Fractional derivatives involving general classes of polynomials and product of two multivariable I-functions defined by Prasad representable by Mellin-Barnes type contour integrals

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ABSTRACT
In this document, first time,we established fractional derivative involving the product of two general classes of polynomials with the product of two multivariable I-functions defined by Prasad [4] and generalized hypergeometric function. Second time, we established fractional derivative involving the product of two general classes of polynomials with the product of three multivariable I-functions defined by Prasad [4] and generalized hypergeometric function.

Keywords: General class of polynomial, fractional derivative, generalized hypergeometric function, multivariable I-function, multivariable H-function, class of polynomials

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1.Introduction and preliminaries.

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,z_2,\cdots,z_r)= I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\middle| \begin{array}{c} (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \\ \\ (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{array}\right) \tag{1.1}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i|<\frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right) \tag{1.3}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_r) = 0(\,|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\,), max(\,|z_1|, \cdots, |z_r|\,) \rightarrow 0$$

$$I(z_1, \cdots, z_r) \;= 0(\,|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\,), min(\,|z_1|, \cdots, |z_r|\,) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

The multivariable I-function of r-variables is defined by Prasad [4] in term of multiple Mellin-Barnes type integral :

$$I(z'_1, \cdots, z'_s) = I^{0,n'_2;0,n'_3;\cdots;0,n'_s:m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_s,q'_s:p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)}} \left(\begin{array}{c|c} \text{z'}_1 & (\text{a'}_{2j};\alpha'^{(1)}_{2j},\alpha'^{(2)}_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \text{z'}_s & (\text{b'}_{2j};\beta'^{(1)}_{2j},\beta'^{(2)}_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{l} (\text{a'}_{sj};\alpha'^{(1)}_{sj},\cdots,\alpha'_{sj}{}^{(s)})_{1,p'_s} : (a'^{(1)}_j,\alpha'^{(1)}_j)_{1,p'^{(1)}};\cdots;(a'_j{}^{(s)},\alpha'_j{}^{(s)})_{1,p'^{(s)}} \\ \\ (\text{b'}_{rj};\beta'^{(1)}_{sj},\cdots,\beta'_{sj}{}^{(s)})_{1,q_s} : (b'^{(1)}_j,\beta'^{(1)}_j)_{1,q'^{(1)}};\cdots;(b'_j{}^{(s)},\beta'_j{}^{(s)})_{1,q'^{(s)}} \end{array}\right) \tag{1.8}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \phi(t_1, \cdots, t_s) \prod_{i=1}^{s} \zeta_i(t_i) z_i'^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.9}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [4]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_i| < \frac{1}{2}\Omega'_i\pi$, where

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'^{(i)}_k - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'^{(i)}_k + \sum_{k=1}^{m'^{(i)}} \beta'^{(i)}_k - \sum_{k=m'^{(i)}+1}^{q'^{(i)}} \beta'^{(i)}_k + \left(\sum_{k=1}^{n'_2} \alpha'^{(i)}_{2k} - \sum_{k=n'_2+1}^{p'_2} \alpha'^{(i)}_{2k} \right) +$$

$$+ \cdots + \left(\sum_{k=1}^{n'_s} \alpha'^{(i)}_{sk} - \sum_{k=n'_s+1}^{p'_s} \alpha'^{(i)}_{sk} \right) - \left(\sum_{k=1}^{q'_2} \beta'^{(i)}_{2k} + \sum_{k=1}^{q'_3} \beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q'_s} \beta'^{(i)}_{sk} \right) \quad (1.10)$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z'_1, \cdots, z'_s) = 0(|z'_1|^{\alpha_1}, \cdots, |z'_s|^{\alpha_r}), max(|z'_1|, \cdots, |z'_s|) \rightarrow 0$$

$$I(z'_1, \cdots, z'_s) = 0(|z'_1|^{\beta_1}, \cdots, |z'_s|^{\beta_r}), min(|z'_1|, \cdots, |z'_s|) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha''_k = min[Re(b'^{(k)}_j / \beta'^{(k)}_j)], j = 1, \cdots, m'_k$ and

