Fractional derivatives involving general classes of polynomials and special functions representable by Mellin-Barnes type contour integrals IV

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ABSTRACT
In this document, first time,we established fractional derivative involving the product of two general classes of polynomials with Multivariable I-function defined by Nambisan et al [6] and generalized Lauricella hypergeometric function. Second time, we established fractional derivative involving the product of two general classes of polynomials with two multivariable I-functions and generalized Lauricella function.

Keywords:Ggeneral class of polynomial, fractional derivative, generalized Lauricella function, multivariable I-function, multivariable H-function

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1.Introduction and preliminaries.

In this document, first time,we established fractional derivative involving the product of two general classes of polynomials with Multivariable I-function defined by Nambisan et al [6] and generalized Lauricella hypergeometric function. Second time, we established fractional derivative involving the product of two general classes of polynomials with two multivariable I-functions and generalized Lauricella function.

The I-function is defined and represented in the following manner.

$$I(z_1,z_2,\cdots,z_r) = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\delta_j^{(i)}s_i\right)} \tag{1.4}$$

For more details, see Nambisan et al [6].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p} A_j\alpha_j^{(i)} - \sum_{j=1}^{q} B_j\beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)}\delta_j^{(i)} \leqslant 0, i = 1,\cdots,r \tag{1.5}$$

The integral (2.1) converges absolutely if

$|arg(z_k)| < \frac{1}{2}\Delta_k\pi, k = 1,\cdots,r$ where

$$\Delta_k = -\sum_{j=n+1}^{p} A_j\alpha_j^{(k)} - \sum_{j=1}^{q} B_j\beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)}\gamma_j^{(k)} > 0 \tag{1.6}$$

Consider the second multivariable I-function.

$$I(z'_1 z_2,\cdots,z'_s) = I^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s}_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}\left(\begin{array}{c|c} \mathrm{z'}_1 & (\mathrm{a'}_j;\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)};A'_j)_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z'}_s & (\mathrm{b'}_j;\beta_j'^{(1)},\cdots,\beta_j'^{(s)};B'_j)_{1,q'} : \end{array}\right.$$

$$\left.\begin{array}{l} (\mathrm{c'}_j^{(1)},\gamma_j'^{(1)};C_j'^{(1)})_{1,p'_1};\cdots;(c_j'^{(s)},\gamma_j'^{(s)};C_j'^{(s)})_{1,p'_s} \\ \\ (\mathrm{d'}_j^{(1)},\delta_j'^{(1)};D_j'^{(1)})_{1,q'_1};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q'_s} \end{array}\right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\psi(t_1,\cdots,t_s)\prod_{i=1}^{s}\xi_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.8}$$

where $\psi(t_1,\cdots,t_s),\xi_i(s_i), i = 1,\cdots,s$ are given by :

$$\psi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'}\Gamma^{A'_j}\left(1-a'_j+\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)}{\prod_{j=n'+1}^{p}{}'\Gamma^{A'_j}\left(a'_j-\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)\prod_{j=1}^{q'}\Gamma^{B'_j}\left(1-b'_j+\sum_{i=1}^{s}\beta_j'^{(i)}t_j\right)} \tag{1.9}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n'_i}\Gamma^{C_j'^{(i)}}\left(1-c_j'^{(i)}+\gamma_j'^{(i)}t_i\right)\prod_{j=1}^{m'_i}\Gamma^{D_j'^{(i)}}\left(d_j'^{(i)}-\delta_j'^{(i)}t_i\right)}{\prod_{j=n'_i+1}^{p'_i}\Gamma^{C_j'^{(i)}}\left(c_j'^{(i)}-\gamma_j'^{(i)}t_i\right)\prod_{j=m'_i+1}^{q'_i}\Gamma^{D_j'^{(i)}}\left(1-d_j'^{(i)}-\delta_j'^{(i)}t_i\right)} \tag{1.10}$$

