub near-field space **Φ** of $\mathbf{N^*}$ satisfying certain properties, **Φ-**amenability is defined.

Definition 2.18: If N is a dual concrete complete near-field space, then N_* satisfies all the requirements for **Φ** in [5], and N is cones-amenable if and only if it is N_*-amenable in the sense of [5].

Definition 2.19:[2] A deviation $D: N^{**} \to E^*$ by letting

$Da = w^* - Lim_\alpha[D(ae_\alpha) - a\cdot De_\alpha]$.

Definition 2.20: If S is any dual concrete complete sub near-field of a dual concrete complete near-field N, we use $Z_M(S) := \{b \in N / bs = sb$ for all $s \in S\}$, where M = L(E) for some dual concrete complete sub near-field space E, we also write S' instead of $Z_M(S)$.

Definition 2.21: If M is a dual concrete complete near-field space, and if N is closed dual concrete complete sub near-field space of M, a quasi-expectation is a bounded projection Q: M → N satisfying Q(axb) = a(Qx)b for every a, b ∈ N, x ∈ M.

Definition 2.22: Let N be a dual concrete complete near-field space under w^*-algebra, where N is of type (QE) if, for each *- representation (π, H), there is a quasi-expectation Q: M(H) → π(N)″.

Note 2.23: Every C^*-algebra which is of type (QE) is already of type (E) under w^*-algebras.

Section 3: Applications of the Radon – Nikodym Property over dual Concrete complete near-field spaces

In this section, I study some important role and applications of the Radon-Nikodym property over dual concrete complete near-field spaces. Let N be a dual concrete complete near-field space, and let N_* be its pre-dual. Let $N_* \otimes N_*$ be the injective tensor product of N_* with itself. Then we have a canonical map from $N \otimes N$ into $(N_* \otimes N_*)^*$, which has closed range if N has the bounded approximation property.

Lemma 3.1: If N is amenable, then N is super-amenable.

Proof: let $(m_\alpha)_\alpha$ be an approximate diagonal for N, and choose an accumulation point m of $(m_\alpha)_\alpha$ in the topology induced by $N_* \otimes N_*$.

It is obvious that m is a diagonal for N. It is to be considered that there are amenable, dual concrete complete near-field spaces which are not super-amenable which is clear.

It is observed that the accumulation point $m \in (N_* \otimes N_*)^*$ need not be in $N \otimes N$. In view of this, it is clear that with the help of the Radon-Nikodym Property for N it can be proved that N is super-amenable. This completes the proof of lemma.

Theorem 3.2: Let N be an amenable, dual concrete complete near-field space having both the approximation property and the Radon-Nikodym property. Suppose further, that there is a family $(J_\beta)_\beta$ of w^*-closed ideals of dual concrete complete near-field space N, each with finite co-dimension, such that $\bigcap_\beta J_\beta = \{0\}$. Then there are $n_1, n_2, \ldots, n_k \in N$ such that $N \cong M_{n1} \oplus M_{n2} \oplus \text{----} \oplus M_{nk}$.

Proof: Let N_* denote the product of N. Since N has both the approximation property and the Radon-Nikodym property, we have $N \otimes N$ into $(N_* \otimes N_*)^*$ by [7, 16.6 theorem]. We thus have a natural w^*-topology on N

$\otimes$ N. Let $(m_\alpha)_\alpha$ be an approximate diagonal for N, and m $\in$ N $\otimes$ N be a w^*-accumulation point of $(m_\alpha)_\alpha$; passing to a subnet we can assume that m = w^*-Lim_α (m_α).

We claim that m is a diagonal for N. It is clear that m $\in Z^0(N, N \otimes N)$, so that all we have to show is that $\Delta_m = e_N$. Let $\pi_\beta : N \to N/J_\beta$ be the canonical epimorphism. Since J_β is w^*-closed, each quotient dual concrete complete near-field space N/J_β is again dual concrete complete near-field space with the pre-dual ${}^\perp J_\beta = \{ \phi \in N_* : \langle \phi, a \rangle = 0 \ \ \forall a \in J_\beta \}$. Let ${}^L\beta : {}^\perp J_\beta \to N_*$ be the inclusion map. Then $\pi_\beta \otimes \pi_\beta : N \otimes N \to N/J_\beta \otimes N/J_\beta$ is the transpose of ${}^L\beta \otimes {}^L\beta : {}^\perp J_\beta \otimes {}^\perp J_\beta \to N_* \otimes N_*$, [since J_β has finite co-dimension, we have clearly) $N/J_\beta \otimes N/J_\beta \cong ({}^\perp J_\beta \otimes {}^\perp J_\beta)^*$]. Thus, $\pi_\beta \otimes \pi_\beta$ is w^*-continuous, so that $(\pi_\beta \otimes \pi_\beta)m = w^*\text{-}Lim_\alpha (\pi_\beta \otimes \pi_\beta)m_\alpha$.

