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COEFFICIENT ESTIMATES AND INCLUSION RELATIONSHIP FOR P-VALENT FUNCTIONS RELATED TO CERTAIN OPERATOR

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Abstract. In this paper, we introduce a new class of multivalent functions defined by a linear operator to study some of the interesting properties like coefficient estimates, inclusion relationship.

1. Introduction, Definitions And Preliminaries

Let $\mathcal{A}_p$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z : | z |< 1\}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \geq 1)\,. \tag{1.1}$$

and let $\mathcal{A} = \mathcal{A}_1$.

For the functions $f(z)$ of the form (1.1) and $g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$, the Hadamard product (or convloution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, b_{n+p}\, z^{n+p}.$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ ([11, 3]).

We denote by $\mathcal{S}_p^*(\eta)$, $\mathcal{C}_p^*(\eta)$ and $\mathcal{K}_p^*(\eta)$ the subclass of $\mathcal{A}_p$ consisting of all analytic functions which are,respectively, p - valent starlike of order η $(0 \leq \eta < p)$ in $\mathbb{U}$, p - valent convex of order η and p - valent close to convex of order η $(0 \leq \eta < p)$ in $\mathbb{U}$ (see, [16]).

Let $\mathcal{N}$ be the class of analytic functions h with $h(0) = 1$, which are convex and univalent in $\mathbb{U}$ and satisfy the following inequality:

$$Re\,\{h(z)\} > 0 \qquad (z \in \mathbb{U})\,.$$

Making use of the aforementioned principle of subordination between analytic functions, we define each of the following subclasses of $\mathcal{A}_p$:

(1.2)

$$\mathcal{S}_p^*\,(\eta; h) = \left\{ f : f \in \mathcal{A}_p \quad and \quad \frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec h(z) \right\} (0 \leq \eta < p; z \in \mathbb{U}; h \in \mathcal{N})\,,$$

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(1.3)
$$\mathcal{C}_p(\eta;h)=\left\{f:f\in\mathcal{A}_p\quad and\quad \frac{1}{p-\eta}\left(1+\frac{zf''(z)}{f'(z)}-\eta\right)\prec h(z)\right\}(0\leq\eta<p;z\in\mathbb{U};h\in\mathcal{N})$$
and
$$\mathcal{K}_p(\eta,\delta;\phi,h)=\left\{f:f\in\mathcal{A}_p:\exists g\in\mathcal{S}_p^*(\eta;h),\frac{1}{p-\delta}\left(\frac{zf'(z)}{g(z)}-\delta\right)\prec\phi(z)\right\},\tag{1.4}$$
where, $0\leq\eta<p,0\leq\delta<p;z\in\mathbb{U};h,\phi\in\mathcal{N}$.

In particular, we set
$$\mathcal{S}_p^*\left(\eta;\left(\frac{1+z}{1-z}\right)^{\alpha}\right)=S_p^*(\eta;h_\alpha)\qquad\left(0\leq\eta<p;z\in\mathbb{U};h_\alpha(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}\in\mathcal{N}\right)\tag{1.5}$$
and
(1.6)
$$\mathcal{C}_p\left(\eta;\left(\frac{1+z}{1-z}\right)^{\alpha}\right)=C_p^*(\eta;h_\alpha)\qquad\left(0\leq\eta<p;z\in\mathbb{U};h_\alpha(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}\in\mathcal{N}\right).$$
It is easily seen from the above definitions that
$$f\in\mathcal{C}_p(\eta;h)\qquad\Leftrightarrow\qquad\frac{zf'(z)}{p}\in\mathcal{S}_p^*(\eta;h)\tag{1.7}$$
and
$$\mathcal{S}_p^*(\eta;h_1)=S_p^*(\eta)\qquad and\qquad\mathcal{C}_p(\eta;h_1)=\mathcal{C}_p(\eta).\tag{1.8}$$

The classes $\mathcal{S}_p^*(\eta;h)$ and $\mathcal{C}_p(\eta;h)$ were studied by Kim et. al. [6] and Ma and Minda [10]. Furthermore, the special classes $\mathcal{S}_1^*(0;h_\alpha)$ and $\mathcal{C}_1(0;h_\alpha)$ of strongly starlike functions of order α in $\mathbb{U}$ and strongly convex functions of order α in $\mathbb{U}$, respectively, were investgated extensively by Mocanu [13] and Nunokawa [14].