For more details, see Nambisan et al [6].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i' = \sum_{j=1}^{p'} A_j' \alpha_j'^{(i)} - \sum_{j=1}^{q'} B_j' \beta_j'^{(i)} + \sum_{j=1}^{p_i'} C_j'^{(i)} \gamma_j'^{(i)} - \sum_{j=1}^{q_i'} D_j'^{(i)} \delta_j'^{(i)} \leqslant 0, i = 1, \cdots, s \tag{1.11}$$

The integral (2.1) converges absolutely if

$|arg(z_k')| < \frac{1}{2}\Delta_k'\pi, k = 1, \cdots, s$ where

$$\Delta_k' = -\sum_{j=n'+1}^{p'} A_j' \alpha_j'^{(k)} - \sum_{j=1}^{q'} B_j' \beta_j'^{(k)} + \sum_{j=1}^{m_k'} D_j'^{(k)} \delta_j'^{(k)} - \sum_{j=m_k'+1}^{q_k'} D_j'^{(k)} \delta_j'^{(k)} + \sum_{j=1}^{n_k'} C_j'^{(k)} \gamma_j'^{(k)} - \sum_{j=n_k'+1}^{p_k'} C_j'^{(k)} \gamma_j'^{(k)} > 0 \tag{1.12}$$

A general of Humbert polynomial is defined by the generating function [5,page 54, eq.(1.5)]

$$\left[C - axt + bt^m(2x-1)^d\right]^{\upsilon} = \sum_{n=0}^{\infty} \phi_n(x) t^n \tag{1.13}$$

where $m, a, d \in N$,and other parameters are unrestricted in general.

The finite series representation for $\phi_n(x)$ is also obtained there [5,page 55 31. eq(2.1)]

$$\phi_n(x) = \sum_{k=0}^{[n/m]} \frac{(-)^k c^{-\upsilon-n+(m-1)k} (\upsilon)_{n+(1-m)k} (ax)^{n-mk} \left[b(2x-1)^d\right]^k}{k!(n-mk)!} \tag{1.14}$$

$$= \sum_{k=0}^{[n/m]} \sum_{l=0}^{[dk]} \frac{(-dk)_l 2^l c^{-\upsilon-n+(m-1)k} (\upsilon)_{n+(1-m)k} (a)^{n-mk} b^k (-)^{(d+1)k} x^{n-mk+l}}{l!k!(n-mk)!} \tag{1.15}$$

The generalized polynomials defined by Srivastava [8], is given in the following manner :

$$S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N_1'/M_1']} \cdots \sum_{K_t=0}^{[N_t'/M_t']} \frac{(-N_1')_{M_1'K_1}}{K_1!} \cdots \frac{(-N_t')_{M_t'K_t}}{K_t!}$$

$$A[N_1', K_1; \cdots; N_t', K_t] y_1^{K_1} \cdots y_t^{K_t} \tag{1.16}$$

Where $M_1', \cdots, M_s'$ are arbitrary positive integers and the coefficients $A[N_1', K_1; \cdots; N_t', K_t]$ are arbitrary constants, real or complex.

The another general class of polynomials of K-variables defined by Srivastava et al [9,page 686,eq.(1.4)] and denoted as follows

$$T_n^{m_1,\cdots,m_k}[x_1,\cdots,x_k] = \sum_{h_1,\cdots,h_k=0}^{M\leqslant n} \left[(-n)_M B(n,h_1,\cdots,h_k) \prod_{i=1}^{k} \frac{x_i^{h_i}}{h_i!}\right] \tag{1.17}$$

The Riemann-Liouville fractional derivative (or integral) of order μ is defined as follows [3 ,page49]

$$D_x^{\mu} f(x) = \begin{cases} \frac{1}{\Gamma(-u)} \int_0^x (x-t)^{-\mu-1} f(t)\mathrm{d}t, Re(\mu) < 0 \\ \quad \cdots \\ \frac{d^m}{dx^m} \left[D_x^{\mu-m} \left\{ f(x) \right\} \right], 0 \leqslant m, m \in \mathbb{N} \end{cases} \tag{1.18}$$

We have the following fractional derivative formula [4,page 67, eq. (4.4.4)] is also required :

$$D_x^{\mu} \left(x^{\lambda} \right) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} x^{\lambda-\mu}, Re(\lambda) > -1 \tag{1.19}$$

The generalized Leibnitz formula for fractional calculus is required in the following form [7, page317]

$$D_x^{\alpha} \left[f(x)g(x) \right] = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\epsilon} D_x^{\alpha-an-\epsilon} \left[f(x) \right] D_x^{an+\epsilon} \left[g(x) \right] \tag{1.20}$$

$0 \leqslant a \leqslant 1; \alpha$ is a real or complex arbitrary number.