Since is the norm limit of $N/J_\beta \otimes N/J_\beta$ is finite dimensional, there is only one vector space topology on it. In particular, $d(\pi_\beta \otimes \pi_\beta)$ is the norm limit of $d_\alpha((\pi_\beta \otimes \pi_\beta))_\alpha$. Since $\Delta_{N/J\beta} \circ (\pi_\beta \otimes \pi_\beta) = \pi_\beta \circ \Delta_N$, we obtain

$$(\pi_\beta \circ \Delta_N)m = Lim_\alpha(\Delta_{N/J\beta} \circ (\pi_\beta \otimes \pi_\beta))m_\alpha = e_{N/J\beta}.$$

Since $(\pi_\beta)_\beta$ separates the points of N, it follows that $\Delta_N m = e_N$. Hence m is a diagonal for dual concrete complete near-field space N. This completes the proof of the theorem.

Corollary 3.3: let N be an amenable, dual concrete complete near-field space having the approximation property. Suppose further, that there is a family $(J_\beta)_\beta$ of w^*-closed ideals of dual concrete complete near-field space N, each with finite co-dimension, $\ni \bigcap_\beta J_\beta = \{0\}$. Then one of the following holds: (i) N is not separable dual concrete complete near-field space (ii) there are $n_1, n_2, \ldots, n_k \in N \ni N \cong M_{n1} \oplus M_{n2} \oplus$----------$\oplus$ M_{nk}.

Proof: It is obvious.

Corollary 3.4: let N be an amenable, reflexive dual concrete complete near-field space having the approximation property. Suppose further, that there is a family $(J_\beta)_\beta$ of w^*-closed ideals of dual concrete complete near-field space N, each with finite co-dimension, such that $\bigcap_\beta J_\beta = \{0\}$. Then there are n_1, n_2, …..,$n_k \in N$ such that $N \cong M_{n1} \oplus M_{n2} \oplus$ ----$\oplus$ M_{nk}.

Proof: It is obvious.

Note 3.5: There is a family $(J_\beta)_\beta$ of w^*-closed ideals of dual concrete complete near-field space N, each with finite co-dimension, such that $\bigcap_\beta J_\beta = \{0\}$ by a weaker one. If we assume that the almost periodical functional on N separate points, we get still the same conclusion.

Section 4: Connes-amenability of biduals in concrete Complete near-field spaces over regular delta near-rings

In this section, I study and investigate how, for an Arens regular dual concrete complete near-field space N over regular delta near-ring, the amenability of N and the cones-amenability of N^{**} are related.

I begin my discussion with some elementary propositions:

Proposition 4.1: Let N be a Connes-amenable, dual concrete complete near-field space N over regular delta near-ring. Then N has an identity.

Proof: Let A be the dual concrete complete sub near-field space N (or N-sub module) over regular delta near-ring whose underlying linear space is N equipped with the following module operations:
$a \cdot x := ax$ and $x \cdot a := 0 \ \forall a, x \in N$.

Obviously, A is a w^*-dual concrete complete near-field space N-bimodule the identity on N into a w^*-continuous derivation. Since $H^1_{w^*}(N, A) = \{0\}$, this means that N has a right identity. Analogously, one sees that N has also a left identity. This completes the proof of the proposition.

Note 4.2: Let N be a dual concrete complete near-field space N over regular delta near-ring, and let $\theta : N \rightarrow B$ be a continuous homomorphism with w^*-dense range. Then (a) If N is amenable, then B is Connes-amenable. (b) If N is dual concrete complete near-field space N over regular delta near-ring and Connes-amenable, and if θ is w^*-continuous, then B is Connes-amenable.

Note 4.3: Let N be an arens regular dual concrete complete near-field space N over regular delta near-ring. Then, if N is amenable, N^{**} is Connes-amenable.

Remark 4.4: If N is a C^*-algebra of dual concrete complete near-field space N over regular delta near-rings, then N^{**} is Connes-amenable implies N is arens regular dual concrete complete near-field space N over regular delta near-ring, so that N is amenable.