Recently, the authors [1] introduced the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}:\mathcal{A}_p\to\mathcal{A}_p$ as follows:
$$\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)=z^p+\frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)}\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+\beta+n)}{\Gamma(p+\alpha+\beta+n-\gamma+1)}\right]a_{n+p}z^{n+p}\tag{1.9}$$
$(\beta>-p;\alpha>\gamma-1;\gamma\in\mathbb{R};p\in\mathbb{N};z\in\mathbb{U})$.

From(1.9), it is easy to verify that
$$z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}f(z)\right)'=(p+\alpha+\beta-\gamma+1)\,\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)-(\alpha+\beta-\gamma+1)\,\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}f(z).\tag{1.10}$$

Remark 1.1. If we let $\gamma=1$, then this operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}$ reduces to $\mathcal{Q}_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [8] and $\mathcal{Q}_{\beta,1}^{\alpha}=\mathcal{Q}_{\beta}^{\alpha}$, was introduced and studied by Jung et al. [5]. If for a choice of α and β, then the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}$ reduces to the familiar other well - known integral operators introduced and discussed by various authors [2, 15, 9, 7].

Definition 1.1. Let $f\in\mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)$, if it satisfies the following relation:
$$\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)=\left\{f\in\mathcal{A}_p:\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f\in\mathcal{S}_p^*(\eta;h)\right\}\tag{1.11}$$
$(0\leq\eta<p,\quad\beta>-p;\alpha>\gamma-1;\gamma\in\mathbb{R};p\in\mathbb{N};z\in\mathbb{U})$.

Definition 1.2. Let $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)$, if it satisfies the following relation:

$$\mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)=\left\{f\in\mathcal{A}_p:\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f\in\mathcal{C}_p(\eta;h)\right\} \tag{1.12}$$

$(0\le\eta<p,\quad \beta>-p;\alpha>\gamma-1;\gamma\in\mathbb{R};p\in\mathbb{N};z\in\mathbb{U})$.

Definition 1.3. Let $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p(\alpha,\beta,\gamma,\eta,\delta,h,\phi)$, if it satisfies the following relation:

$$\mathcal{K}_p(\alpha,\beta,\gamma,\eta,h,\phi)=\left\{f\in\mathcal{A}_p:\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f\in\mathcal{K}_p(\eta,\delta;h,\phi)\right\} \tag{1.13}$$

$(0\le\eta<p, 0\le\delta<p\quad \beta>-p;\alpha>\gamma-1;\gamma\in\mathbb{R};p\in\mathbb{N};z\in\mathbb{U})$.

Remark 1.2. If $f\in\mathcal{K}_p(\alpha,\beta,\gamma,\eta,\delta,h,\phi)$, then $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f\in\mathcal{K}_p(\eta,\delta;h,\phi)$, hence there exists a function $g\in\mathcal{S}_p^*(\eta;h)$ such that

$$\frac{1}{p-\delta}\left(\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}g(z)}-\delta\right)\prec\phi(z).$$

It is easy to check that

$$\frac{z}{p}\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)\right)'=\mathfrak{R}_{\beta,p}^{\alpha,\gamma}\left(\frac{zf'(z)}{p}\right) \tag{1.14}$$

and according to this formula, we have

$$f\in\mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)\quad\Leftrightarrow\quad\frac{zf'(z)}{p}\in\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h) \tag{1.15}$$

In this paper, we investigate inclusion properties of the classes $\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)$, $\mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)$, $\mathcal{K}_p(\alpha,\beta,\gamma,\eta,h)$ and coefficient estimation for the classes $\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)$, $\mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)$.

2. Inclusion relationships

The following results are needed for our investigation.