The following generalized Lauricella function in terms of multiple contour integrals is also required [10,page 253]

We have $F_{Q;Q_1,\cdots,Q_t}^{P;P_1,\cdots,P_t}(-z_1,\cdots,-z_t) = \dfrac{\prod_{j=1}^{Q}\Gamma(B_j)\prod_{j=1}^{Q_1}\Gamma(g'_j)\cdots\prod_{j=1}^{Q_t}\Gamma(g_j^{(t)})}{\prod_{j=1}^{P}\Gamma(A_j)\prod_{j=1}^{P_1}\Gamma(e'_j)\cdots\prod_{j=1}^{P_t}\Gamma(e_j^{(t)})} \dfrac{1}{(2\pi\omega)^t} \int_{L_1} \cdots \int_{L_t}$

$$\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_t)\prod_{j_1}^{P_1}\Gamma(e'_j+s_1)\cdots\prod_{j_1}^{P_t}\Gamma(e_j^{(t)}+s_t)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_t)\prod_{j=1}^{Q_1}\Gamma(g'_j+s_1)\cdots\prod_{j=1}^{Q_t}\Gamma(g_j^{(t)}+s_t)}\Gamma(-s_1)\cdots\Gamma(-s_t)z_1^{s_1}\cdots z_t^{s_t}\,\mathrm{d}s_1\cdots\mathrm{d}s_t \tag{1.21}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_t)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,t$. The above result (1.21) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,t$

2. Required formula

The fractional derivative of multivariable I-function is as follows :

Lemme

$$D_z^{\alpha}\left[z^{\lambda} I\big(c_1 z^{\sigma_1},\cdots,c_r z^{\sigma_r}\big)\right] = z^{\lambda-\alpha} I_{p+1,q:p_1,q_1;\cdots;p_r,q_r}^{0,n+1:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|l} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}, \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}, \end{array}\right.$$

$$\left.\begin{array}{l} (\text{-}\lambda;\sigma_1,\cdots,\sigma_r;1) : (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\alpha-\lambda;\sigma_1,\cdots,\sigma_r;1) : (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{2.1}$$

To prove the Lemme ,express the multivariable I-function in Mellin-Barnes contour integral with the help of (1.2) and use the formula (1.19). Now interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

Where : $\sigma_i > 0, i = 1, \cdots, r$; $Re\left[\lambda + \sum_{i=1}^{r} \sigma_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + 1\right] > 0$

3. Main results

We will use these notations for the two following theorems:

$$X = m_1, n_1; \cdots; m_r, n_r; 1, P_1; \cdots; 1, P_t \tag{3.1}$$

$$V = p_1, q_1; \cdots; p_r, q_r; P_1, Q_1 + 1; \cdots; P_t, Q_t + 1 \tag{3.2}$$

$$A = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}, 0, \cdots, 0; A_j)_{1,p} \tag{3.3}$$

$$B = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}, 0, \cdots, 0; B_j)_{1,q} \tag{3.4}$$

$$\mathrm{C} = (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r};$$

$$(1, 0; 1), (1 - e'_j, f'_j; 1)_{1,P_1}, \cdots, (1, 0; 1), (1 - e_j^{(t)}, f_j^{(t)}; 1)_{1,P_t} \tag{3.5}$$

$$D = (\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r};$$

$$(0, 1; 1), (1 - g'_j, fk'_j; 1)_{1,Q_1}, \cdots, (0, 1; 1), (1 - eg_j^{(t)}, k_j^{(t)}; 1)_{1,Q_t} \tag{3.6}$$