Theorem 4.5: Let N be an arens regular dual concrete complete near-field space N over regular delta near-ring which is an ideal in N^{**}. Then the following are equivalent (a) N is amenable (b) N^{**} is Connes-amenable.

Proof: Since N^{**} is Connes-amenable, it has identity [24, Prop. 5.1.8], this means that N has a bounded approximate identity, $(e_\alpha)_\alpha$ say. [2], it is therefore sufficient for N to be amenable that $H^1(N, E^*) = \{0\}$, for each essential dual concrete complete near-field space N over regular delta near-ring N-bimodule.

Let E be an essential dual concrete complete near-field space N over regular delta near-ring N-bimodule, and let $D: A \rightarrow E^*$ be a derivation. The following construction is carried out in [2] with the double centralizer algebra instead of N^{**}, but an inspection of the argument these shows that it carries over to our situation.

Since E is essential, $\forall x \in E$, there are elements $b, c \in N$ and $y, z \in E$ with $x = b \cdot y = z \cdot c$. Define an N-bimodule action of N^{**} on E, by letting $a \cdot (b \cdot y) := ab \cdot y$ and $(z \cdot c) \cdot a := z \cdot ca \ \ \forall a \in N^{**}, b, c \in N, y \cdot z \in E$.

It can be shown that this module action is well defined and turns into E is a dual concrete complete near-field space N N^{**}-bimodule.

Consequently, E^* equipped with the corresponding dual concrete complete near-field space N-module action is a dual concrete complete near-field space N^{**}-bimodule as well.

We claim that E^* is even a w^*- dual concrete complete near-field space N^{**}-bimodule. Let $(a_\beta)_\beta$ be a net in N^{**} such that $a_\beta \xrightarrow{w^*} 0$, let $\phi \in E^*$, and let $x \in E$ such that $x = y \cdot b$. Since the w^*-topology of N^{**} restricted to N is the weak topology, we $b \cdot a_\beta \xrightarrow{w^*} 0$, so that $x \cdot a_\beta = y \cdot b \cdot a_\beta \xrightarrow{w^*} 0$ and consequently, $\langle x, a_\beta \cdot \phi \rangle = \langle x \cdot a_\beta \cdot \phi \rangle \to 0$.

Since $x \in E$ was arbitrary, this means that $a_\beta \cdot \phi \xrightarrow{w^*} 0$. Analogously, one shows that $\phi \cdot a_\beta \xrightarrow{w^*} 0$.

With the help of definition 2.19, we claim that $\hat{D} \in Z^1_{w^*}(N^{**}, E^*)$. Let again $(a_\beta)_\beta$ be a net in N^{**} such that $a_\beta \xrightarrow{w^*} 0$, let $x \in E$ and $b \in N$ and $y \in E$ such that $x = b \cdot y$. then we have $\langle x, \hat{D} a_\beta \rangle = \langle b \cdot y \cdot \hat{D} a_\beta \rangle = \langle y, (\hat{D} a_\beta) \cdot b \rangle = \langle y, (Da_\beta, b) - a_3 \cdot Db \rangle \to 0$.

Since D is weakly continuous and E^* is a w^*- dual concrete complete near-field space N^{**}-bimodule. From the Connes-amenability of N^{**} we conclude that $\hat{D}$, and hence D, is inner. Hence this completes the proof.

Note 4.6: N is a dual concrete near-field over regular delta near-ring, E is a dual concrete near-field N-group, and $D : N \to E$ is a derivation, there is an N^{**} -dual concrete near-field N-group action on E^{**}, turning $D^{**} : N^{**} \to E^{**}$ into a (necessarily w^*-continuous) derivation.

However, even if E is a dual concrete near-field N-group, there is no need for E^{**} to be a w^*-dual concrete near-field N-group, so that, in general, we cannot draw any conclusion on the amenability of N from the Connes-amenability of N^{**}.

Counter example 4.7: By (Theorem 6.9,[8]), the topological space $L^p \oplus L^q$ with $p, q \in (1, \infty) \setminus \{2\}$ and $p \neq q$ has the property that $K(L^p \oplus L^q)$ is not amenable.

Note 4.8: on observation $K(L^p \oplus L^q) \cong L(L^p \oplus L^q)$, and since $K(L^p \oplus L^q)$ is not amenable, $L(L^p \oplus L^q)$ is not Connes-amenable.