$$\beta''_k = max[Re((a'^{(k)}_j - 1)/\alpha'^{(k)}_j)], j = 1, \cdots, n'_k$$

We will use these following notations in this section :

$$I(z''_1, \cdots, z''_u) = I^{0,n''_2;0,n''_3;\cdots;0,n''_u:m''^{(1)},n''^{(1)};\cdots;m''^{(u)},n''^{(u)}}_{p''_2,q''_2,p''_3,q''_3;\cdots;p''_u,q''_u:p''^{(1)},q''^{(1)};\cdots;p''^{(u)},q''^{(u)}} \left(\begin{array}{c} \text{z"}_1 \\ \cdot \\ \cdot \\ \cdot \\ \text{z"}_u \end{array} \middle| \begin{array}{c} (\text{a"}_{2j}; \alpha''^{(1)}_{2j}, \alpha''^{(2)}_{2j})_{1,p''_2}; \cdots ; \\ \\ \\ (\text{b"}_{2j}; \beta''^{(1)}_{2j}, \beta''^{(2)}_{2j})_{1,q''_2}; \cdots ; \end{array} \right.$$

$$\left. \begin{array}{c} (\text{a"}_{uj}; \alpha''^{(1)}_{uj}, \cdots, \alpha''^{(u)}_{uj})_{1,p''_u} : (a''^{(1)}_j, \alpha''^{(1)}_j)_{1,p''^{(1)}}; \cdots ; (a''^{(u)}_j, \alpha''^{(u)}_j)_{1,p''^{(u)}} \\ \\ (\text{b"}_{uj}; \beta''^{(1)}_{uj}, \cdots, \beta''^{(u)}_{uj})_{1,q''_u} : (b''^{(1)}_j, \beta''^{(1)}_j)_{1,q'^{(1)}}; \cdots ; (b''^{(u)}_j, \beta''^{(u)}_j)_{1,q''^{(u)}} \end{array} \right) \quad (1.11)$$

$$= \frac{1}{(2\pi\omega)^u} \int_{L''_1} \cdots \int_{L''_u} \psi(x_1, \cdots, x_u) \prod_{i=1}^{u} \xi_i(x_i) z''^{x_i}_i \mathrm{d}x_1 \cdots \mathrm{d}x_u \quad (1.12)$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

where $|argz''_i| < \frac{1}{2}\Omega''_i\pi$,

$$\Omega''_i = \sum_{k=1}^{n''^{(i)}} \alpha''^{(i)}_k - \sum_{k=n''^{(i)}+1}^{p''^{(i)}} \alpha''^{(i)}_k + \sum_{k=1}^{m''^{(i)}} \beta''^{(i)}_k - \sum_{k=m''^{(i)}+1}^{q''^{(i)}} \beta''^{(i)}_k + \left(\sum_{k=1}^{n''_2} \alpha''_{2k}{}^{(i)} - \sum_{k=n''_2+1}^{p''_2} \alpha''_{2k}{}^{(i)}\right)$$

$$+\cdots+\left(\sum_{k=1}^{n''_u} \alpha''_{uk}{}^{(i)} - \sum_{k=n''_u+1}^{p''_u} \alpha''_{uk}{}^{(i)}\right) - \left(\sum_{k=1}^{q''_2} \beta''_{2k}{}^{(i)} + \sum_{k=1}^{q''_3} \beta''_{3k}{}^{(i)} + \cdots + \sum_{k=1}^{q''_u} \beta''_{uk}{}^{(i)}\right) \tag{1.13}$$

where $i = 1, \cdots, u$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z''_1, \cdots, z''_u) = 0(\,|z''_1|^{\alpha'_1}, \cdots, |z''_u|^{\alpha'_s}\,), max(\,|z''_1|, \cdots, |z''_u|\,) \to 0$$

$$I(z''_1, \cdots, z''_u) \; = 0(\,|z''_1|^{\beta'_1} \;, \cdots, |z''_u|^{\beta'_s}\,), min(\,|z''_1|, \cdots, |z''_u|\,) \to \infty$$

where $k = 1, \cdots, z : \alpha''_k = min[Re(b''^{(k)}_j / \beta''^{(k)}_j)], j = 1, \cdots, m''_k$ and

$$\beta''_k = max[Re((a''^{(k)}_j - 1)/\alpha''^{(k)}_j)], j = 1, \cdots, n''_k$$