$$K_1 = [1 - A_j; 0, \cdots, 0, a'_j, \cdots, a_t^{(t)}; 1]_{1,P}, [1 - l_i + \sigma_i + \sum_{j=1}^{k} \delta_i^{(j)} h_j; \lambda'_i, \cdots, \lambda_i^{(s)}, \mu'_i, \cdots, \rho_i^{(t)}); 1]_{1,r'}$$

$$, (Mh - \mu - N - l - \sum_{j=1}^{r} \upsilon l_j - \sum_{j=1}^{k} \rho_j h_i; \mu_1, \cdots, \mu_s, \lambda_1, \cdots, \lambda_t; 1) \tag{3.7}$$

$$K_2 = [1 - B_j; 0, \cdots, 0, b'_j, \cdots, b_t^{(t)}; 1]_{1,Q}, [1 + \sigma_i + \sum_{j=1}^{k} \delta_i^{(j)} h_j; \lambda'_i, \cdots, \lambda_i^{(s)}, \mu'_i, \cdots, \mu_i^{(t)}); 1]_{1,r'},$$

$$(Mh - \mu - N - l + \alpha - \sum_{j=1}^{r} \upsilon l_j - \sum_{j=1}^{k} \rho_j h_i; \mu_1, \cdots, \mu_s, \lambda_1, \cdots, \lambda_t; 1) \tag{3.8}$$

Theorem 1

$$D_x^a \left\{ x^\mu \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\sigma_i} \, \phi_N(zx) \, S_{N_1, \cdots, N_k}^{M_1, \cdots, M_k} \left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\delta'_i} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \mathrm{z}_k x^{\rho_k} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\delta_i^{(k)}} \end{array} \right) \right.$$

$$I\left(\begin{array}{c}\mathrm{Z}_1 x^{\mu_1}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\lambda'_i}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{Z}_r x^{\mu_r}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\lambda_i^{(r)}}\end{array}\right)$$

$$F_{Q;Q_1,\cdots,Q_t}^{P;P_1,\cdots,P_t}\left[-Z'_1x^{\lambda_1}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\mu'_i},\cdots,-Z'_tx^{\lambda_t}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\mu_i^{(t)}})\right]\Bigg\}$$

$$=x^{\mu-a}\prod_{i=1}^{r'}\epsilon_i^{\sigma_i}\frac{\prod_{j=1}^{Q}\Gamma(B_j)\prod_{j=1}^{Q_1}\Gamma(g'_j)\cdots\prod_{j=1}^{Q_t}\Gamma(g_j^{(t)})}{\prod_{j=1}^{P}\Gamma(A_j)\prod_{j=1}^{P_1}\Gamma(e'_j)\cdots\prod_{j=1}^{P_t}\Gamma(e_j^{(t)})}$$

$$\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\frac{(-dh)_l c^{-\upsilon'-N+(M-1)h}(\upsilon')_{N+(1-M)h}(a)^{N-Mh}b^h(-)^{(d+1)h}2^l x^{N-Mh+1}}{l!h!(N-Mh)!}$$

$$\sum_{h_1=0}^{[N'_1/M'_1]}\cdots\sum_{h_k=0}^{[N'_k/M'_k]}A(h_1,\cdots,h_k)\prod_{j=1}^{k}\left[\frac{(-N_j)_{M_jh_j}}{h_j!}(Z_jx^{\rho_j})^{h_j}\left(\epsilon_1^{\delta_1^{(j)}}\cdots\epsilon_{r'}^{\delta_{r'}^{(j)}}\right)^{h_j}\right]$$

$$\sum_{l_1,\cdots,l_{r'}=0}^{\infty}\prod_{i=1}^{r'}\left[\frac{\left(-\frac{x^{\upsilon}}{\epsilon_i}\right)^{l_i}}{l_i!}\right]I_{p+P+r'+1,q+Q+r'+1;V}^{0,n+P+r'+1;X}\left(\begin{array}{c|c}\mathrm{Z}_1x^{\mu_1}\prod_{i=1}^{r'}\epsilon_i^{-\lambda'_i} & \mathrm{A}\ ,\ \mathrm{K}_1:C\\ \cdot & \\ \cdot & \\ \mathrm{Z}_rx^{\mu_s}\prod_{i=1}^{r'}\epsilon_i^{-\lambda_i^{(s)}} & \\ \mathrm{Z'}_1x^{\lambda_1}\prod_{i=1}^{r'}\epsilon_i^{-\mu'_i} & \\ \cdot & \\ \cdot & \\ \mathrm{Z'}_tx^{\lambda_t}\prod_{i=1}^{r'}\epsilon_i^{-\mu_i^{(t)}} & \mathrm{B},\ \mathrm{K}_2:D\end{array}\right) \tag{3.9}$$