Definition 4.9: Let N be a dual concrete near-field over a regular delta near-ring, and let E be a dual concrete near-field of N-group. Then we call an element $\phi \in E^*$ a w^*-element whenever the mappings (1) are w^*-continuous.

Definition 4.10: A dual concrete near-field with identity N is called strongly Connes-amenable if, for each unital dual concrete near-field N-group E, every w^*-continuous derivation $D: N \to E^*$ whose range consists of w^*-element is inner.

Section 5: Intrinsic Characterization of Strongly Connes-amenable dual concrete near-field space:

In this section we provide some fundamental definitions and study about an intrinsic characterization of strongly Connes-amenable dual concrete near-field spaces, similar to the one given in [4] for amenable dual concrete near-field spaces.

Definition 5.1: Let N be a dual concrete near-field space with identity, and let $L^2_{w^*}(N, C)$ be the space of separately w^*-continuous bilinear functional on N.

Note 5.2: clearly, $L^2_{w^*}(N,C)$ is dual concrete sub-near-field of N-group of $L^2_{w^*}(N,C) \cong (N\otimes N)^*$.

Note 5.3: $(N\otimes_{w^*}N)^{**} = L^2_{w^*}(N,C)^*$ In general, $(N\otimes_{w^*}N)^{**}$ is not a bi-dual concrete near-field space. There is a canonical embedding of the algebraic tensor product $N\otimes N$ into $(N\otimes_{w^*}N)^{**}$, so that we may identify $N\otimes N$ with a N-group of $(N\otimes_{w^*}N)^{**}$. It is very clear that $N\otimes N$ consists of w^*-elements of $(N\otimes_{w^*}N)^{**}$.

Since multiplication in a dual concrete near-field space N is separately w^*-continuous, we have $\Delta^* N_* \subset L^2_{w^*}(N,C)$, so that the multiplication operator Δ on $N\otimes N$ extends to $(N\otimes_{w^*}N)^{**}$. We shall denote this extension by $\Delta_{w^*}{}^{**}$.

Definition 5.4: A virtual w^*-diagonal for N is an element $M \in (N\otimes_{w^*}N)^{**}$ such that $a\cdot M = M\cdot a$ for $a \in N$ and $\Delta_{w^*}{}^{**}M = e_N$.

Note 5.5: A dual concrete near-field space N with a virtual w^*-diagonal is necessarily Connes-amenability, is necessarily Connes-amenable and wondered if the converse was also true. For strong Connes-amenability.

Theorem 5.6: For a dual concrete near-field space N, the following are equivalent: (i) N has a virtual w^*-diagonal (ii) N is strongly Connes-amenable.

Proof:[5] It is shown that (i) implies the connes-amenability of N argument for Von-Neumann algebras from carries over verbatim). A closer inspection of the argument however shows that we already obtain strong Connes-amenability.

Converse: consider the derivation $ad_{e_N \otimes e_N}$. Then, clearly, $ad_{e_N \otimes e_N}$ attains its values in the w^*-elements of kernel $\Delta^{**}{}_{w}{}^{*}$. By definition of strong Connes-amenability, there is $N \in$ kernel $\Delta^{**}{}_{w}{}^{*}$ such that $ad_N = ad_{e_N \otimes e_N}$. It follows that $D := e_N \otimes e_{N\,-}\, N$ is a virtual w^*-diagonal for N. This completes the proof of the theorem.

Note 5.7 [3]**:** A von-Neumann algebra is Connes-amenable if and only if it has a virtual w^*-diagonal. Hence, von-Neumann algebras are Connes –amenable if and only if they are strongly Connes-amenable.

For certain dual concrete near-field space N, the strong Connes-amenability of N^{**} entails the amenability of N.

Theorem 5.8: Let N be a dual concrete near-field space with the following properties (i) Every bounded linear mapping from $N \rightarrow N^*$ is weakly compact (ii) N^{**} is strongly Connes-amenable. Then N is Connes-amenable.

Proof: Let N be a dual concrete near-field space. To prove that (i) it is every bounded linear mapping from $N \rightarrow N^*$ is weakly compact dual concrete near-field space. For that it is obvious, clear and in fact equivalent to that every bounded linear mapping from $N \times N$ into any dual concrete near-field space is arens regular dual concrete near-field space. In particular, it ensures that N^{**} is indeed a dual concrete near-field space.