A general of Humbert polynomial is defined by the generating function [3,page 54, eq.(1.5)]

$$\left[C - axt + bt^m(2x-1)^d\right]^v = \sum_{n=0}^{\infty} \phi_n(x)t^n \tag{1.17}$$

where $m, a, d \in N$,and other parameters are unrestricted in general.

The finite series representation for $\phi_n(x)$ is also obtained there [3,page 55 31. eq(2.1)]

$$\phi_n(x) = \sum_{k=0}^{[n/m]} \frac{(-)^k c^{-v-n+(m-1)k}(v)_{n+(1-m)k}(ax)^{n-mk}\left[b(2x-1)^d\right]^k}{k!(n-mk)!} \tag{1.18}$$

$$= \sum_{k=0}^{[n/m]} \sum_{l=0}^{[dk]} \frac{(-dk)_l 2^l c^{-v-n+(m-1)k}(v)_{n+(1-m)k}(a)^{n-mk} b^k (-)^{(d+1)k} x^{n-mk+l}}{l!k!(n-mk)!} \tag{1.19}$$

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N'_1/M'_1]}\cdots\sum_{K_t=0}^{[N'_t/M'_t]}\frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}$$

$$A[N'_1,K_1;\cdots;N'_t,K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.20}$$

Where $M'_1,\cdots,M'_s$ are arbitrary positive integers and the coefficients $A[N'_1,K_1;\cdots;N'_t,K_t]$ are arbitrary constants, real or complex.

The another general class of polynomials of K-variables defined by Srivastava et al [7,page 686,eq.(1.4)] and denoted as follows

$$T_n^{m_1,\cdots,m_k}[x_1,\cdots,x_k] = \sum_{h_1,\cdots,h_k=0}^{M\leqslant n}\left[(-n)_M B(n,h_1,\cdots,h_k)\prod_{i=1}^k\frac{x_i^{h_i}}{h_i!}\right] \tag{1.21}$$

The Riemann-Liouville fractional derivative (or integral) of order μ is defined as follows [1 ,page49]

$$D_x^\mu f(x) = \begin{cases} \frac{1}{\Gamma(-u)}\int_0^x(x-t)^{-\mu-1}f(t)\mathrm{d}t, Re(\mu)<0 \\ \cdots \\ \frac{d^m}{dx^m}\left[D_x^{\mu-m}\{f(x)\}\right], 0\leqslant m, m\in\mathbb{N}\end{cases} \tag{1.22}$$

We have the following fractional derivative formula [2,page 67, eq. (4.4.4)] is also required :

$$D_x^\mu\left(x^\lambda\right) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}x^{\lambda-\mu}, Re(\lambda)>-1 \tag{1.23}$$

The generalized Leibnitz formula for fractional calculus is required in the following form [5, page317]

$$D_x^\alpha[f(x)g(x)] = \sum_{n=-\infty}^{\infty}a\binom{\alpha}{an+\epsilon}D_x^{\alpha-an-\epsilon}[f(x)]\,D_x^{an+\epsilon}[g(x)] \tag{1.24}$$

$0\leqslant a\leqslant 1;\alpha$ is a real or complex arbitrary number.