The quantities A,B,K_1,K_2,C and D are defined above, where the I-function on the right hand side is of (r+t) variables and $F_{Q;Q_1,\cdots,Q_t}^{P;P_1,\cdots,P_t}(.)$ is the generalized Lauricella function [10, page 253-254].

Provided that

a) $\upsilon,\mu,\sigma_i,\rho_j,\delta_i^{(j)},\mu_l,\lambda_i^{(l)},\lambda_m,\mu_i^{(m)}>0;i=1,\cdots,r';j=1,\cdots,k;l=1,\cdots,s;m=1,\cdots,t$

b) $Re\big[\mu+1+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\big]>0$

Remark: **If** $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$ **, the multivariable I-function defined by Nambisan et al [6] degenere in multivariable H-function defined by Srivastava et al [11], see Jaimini et al [2, page 13].**

Concerning the following theorem, the quantities A, B, C and D are similar.

Let $K_3 = (-\mu; \sigma_1', \cdots, \sigma_l'; 1) and K_4 = (-\alpha - em' - \epsilon - \mu; \sigma_1', \cdots, \sigma_l'; 1)$ (3.10)

$$K_5 = [1 - A_j; 0, \cdots, 0, \phi_j', \cdots, \phi_t^{(t)}; 1]_{1,P}, [1 + \sigma_i + \sum_{j=1}^{k} \delta_i^{(j)} h_j - l_i; \lambda_i', \cdots, \lambda_i^{(s)}, \mu_i', \cdots, \rho_i^{(t)}); 1]_{1,r'}$$

$$(Mh - N - l - \sum_{j=1}^{r} \upsilon l_j - \sum_{j=1}^{k} \rho_j h_i; \mu_1, \cdots, \mu_s, \lambda_1, \cdots, \lambda_t; 1) \tag{3.11}$$

$$K_6 = [1 - B_j; 0, \cdots, 0, \psi_j', \cdots, \psi_t^{(t)}; 1]_{1,Q}, [1 + \sigma_i + \sum_{j=1}^{k} \lambda_i^{(j)} h_j; \lambda_i', \cdots, \lambda_i^{(s)}, \mu_i', \cdots, \mu_i^{(s)}); 1]_{1,r},$$

$$(Mh + em' + \epsilon - N - l' - \sum_{j=1}^{r} \upsilon l_j - \sum_{j=1}^{k} \rho_j h_i; \mu_1, \cdots, \mu_s, \lambda_1, \cdots, \lambda_t; 1) \tag{3.12}$$

$$A' = (a_j'; \alpha_j'^{(1)}, \cdots, \alpha_j'^{(s)}, 0, \cdots, 0; A_j')_{1,p'} \tag{3.13}$$

$$B' = (b_j'; \beta_j'^{(1)}, \cdots, \beta_j'^{(s)}, 0, \cdots, 0; B_j')_{1,q'} \tag{3.14}$$

$$\text{C'} = (\text{c'}_j^{(1)}, \gamma_j'^{(1)}; C_j'^{(1)})_{1,p_1'}; \cdots; (c_j'^{(s)}, \gamma_j'^{(s)}; C_j'^{(s)})_{1,p_s'} \tag{3.15}$$

$$D' = (\text{d'}_j^{(1)}, \delta_j'^{(1)}; D_j'^{(1)})_{1,q_1'}; \cdots; (d_j'^{(s)}, \delta_j'^{(s)}; D_j'^{(s)})_{1,q_s'} \tag{3.16}$$