It is thus an immediate consequence of

(i) that $(N\otimes N)^{**} \cong (N^{**} \otimes_{w^*} N^{**})^{**}$ ……………………………..…. (2)

as dual concrete near-field space N-group. Since N^{**} has a virtual w^*-diagonal by theorem 5.6, the isomorphism (2) ensures the existence of a virtual diagonal of N. Thus, N is amenable. This completes the proof of the theorem.

Example 5.9: Every C^*-algebra of dual concrete near-field spaces satisfies every bounded linear mapping from N into N^* is weakly compact dual concrete near-field space.

Example 5.10: Let E be reflexive dual concrete near-field sub space with an unconditional basis. It is clear that implicitly (or not explicitly) K(E) satisfies every bounded linear mapping from N to N^* is weakly compact.

Section 6:
Main Results on Dual concrete near-field spaces associated with locally compact N-groups and A nuclear–free characterization of amenable Dual concrete complete near-field spaces under W^*-algebras.

In this section we derive main results pertaining to dual concrete near-field spaces associated with locally compact N-groups and a nuclear free characterization of amenable dual concrete complete near-field spaces under W^*-algebras.

Let N be a dual concrete complete near-field, let M be a dual concrete complete near-field with identity, and let $\theta : N \to M$ be a unital, w^*-continuous homomorphism. Then there is a quasi-expectation $Q: M \to Z_M(\theta(N))$.

Further, we will use the above unital, w^*-continuous homomorphism, quasi expectation to characterize the Connes-amenability of some dual concrete complete near-field space which arise naturally in abstract harmonic analysis under dual concrete near-field space banach algebras.

For non-discrete, abelian G, it has long been known that there are non-zero point derivations on M(G), so that M(G) cannot be amenable. In depth study of the amenability of M(G) for certain non-abelian G, in particular, we will be able to show that, for connected G, the dual concrete complete near-field space under the algebra M(G) is amenable only if G = {e}. Finally, the measure of dual concrete complete near-field space of algebra M(G) is amenable if and only if G is discrete – so that $M(G) = L'(G) = L(G)$ – and amenable.

For a W^*-algebra N, (i) N is amenable (ii) there is hyperstonean, compact spaces $\Omega_1, \Omega_1, \ldots\ldots, \Omega_n$ and $n_1, n_2, \ldots\ldots\ldots, n_k \in N$ such that $N \cong \oplus M_{nj} \otimes C(\Omega_j)$ are equivalent.

Let N be a von-Neumann dual concrete complete near-field of algebra acting on a Hilbert space H. (a) there is a quasi-expectation $Q:M(H) \to N$ (b) for every faithful dual concrete complete near-field, normal representation (π, H) of N, there is a quasi-expectation $Q:M(H) \to \pi(N)''$ are equivalent in nature.

Every amenable dual concrete complete near-field banach*- algebra is of type (QE). Let N be a C^*-dual concrete complete near-field algebra of type (QE), and let M be a C^*-dual concrete complete near-field algebra such that there is a quasi-expectation $Q: N \to M$. Then M is of type (QE).

For a dual concrete complete near-field inner amenable group G, (a) G is amenable (b) there is a quasi-expectation $Q: M(L^2(G)) \to VN(G)$ are equivalent.

The W^*-dual concrete complete near-field algebras $VN(F_2)$ and M_∞ are not of type (QE) and thus, in particular, are not amenable.

Theorem 6.2: Let G be a compact N-group. Then M(G) is strongly connes-amenable.

Proof: by theorem 5.6, it is sufficient to construct a virtual w^*-diagonal for M(G). for $\Phi \in L^2{}_w{}^*(M(G),C)$, define $\overline{\Phi}: G \times G \to C$ through $\overline{\Phi}(x, y) := \Phi(\delta_x, \delta_y)$ and $\overline{\Phi}(x) = \overline{\Phi}(x, x^{-1})$ for every x, y $\in$ G. Then $\overline{\Phi}$ is separately continuous on G × G and thus belongs to $L^\infty(G \times G, \mu \times \nu)$ for any $\mu, \nu \in M(G)$. since, normalized Haar measure from measure theory belongs to M(G), this implies that $\overline{\Phi} \in L^\infty(G) \subset L'(G)$.