Lemma 2.1. [4] *Let h be convex univalent in $\mathbb{U}$ with $h(0)=1$ and $Re(\kappa h(z)+\nu)>0$ $(\kappa,\nu\in\mathbb{C};z\in\mathbb{U})$. If q is analytic in $\mathbb{U}$ with $q(0)=1$, then the subordination*

$$q(z)+\frac{zq'(z)}{\kappa q(z)+\nu}\prec h(z)\qquad(z\in\mathbb{U})$$

implies that

$$q(z)\prec h(z)\qquad(z\in\mathbb{U}).$$

Lemma 2.2. [12] *Let h be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with $Re\omega(z)\ge0$ $(z\in\mathbb{U})$. If q is analytic in $\mathbb{U}$ with $q(0)=h(0)$, then the subordination*

$$q(z)+\omega(z)zq'(z)\prec h(z)\qquad(z\in\mathbb{U})$$

implies that

$$q(z)\prec h(z)\qquad(z\in\mathbb{U}).$$

Theorem 2.3. *Let $\gamma\in\mathbb{R},\beta>-p,\alpha>\gamma-1,0\le\eta<p$. Then*

$$\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)\subset\mathcal{S}_p^*(\alpha+1,\beta,\gamma,\eta,h).$$

Proof. Let $f \in \mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)$ and set

$$q(z) = \frac{1}{p-\eta}\left(\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} f(z)} - \eta\right) \tag{2.1}$$

where q is analytic in $(z \in \mathbb{U})$ with $q(0) = 1$ and $q(z) \neq 0$ for all $(z \in \mathbb{U})$. Applying (1.10) and (2.1), we obtain

$$(p+\alpha+\beta-\gamma+1)\frac{\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f(z)}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} f(z)} = (p-\eta)\,q(z) + \eta + \alpha + \beta - \gamma + 1 \tag{2.2}$$

By logarithmically differentiating both sides of (2.2) and multiplying the resulting equation by z, we have

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z)+\eta+\alpha+\beta-\gamma+1} \qquad (z \in \mathbb{U}). \tag{2.3}$$

By applying Lemma 2.1 to (2.4), it follows that $q \prec h$ in $(z \in \mathbb{U})$, that is, $f \in \mathcal{S}_p^*(\alpha+1,\beta,\gamma,\eta,h)$. □

Theorem 2.4. *Let $\gamma \in \mathbb{R}, \beta > -p, \alpha > \gamma - 1, 0 \leq \eta < p$. Then*

$$\mathcal{C}_p(\alpha,\beta,\gamma,\eta,h) \subset \mathcal{C}_p(\alpha+1,\beta,\gamma,\eta,h). \tag{2.4}$$

Proof. If $f \in \mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)$, then by definition we have, $\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f \in \mathcal{C}_p(\eta,h)$.

According to (1.14), (1.15) together with Theorem (2.3), we get the desired estimation. □

Taking

$$\psi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}, \qquad (-1 \leq B < A \leq 1, 0 < \alpha \leq 1),$$

in Theorem(2.3) and Theorem(2.4), we have the following corollary.

Corollary 2.5. *Let $\gamma \in \mathbb{R}, \beta > -p, \alpha > \gamma - 1, 0 \leq \eta < p$, $0 < \alpha \leq 1$ and $-1 \leq B < A \leq 1$. Then*

$$\mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,A,B) \subset \mathcal{S}_p^*(\alpha+1,\beta,\gamma,\eta,A,B).$$

Corollary 2.6. *Let $\gamma \in \mathbb{R}, \beta > -p, \alpha > \gamma - 1, 0 \leq \eta < p$, $0 < \alpha \leq 1$ and $-1 \leq B < A \leq 1$. Then*

$$\mathcal{C}_p(\alpha,\beta,\gamma,\eta,A,B) \subset \mathcal{C}_p(\alpha+1,\beta,\gamma,\eta,A,B).$$

Theorem 2.7. *Let $\gamma \in \mathbb{R}, \beta > -p, \alpha > \gamma - 1, 0 \leq \eta < p$.*

$$\mathcal{K}_p(\alpha,\beta,\gamma,\eta,h,\phi) \subset \mathcal{K}_p(\alpha+1,\beta,\gamma,\eta,h,\phi). \tag{2.5}$$

Proof. If $f \in \mathcal{K}_p(\alpha,\beta,\gamma,\eta,h,\phi)$, related to $g \in \mathcal{S}_p^*(\alpha,\beta,\gamma,\eta,h)$, according to the definition of these classes, we have

$$\frac{1}{p-\delta}\left(\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha,\gamma} f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha,\gamma} g(z)} - \delta\right) \prec \phi(z). \tag{2.6}$$