Theorem 2

$$D_x^a \left\{ x^\mu \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\sigma_i} \phi_N(zx) \, S_{N_1, \cdots, N_k}^{M_1, \cdots, M_k} \begin{pmatrix} \text{z}_1 x^{\rho_1} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\delta_i'} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \text{z}_k x^{\rho_k} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\delta_i^{(k)}} \end{pmatrix} \right.$$

$$I \begin{pmatrix} \text{Z}_1 x^{\mu_1} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{-\lambda_i'} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \text{Z}_r x^{\mu_r} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{-\lambda_i^{(r)}} \end{pmatrix} I \left(\begin{array}{c|c} \text{w}_1 x^{\lambda_1} & \text{A' : C'} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \text{w}_s x^{\lambda_s} & \text{B' : D'} \end{array} \right)$$

$$\left. F_{Q;Q_1, \cdots, Q_t}^{P;P_1, \cdots, P_t} \left[-Z_1' x^{\lambda_1} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{-\mu_i'}, \cdots, -Z_t' x^{\lambda_t} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{-\mu_i^{(t)}}) \right] \right\}$$

$$= x^{\mu-a} \prod_{i=1}^{r'} \epsilon_i^{\sigma_i} \frac{\prod_{j=1}^{Q} \Gamma(B_j) \prod_{j=1}^{Q_1} \Gamma(g'_j) \cdots \prod_{j=1}^{Q_t} \Gamma(g_j^{(t)})}{\prod_{j=1}^{P} \Gamma(A_j) \prod_{j=1}^{P_1} \Gamma(e'_j) \cdots \prod_{j=1}^{P_t} \Gamma(e_j^{(t)})}$$

$$\sum_{h=0}^{[N/M]} \sum_{l=0}^{[dh]} \frac{(-dh)_l c^{-\upsilon'-N+(M-1)h} (\upsilon')_{N+(1-M)h} (a)^{N-Mh} b^h (-)^{(d+1)h} 2^l x^{N-Mh+1}}{l! h! (N-Mh)!} \sum_{h_1=0}^{[N'_1/M'_1]} \cdots \sum_{h_k=0}^{[N'_k/M'_k]}$$

$$A(h_1, \cdots, h_k) \prod_{j=1}^{k} \left[\frac{(-N_j)_{M_j h_j}}{h_j!} (Z_j x^{\rho_j})^{h_j} \left(\epsilon_1^{\delta_1^{(j)}} \cdots \epsilon_{r'}^{\delta_{r'}^{(j)}} \right)^{h_j} \right] \quad \sum_{l_1, \cdots, l_{r'}=0}^{\infty} \prod_{i=1}^{r'} \left[\frac{\left(-\frac{x^\upsilon}{\epsilon_i} \right)^{l_i}}{l_i!} \right]$$

$$I_{p+P+r+1,q+Q+r+1;V}^{0,n+P+r+1;X} \left(\begin{array}{c|c} \mathrm{Z}_1 x^{\mu_1} \prod_{i=1}^{r'} \epsilon_i^{-\lambda'_i} & \\ \cdot & \mathrm{A}\ ,\mathrm{K}_5 : C \\ \cdot & \\ \mathrm{Z}_r x^{\mu_s} \prod_{i=1}^{r'} \epsilon_i^{-\lambda_i^{(s)}} & \\ \mathrm{Z'}_1 x^{\lambda_1} \prod_{i=1}^{r'} \epsilon_i^{-\mu'_i} & \\ \cdot & \\ \cdot & \mathrm{B},\ \mathrm{K}_6 : D \\ \mathrm{Z'}_t x^{\lambda_t} \prod_{i=1}^{r'} \epsilon_i^{-\mu_i^{(t)}} & \end{array} \right) I_{p'+1,q';V'}^{0,n'+1;X'} \left(\begin{array}{c|c} \mathrm{w}_1 x^{\lambda_1} & \mathrm{A'},\mathrm{K}_3 : C' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{w}_s x^{\lambda_s} & \mathrm{B'},\mathrm{K}_4 : D' \end{array} \right) \qquad \textbf{(3.17)}$$