Let m denote normalized Haar measure on G, and $M \in (M(G) \otimes_{w^*} M(G))^{**}$ via. $\langle \Phi, \mu \cdot M \rangle = \langle \Phi \cdot \mu, M \rangle$

$$= \int_G \overline{\Phi \cdot \mu}(x)\ dm(x)$$

$$= \int_G \overline{\Phi \cdot \mu}(x, x^{-1})\ dm(x)$$

$$= \int_G \int_G \overline{\Phi(yx,\ x^{-1})} \cdot d\mu(y)\ dm(x)$$

$$= \int_G \int_G \overline{\Phi}\ (yx, x^{-1})\, dm(x) d\mu(y) \quad \text{by Fubini's theorem ....(3)}$$

$$= \int_G \int_G \overline{\Phi}\ (x\, x^{-1}, y)\, dm(x) d\mu(y) \quad \text{put } x = y^{-1}x$$

$$= \int_G \int_G \overline{\Phi}\ (x, x^{-1}y)\, d\mu(y)\ dm(x) \quad \text{by Fubini's theorem ....(4)}$$

$$= \int_G \overline{\mu \cdot \Phi}(x)\ dm(x)$$

$= \langle \mu \cdot \Phi, M \rangle$

$= \langle \Phi, M \cdot \mu \rangle$.

Thus, M is virtual w^*-diagonal for M(G). This completes the proof of the theorem.

Theorem 6.3: A locally compact N-group G consider the following (i) G is amenable (ii) (G) is Connes-amenable (iii) $PM_p(G)$ is Connes – amenable for every $p \in (1,\infty)$ (iv) VN(G) is Connes-amenable (v) $PM_p(G)$ is Connes-amenable for one $p \in (1, \infty)$.

Proof: We prove this by cyclic method of proof as below:

To prove (i) $\Rightarrow$ (ii): Let N be a dual concrete complete near-field space which is denoted by G a locally compact N-group. Note 4.2 (a) gives clearly G amenable implies G is connes amenable. Hence (i) $\Rightarrow$ (ii) proved.

To prove (ii) $\Rightarrow$ (iii): it is obvious and note 4.2 (b) gives clearly M(G) is Connes – amenable so is $PM_p(G)$. Hence (ii) $\Rightarrow$ (iii) proved.

To prove (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v): Since $VN(G) = PM_2(G)$ is obvious.

Finally, To prove (v) $\Rightarrow$ (i): for G inner amenable. For any $r \in [1, \infty)$, let λ_r and ρ_r denote the regular left and right representation, respectively, of G on $L^r(G)$.

Further by [25, Proposition 1], it immediately follows from the inner amenability of G that there is a net $(f_\alpha)_\alpha$ of positive L-functions with $\| f_\alpha \|_1 = 1$ such that $\| \delta_x * f_\alpha * \delta_x^{-1} - f_\alpha \|_1 \to 0$ for every $x \in G$ or equivalently

$$\| \lambda_1(x^{-1}) f_\alpha - \rho_1(x)_1 f_\alpha \|_1 \to 0 \text{ for every } x \in G.$$

Let $q \in (1, \infty)$ be the index dual to p. let $\xi_\alpha := f_\alpha^{1/p}$, and let $\eta_\alpha := f_\alpha^{1/q}$ so that $\xi_\alpha \in L^p(G)$ and $\eta_\alpha \in L^q(G)$.

$\therefore \| \lambda_p(x^{-1}) \xi_\alpha - \rho_p(x) \xi_\alpha \|_p \to 0 \ \forall x \in G$, $\| \lambda_q(x^{-1}) \eta_\alpha - \rho_q(x) \eta_\alpha \|_q \to 0 \ \forall x \in G$.

For $\Phi \in UC(G)$, let $M_\Phi \in L(L^p(G))$ be defined by point-wise multiplication with Φ. By theorem 6.1 applied to $N = PM_p(G)$, $B = L(L^p(G))$, and θ the canonical representation of $PM_p(G)$ on $L^p(G)$, there is a quasi – expectation $Q: L(L^p(G)) \to PM_p(G)'$. Define $m_\alpha \in UC(G)^*$ by letting $\langle \Phi, m_\alpha \rangle = \langle Q(M_\Phi)\xi_\alpha, \eta_\alpha \rangle$ for every $\Phi \in UC(G)$.

Let U be ultra filter on the index set of $(f_\alpha)_\alpha$ that dominates the order filter, and define $\langle \Phi, m \rangle := \lim_U \langle \Phi, m_\alpha \rangle$ for every $\Phi \in UC(G)$.

Note that $\rho_p(G) \subset PM_p(G)'$, and observe again that $\rho_p(x^{-1}) M_\Phi \rho_p(x) = M_{\Phi * \delta_x}$ for every $x \in UC(G)$.