Now, if we let

$$q(z) = \frac{1}{p-\delta}\left(\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma} g(z)} - \delta\right) \tag{2.7}$$

then q is analytic in $\mathbb{U}$, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. Using (1.10), we obtain
(2.8)
$$[(p-\delta)q(z)+\delta]\,\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}g(z)+(\alpha+\beta-\gamma+1)\,\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}f(z) = (p+\alpha+\beta-\gamma+1)\,\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z).$$
Differentiating (2.8) and multiplying by z, we have

$$\begin{aligned}(p+\alpha+\beta-\gamma+1)\,z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)' &= z\,(\alpha+\beta-\gamma+1)\left(\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}f(z)\right)' \\ &\quad + z\,[(p-\delta)q(z)+\delta]\left(\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}g(z)\right)' \\ &\quad + z(p-\delta)q'(z)\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}g(z).\end{aligned} \tag{2.9}$$

Since $g \in \mathcal{S}^*_p(\alpha,\beta,\gamma,\eta,h)$, by Theorem (2.3), we have $g \in \mathcal{S}^*_p(\alpha+1,\beta,\gamma,\eta,h)$. Letting

$$H(z) = \frac{1}{p-\eta}\left(\frac{z\left(\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}g(z)\right)'}{\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}g(z)} - \eta\right) \tag{2.10}$$

then $H(z) \prec h(z)$, and using (1.10) once again, we have

$$(p+\alpha+\beta-\gamma+1)\frac{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}g(z)}{\mathfrak{R}^{\alpha+1,\gamma}_{\beta,p}g(z)} = (p-\eta)H(z)+\eta+(\alpha+\beta-\gamma+1). \tag{2.11}$$

From (2.9) and (2.11), we obtain

$$\frac{1}{p-\delta}\left(\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}g(z)} - \delta\right) = q(z) + \frac{zq'(z)}{(p-\eta)H(z)+\eta+(\alpha+\beta-\gamma+1)}.$$

and combaining with (2.6), we deduce that

$$q(z)+\omega(z)zq'(z) \prec \phi(z), \quad \text{where} \quad \omega(z) = \frac{1}{(p-\eta)H(z)+\eta+(\alpha+\beta-\gamma+1)}. \tag{2.12}$$

According to Lemma(2.2), the subordination (2.12) gives the required result. □

3. Coefficient Estimates

Theorem 3.1. *Let $f \in \mathcal{S}^*_p(\alpha,\beta,\gamma,\eta,h)$, then*

$$\begin{aligned}|a_{p+1}| &\leq \frac{2(p-\eta)(p+\alpha+\beta-\gamma+1)}{(p+\beta)} \qquad \text{and} \\ |a_{p+2}| &\leq \frac{(p-\eta)\,[2(p-\eta)+1]\,(p+\alpha+\beta-\gamma+1)(p+\alpha+\beta-\gamma+2)}{(p+\beta)(p+\beta+1)}.\end{aligned} \tag{3.1}$$

where $\gamma \in \mathbb{R}, \beta > -p, \alpha > \gamma - 1, 0 \leq \eta < p$.

Proof. Let $f \in \mathcal{S}^*_p(\alpha,\beta,\gamma,\eta,h)$.

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)'}{\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)} - \eta\right) = q(z) \prec h(z) \tag{3.2}$$

where q is analytic in $z \in \mathbb{U}$ with $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$.

From (3.2),

$$z\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right)' - \eta\left(\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z)\right) = (p-\eta)q(z)\mathfrak{R}^{\alpha,\gamma}_{\beta,p}f(z) \tag{3.3}$$

A simple computation shows that

$$\frac{(\beta+p)(p-\eta+1)}{(p+\alpha+\beta-\gamma+1)}a_{p+1} = \frac{(p-\eta)(\beta+p)}{(p+\alpha+\beta-\gamma+1)}a_{p+1} + (p-\eta)q_1$$

and

$$\frac{2(\beta+p)(\beta+p+1)}{(p+\alpha+\beta-\gamma+1)\,(p+\alpha+\beta-\gamma+2)}a_{p+2} = \frac{(p-\eta)q_1(\beta+p)}{(p+\alpha+\beta-\gamma+1)}a_{p+1} + (p-\eta)q_2$$