Provided that

a) $\mu, \sigma_i, \rho_j, \delta_i^{(j)}, \mu_p, \lambda_i^{(p)}, \lambda_k, \mu_i^{(k)}, \sigma'_q > 0; i = 1, \cdots, r'; j = 1, \cdots, k; p = 1, \cdots, s; q = 1, \cdots, l; k = 1, \cdots, t$

b) $Re\left[\mu + 1 + \sum_{i=1}^{s} \lambda_i \min_{1 \leqslant j \leqslant m'_i} \frac{d_j'^{(i)}}{\delta_j'^{(i)}}\right] > 0 \,; Re\left[1 + \sum_{i=1}^{r} \mu_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$

Remark concerning the theorems 3 and 4 : If ; $A_j = B_j = C_j^{(i)} = D_j^{(i)} = A'_j = B'_j = C'^{(i)}_j = D'^{(i)}_j = 1$ **The multivariable I-function defined by Nambisan et al [6] degenere in the multivariable H-functions defined by Srivastava et al [11] and we obtain the H-function of** $(r+t)$ **variables and H-function of s variables, see Jaimini et al [2, page 268].**

To prove the theorem 1 ,we have

$$\text{L.H.S} = D_x^a \left\{ x^\mu \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\sigma_i} \phi_N(zx)\, S_{N_1,\cdots,N_k}^{M_1,\cdots,M_k} \left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\delta'_i} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_k x^{\rho_k} \prod_{i=1}^{r'} (x^\upsilon + \epsilon_i)^{\delta_i^{(k)}} \end{array} \right) \right.$$

$$I\left(\begin{array}{c} \mathrm{Z}_1 x^{\mu_1}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\lambda'_i} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_r x^{\mu_r}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\lambda_i^{(r)}} \end{array}\right)$$

$$F_{Q;Q_1,\cdots,Q_t}^{P;P_1,\cdots,P_t}\left[-Z'_1 x^{\lambda_1}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\mu'_i},\cdots,-Z'_t x^{\lambda_t}\prod_{i=1}^{r'}(x^{\upsilon}+\epsilon_i)^{-\mu_i^{(t)}})\right]\Bigg\} \tag{3.18}$$

First, using the definitions of general class of polynomials $\phi_n(.)$, $S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[.]$ given in (1.15) and (1.16) respectively and expressing the I-function of r variables and generalized Lauricella function in Mellin-Barnes contour integral with the help of equations (1.2) and (1.21) respectively, changing the order of summations and integrations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process), we have :

$$\text{L.H.S } = x^{\mu-a}\prod_{i=1}^{r'}\epsilon_i^{\sigma_i}\frac{\prod_{j=1}^{Q}\Gamma(B_j)\prod_{j=1}^{Q_1}\Gamma(g'_j)\cdots\prod_{j=1}^{Q_t}\Gamma(g_j^{(t)})}{\prod_{j=1}^{P}\Gamma(A_j)\prod_{j=1}^{P_1}\Gamma(e'_j)\cdots\prod_{j=1}^{P_t}\Gamma(e_j^{(t)})}$$

$$\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\frac{(-dh)_l x^{N-Mh+l}c^{-\upsilon'-N+(M-1)h}(\upsilon')_{N+(1-M)h}(a)^{N-Mh}b^h(-)^{(d+1)h}2^l x^{N-Mh+1}}{l!h!(N-Mh)!}$$

$$\sum_{h_1=0}^{[N_1/M_1]}\cdots\sum_{h_k=0}^{[N_k/M_k]}A(h_1,\cdots,h_k)\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_j h_j}}{h_j!}y_j^{h_j}\right]\frac{1}{(2\pi\omega)^{r+t}}\int_{L_1}\cdots\int_{L_r}\int_{L'_1}\cdots\int_{L'_t}\xi(S_1,\cdots,S_r)$$

$$\prod_{i=1}^{r}\phi_i(S_i)z_j^{S_j}\quad\psi(s_1,\cdots,s_t)\prod_{j=1}^{t}\left[\phi'_j(s_j)\Gamma(-s_j)Z_j'^{s_j}\right]\ D_x^{\alpha}\Big\{x^{\mu+N-Mh+l'+\sum_{j=1}^{k}\rho_j h_j+\sum_{j=1}^{s}\mu_j S_j+\sum_{j=1}^{t}\lambda_j s_j}$$