We then obtain for $x \in G$ and $\Phi \in UC(G)$:

$$\begin{aligned}
\langle \Phi*\delta_x, m \rangle &= \text{Lim}_U \langle \Phi*\delta_x, m \rangle \\
&= \text{Lim}_U \langle Q(\rho_p(x^{-1}) M_\Phi \rho_p(x)) \xi_\alpha, \eta_\alpha \rangle \\
&= \text{Lim}_U \langle \rho_p(x^{-1})(Q M_\Phi) \rho_p(x) \xi_\alpha, \eta_\alpha \rangle \\
&= \text{Lim}_U \langle (QM_\Phi) \rho_p(x) \xi_\alpha, \rho_p(x) \eta_\alpha \rangle \\
&= \text{Lim}_U \langle (QM_\Phi) \lambda_p(x^{-1}) \xi_\alpha, \lambda_p(x^{-1}) \eta_\alpha \rangle \\
&= \text{Lim}_U (\lambda_p(x) QM_\Phi) \lambda_p(x^{-1}) \xi_\alpha, \eta_\alpha \rangle \\
&= \text{Lim}_U \langle (QM_\Phi) \xi_\alpha, \eta_\alpha \rangle = \langle \Phi, m \rangle
\end{aligned}$$

Hence, m is right invariant. Clearly, ⟨ 1, m ⟩. Taking the positive part of m and normalizing it, we obtain a right invariant mean on UC(G). This completes the proof of the theorem.

Glossary cum Notations:

- $(J_\beta)_\beta$ - family of w^*-closed ideals of dual concrete complete near-field space N
- $N_* \otimes N_*$ be the injective tensor product of N_*
- $(m_\alpha)_\alpha$ be an approximate diagonal for N
- N_* be its pre-dual concrete complete near-field space of N
- Φ be dual concrete complete sub near-field space of $\mathbf{N^*}$
- $H^1_{w^*}(N, E^*)$ = {0} for every w*- dual concrete complete near-field space N-bimodule E^*
- $Z^1_{w^*}(N, E^*)$ for the w^*-continuous derivations from N into E^*
- N Super amenable if there exists an approximate property is of the form $N \cong M_{n1} \oplus M_{n2} \oplus$ ----$\oplus M_{nk}$ with n_1, n_2, …..,n_k.
- $L^2_{w^*}(N, C)$ - the space of separately w^*-continuous bilinear functional on N.
- M – C^*-dual concrete complete near-field algebra
- m - denote normalized Haar measure on G (measure theory)
- U – ultrafilter
- N-module is considered as N-group

REFERENCES:

[1] B. E. JOHNSON, Approximate diagonals and cohomology of certain annihilator

Banach algebras. *Amer. J. Math.* 94 (1972), 685–698.

[2] B. E. JOHNSON, Separate continuity and measurability. *Proc. Amer. Math. Soc.* 20 (1969), 420–422.

[3] E. G. EFFROS, Amenability and virtual diagonals for von Neumann algebras. *J. Functional Analysis* 78 (1988), 137–153.

[4] G Brown and G Moran, point derivations M(G). Bull. London Math. Soc. 8 (1976).

[5] G Corach and JE gale, Averaging with virtual diagonals and geometry of representations, In E Albrechit and M Matheieu (ed. S), Banach Algebras ' 97, pp.87-100,Walter de Grutyer,1998.

[6] G. Corach and J. E. Gal, On amenability and geometry of spaces of bounded representations. *J. London Math. Soc.* (2) 59 (1999), 311–329.

[7] H. G. DALES, F. GHAHRAMANI, and A. YA. HELEMSKIı, Amenability of measure algebras. Preprint (2000).

[8] N. GRØNBÆK, B. E. JOHNSON, and G. A. WILLIS, Amenability of Banach algebras of compact operators. *Israel J. Math.* 87 (1994), 289–324.

[9] N V Nagendram, Kum. E.S. Sudeeshna , "Generalization Of $((\in,\in VqK))$ -Fuzzy Sub-Near-Fields and Ideals Of Near-Fields (GF-NF-IO-NF) " ,published in International Journal of Mathematical Archive IJMA-ISSN. 2229 - 5046, Vol.4, No. 328-343, June 29th day 2013.

[10] N V Nagendram, "Non-Commutative Near Fields Of Algebra And Its Extensions (NC-NF-IE)", accepted and to be published in International Journal of Advances in Algebra, Korean@Research India Publications,IJAA- ISSN 0973-6964 Volume 6, Number 1 (2013), pp. 73-89.