Using the inequality, $|q_n| \leq 2$ for all $n \geq 1$, we get

$$|a_{p+1}| \leq \frac{2(p-\eta)(p+\alpha+\beta-\gamma+1)}{(p+\beta)}$$

and

$$|a_{p+2}| \leq \frac{(p-\eta)\left[2(p-\eta)+1\right](p+\alpha+\beta-\gamma+1)(p+\alpha+\beta-\gamma+2)}{(p+\beta)(p+\beta+1)}.$$

□

Theorem 3.2. *Let $f \in \mathcal{C}_p(\alpha,\beta,\gamma,\eta,h)$, then*

$$\begin{aligned} |a_{p+1}| &\leq \frac{2p(p-\eta)(p+\alpha+\beta-\gamma+1)}{(p+1)(p+\beta)} \qquad \text{and} \\ |a_{p+2}| &\leq \frac{p(p-\eta)\left[2(p-\eta)+1\right](p+\alpha+\beta-\gamma+1)(p+\alpha+\beta-\gamma+2)}{(p+\beta)(p+\beta+1)}. \end{aligned} \tag{3.4}$$

where $\gamma \in \mathbb{R}, \beta > -p, \alpha > \gamma - 1, 0 \leq \eta < p$

We adopt the same technique of Theorem (3.1) for prooving this result. So, the proof is omitted.

References

[1] On certain subclass of p-valent fucntions, M.K.Aouf, El-Ashwah and Ahmed M Abd Eltawab, Journal of classical Analyis, 4, 1(2014), 63 - 68.

[2] Bernardi, S. D. Convex and starlike univalent functions. Trans. Amer. Math. Soc. 135 1969 429–446.

[3] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[4] Eenigenburg, Paul; Miller, Sanford S.; Mocanu, Petru T.; Reade, Maxwell O. On a Briot-Bouquet differential subordination. General inequalities, 3 (Oberwolfach, 1981), 339–348, Internat. Schriftenreihe Numer. Math., 64, Birkh?user, Basel, 1983

[5] Jung, Il Bong; Kim, Yong Chan; Srivastava, H. M. The Hardy space of analytic functions associated with certain one-parameter families of integral operators. J. Math. Anal. Appl. 176 (1993), no. 1, 138–147.

[6] Kim, Yong Chan; Choi, Jae Ho; Sugawa, Toshiyuki. Coefficient bounds and convolution properties for certain classes of close-to-convex functions. Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), no. 6, 95–98.

[7] Libera, R. J. Some classes of regular univalent functions. Proc. Amer. Math. Soc. 16 1965 755–758.

[8] Liu, Jin-Lin; Owa, Shigeyoshi. Properties of certain integral operator. Int. J. Math. Sci. 3 (2004), no. 1, 69–75.

[9] Livingston, A. E. On the radius of univalence of certain analytic functions. Proc. Amer. Math. Soc. 17 1966 352–357.

[10] Ma, W.; Minda, D. An internal geometric characterization of strongly starlike functions. Ann. Univ. Mariae Curie-Sk?odowska Sect. A 45 (1991), 89–97 (1992).

[11] Miller, Sanford S.; Mocanu, Petru T. Differential subordinations. Theory and applications. Monographs and Textbooks in Pure and Applied Mathematics, 225. Marcel Dekker, Inc., New York, 2000.

[12] Miller, Sanford S.; Mocanu, Petru T. Differential subordinations and univalent functions. Michigan Math. J. 28 (1981), no. 2, 157–172.

[13] Mocanu, Petru T. Alpha-convex integral operator and strongly-starlike functions. Studia Univ. Babes,-Bolyai Math. 34 (1989), no. 2, 18–24.

[14] Nunokawa, Mamoru. On the order of strongly starlikeness of strongly convex functions. Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), no. 7, 234–237.

[15] Saitoh, Hitoshi; Owa, Shigeyoshi; Sekine, Tadayuki; Nunokawa, Mamoru; Yamakawa, Rikuo. An application of a certain integral operator. Appl. Math. Lett. 5 (1992), no. 2, 21–24.

[16] Yaguchi, Teruo. The radii of starlikeness and convexity for certain multivalent functions. Current topics in analytic function theory, 375–386, World Sci. Publ., River Edge, NJ, 1992.

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