$$\prod_{i=1}^{r'}\left[(x^{\upsilon}+\epsilon_j)^{\sigma_j+\sum_{j=1}^{k}\delta_j h_j-\sum_{j=1}^{s}\lambda_i^{(j)}S_j-\sum_{j=1}^{t}\mu_i^{(j)}s_j}\right]\Big\}\mathrm{d}S_1\cdots\mathrm{d}S_r\mathrm{d}s_1\cdots\mathrm{d}s_t \tag{3.19}$$

Now use the binomial formula : $(x^{\upsilon}+a)^{\lambda}=a^{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(\frac{x^{\upsilon}}{a}\right)^m$; $\left|\frac{x^{\upsilon}}{a}\right|<1$ (3.20)

twice and then the result (1.19) therein. On interpreting the contour integrals into I-function of (s+t) variables with the help of (1.2), we obtain the desired result.

To prove the theorem 2, we set $f(x), g(x)$ in equation (1.20) as the following

$$f(x)=x^{\mu}I\left(\begin{array}{c|c} \mathrm{w}_1 x^{\lambda_1} & \text{A' :C'} \\ . & . \\ . & . \\ . & . \\ \mathrm{w}_s x^{\lambda_s} & \text{B':D'} \end{array}\right) \tag{3.21}$$

and $g(x) = \left\{ x^{\mu} \prod_{i=1}^{r'} (x^{\upsilon} + \epsilon_i)^{\sigma_i} \, \phi_N(zx) \, S_{N_1,\cdots,N_k}^{M_1,\cdots,M_k} \begin{pmatrix} \mathrm{z}_1 x^{\rho_1} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{\delta'_i} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \mathrm{z}_k x^{\rho_k} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{\delta_i^{(k)}} \end{pmatrix} \right.$

$$I \begin{pmatrix} \mathrm{Z}_1 x^{\mu_1} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\lambda'_i} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \mathrm{Z}_r x^{\mu_r} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\lambda_i^{(r)}} \end{pmatrix}$$

$$\left. F_{Q;Q_1,\cdots,Q_t}^{P;P_1,\cdots,P_t} \left[-Z'_1 x^{\lambda_1} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\mu'_i}, \cdots, -Z'_t x^{\lambda_t} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\mu_i^{(t)}}) \right] \right\} \tag{3.22}$$

We have L.H.S $= \sum_{m'=-\infty}^{\infty} e \binom{\alpha}{em'+\epsilon} D_x^{\alpha-en'-\epsilon} \left[x^{\mu} \aleph(w_1 x^{\lambda_1}, \cdots, w_s x^{\lambda_s}\right] D_x^{em'+\epsilon} \left\{ x^{\mu} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{\sigma_i} \phi_N(zx) \right.$

$$S_{N_1,\cdots,N_k}^{M_1,\cdots,M_k} \begin{pmatrix} \mathrm{z}_1 x^{\rho_1} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{\delta'_i} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \mathrm{z}_k x^{\rho_k} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{\delta_i^{(k)}} \end{pmatrix} I \begin{pmatrix} \mathrm{Z}_1 x^{\mu_1} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\lambda'_i} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \mathrm{Z}_r x^{\mu_r} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\lambda_i^{(r)}} \end{pmatrix}$$

$$\left. F_{Q;Q_1,\cdots,Q_t}^{P;P_1,\cdots,P_t} \left[-Z'_1 x^{\lambda_1} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\mu'_i}, \cdots, -Z'_t x^{\lambda_t} \prod_{i=1}^{r'} (x^{\upsilon}+\epsilon_i)^{-\mu_i^{(t)}}) \right] \right\} \tag{3.23}$$

Now use the Lemme and the theorem 1, we obtain the theorem 2

4. Conclusion

In this paper we have evaluated a generalized contour integral involving the multivariable Aleph-function, two classes of polynomials, the multivariable I-function defined by Nambisan et al [6] and the generalized Lauricella function. The four formulaes established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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