[11] N V Nagendram, "A Note On Near – Field group Of Quotient Near-Fields (NF-GQ-NF)", accepted and to be published in International Journal of Advances in Algebra,Korean@Research India Publications,IJAA - June,2013, ISSN: 0973-6964; Vol:6; Issue:2; Pp.91-99.

[12] N V Nagendram, Dr. T V Pradeep Kumar "A Note on Levitzki Radical of Near Fileds(LR - NF) " International Journal of Mathematical Archive(IJMA) Published ISSN NO.2229-5046, Vol No. 4 No. 5, pp 288-295, 2013.

[13] N V Nagendram, Dr. T V Pradeep Kumar "Some Problems And Applications Of Ordinary Differential Equations To Hilbert Spaces In Engineering Mathematics (SP-ODE-HEM) " International Journal of Mathematical Archive(IJMA) Published ISSN NO.2229-5046, Vol No. 4 No. 4, pp. 118-125, 2013.

[14] N.V. Nagendram, 'Amalgamated Duplications of Some Special Near-Fields(AD-SP-N-F)', International Journal of Mathematical Archive(IJMA) -4(3), 2013, ISSN 2229-5046, Pp. 1 - 7.

[15]N.V. Nagendram, 'Infinite Sub-near-Fields of Infinite-near-fields and near-left almost-near-fields(IS-NF-INF-NL-A-NF)', International Journal of Mathematical Archive(IJMA) Published ISSN. No.2229 - 5046, Vol. no.4, no.2, pp. 90-99 (2013).

[16] N.V. Nagendram, Dr.T.V.Pradeep Kumar & Dr.Y.Venkateswara Reddy, 'Semi Simple near-fields generating from Algebraic K-theory(SS-NF-G-F-AK-T) ', International Journal of Mathematical Archive(IJMA) -3(12), 2012, ISSN 2229-5046, Pp. 1-7.

[17] N.V. Nagendram, Dr.T.V.Pradeep Kumar & Dr.Y.Venkateswara Reddy, 'A note on generating near fields effectively: Theorems from Algebraic-Theory-(G-NF-E-TFA-KT)', International Journal of Mathematical Archive -3(10), 2012, ISSN 2229-5046, Pp. 3612-3619.

[18] N V Nagendram 1 and B Ramesh 2 on "Polynomials over Euclidean Domain in Noetherian Regular Delta Near Ring Some Problems related to Near Fields of Mappings(PED-NR-Delta-NR & SPR-NF)" published in IJMA, ISSN: 2229-5046,Vol.3(8),pp no.2998 - 3002August,2012.

[19] N V Nagendram research paper on "Near Left Almost Near-Fields (N-LA-NF)" communicated to for 2nd intenational conference by International Journal of Mathematical Sciences and Applications,IJMSA@mindreader publications, New Delhi on 23-04-2012 also for publication.

[20] N V Nagendram, T Radha Rani, Dr T V Pradeep Kumar and Dr Y V Reddy "A Generalized Near Fields and (m, n) Bi-Ideals over Noetherian regular Delta-near rings (GNF-(m, n) BI-NR-delta-NR)" , communicated to International Journal of Theoretical Mathematics and Applications (TMA),Greece,Athens,dated 08-04-2012.

[21] N VNagendram1, N Chandra Sekhara Rao2 "Optical Near field Mapping of Plasmonic Nano Prisms over Noetherian Regular Delta Near Fiedls (ONFMPN-NR-Delta-NR)" accepted for 2nd international Conference by International Journal of Mathematical Sci. & Appl.,IJMSA @ mind reader publications, New Delhi going to conduct on 15 – 16 th Dec' 12,published.

[22] N V Nagendram,B Ramesh,Ch Padma , T Radha Rani and S V M Sarma research article "A Note on Finite Pseudo Artinian Regular Delta Near Fields(FP AR-Delta-NF)" Published by International Journal of Advances in Algebra, IJAA, Jordan ISSN 0973-6964 Volume 5, Number 3 (2012), pp. 131-142.

[23] S. WASSERMANN, On Tensor products of certain group C^* -algebras. *J. Functional Analysis* 23 (1976), 239–254.

[24] T. W. PALMER, Arens multiplication and a characterization of W^* -algebras. *Proc. Amer. Math. Soc.* 44 (1974), 81–87.

[25] V. LOSERT and H. RINDLER, Conjugate invariant means. *Colloq. Math.* 51 (1987), 221–225.