The following generalized hypergeometric function in terms of multiple contour integrals is also required [8 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P\Gamma(A_j)}{\prod_{j=1}^Q\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_t)\right]$$

$$= \frac{1}{(2\pi\omega)^t}\int_{L_1}\cdots\int_{L_t}\frac{\prod_{j=1}^P\Gamma(A_j+s_1+\cdots+s_t)}{\prod_{j=1}^Q\Gamma(B_j+s_1+\cdots+s_t)}\Gamma(-s_1)\cdots\Gamma(-s_t)x_1^{s_1}\cdots x_t^{s_t}\mathrm{d}s_1\cdots\mathrm{d}s_t \tag{1.25}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_t)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,t$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,t$

2. Required formula

The fractional derivative of multivariable I-function is as follows :

Lemme

$$D_z^{\alpha}\left[z^{\lambda}I\big(c_1z^{\sigma_1},\cdots,c_rz^{\sigma_r}\big)\right]=z^{\lambda-\alpha}I^{0,n_2;0,n_3;\cdots;0,n_r+1:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r+1,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)})_{1,p_2}; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)})_{1,q_2}; \end{array}\right.$$

$$\left.\begin{array}{c} \cdots;(-\lambda;\sigma_1,\cdots,\sigma_r),(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)})_{1,p_r}:(a_j^{(1)},\alpha_j^{(1)})_{1,p^{(1)}};\cdots;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ \\ \cdots;(\alpha-\lambda;\sigma_1,\cdots,\sigma_r),(b_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)})_{1,q_r}:(b_j^{(1)},\beta_j^{(1)})_{1,q^{(1)}};\cdots;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}} \end{array}\right) \tag{2.1}$$

To prove the Lemme ,express the multivariable I-function defined by Prasad [4] in Mellin-Barnes contour integral with the help of (1.2) and use the formula (1.23). Now interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

with : $\sigma_i>0, i=1,\cdots,r$; $Re\big[\lambda+\sum_{i=1}^{r}\sigma_i\min_{1\leqslant j\leqslant m^{(i)}}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}+1\big]>0$

3. Main results

We will use these notations for the two following theorems:

$$U=p_2',q_2';p_3',q_3';\cdots;p_{s-1}',q_{s-1}';p_2'',q_2'';p_3'',q_3'';\cdots;p_{u-1}'',q_{u-1}'';0,0;\cdots;0,0 \tag{3.1}$$

$$V=0,n_2';0,n_3';\cdots;0,n_{s-1}';0,n_2'';0,n_3'';\cdots;0,n_{u-1}'';0,0;\cdots;0,0 \tag{3.2}$$

$$X=m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)};m''^{(1)},n''^{(1)};\cdots;m''^{(u)},n''^{(u)};1,0;\cdots;1,0 \tag{3.3}$$

$$Y=p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)};p''^{(1)},q''^{(1)};\cdots;p''^{(u)},q''^{(u)};0,1;\cdots;0,1 \tag{3.4}$$

$$A=(a_{2k}';\alpha_{2k}'^{(1)},\alpha_{2k}'^{(2)});\cdots;(a_{(s-1)k}';\alpha_{(s-1)k}'^{(1)},\alpha_{(s-1)k}'^{(2)},\cdots,\alpha_{(s-1)k}'^{(s-1)});(a_{2k}'';\alpha_{2k}''^{(1)},\alpha_{2k}''^{(2)});\cdots;$$

$$(a_{(u-1)k}'';\alpha_{(u-1)k}''^{(1)},\alpha_{(u-1)k}''^{(2)},\cdots,\alpha_{(u-1)k}''^{(u-1)}) \tag{3.5}$$

$$B=(b_{2k}';\beta_{2k}'^{(1)},\beta_{2k}'^{(2)});\cdots;(b_{(s-1)k}';\beta_{(s-1)k}'^{(1)},\beta_{(s-1)k}'^{(2)},\cdots,\beta_{(s-1)k}'^{(s-1)});(b_{2k}'';\beta_{2k}''^{(1)},\beta_{2k}''^{(2)});\cdots;$$

$$(b''_{(u-1)k};\beta''^{(1)}_{(u-1)k},\beta''^{(2)}_{(u-1)k},\cdots,\beta''^{(u-1)}_{(u-1)k}) \tag{3.6}$$

$$\mathfrak{A} =(a'_{sk};\alpha'^{(1)}_{sk},\alpha'^{(2)}_{sk},\cdots,\alpha'^{(s)}_{sk},0,\cdots,0,0\cdots,0) \tag{3.7}$$

$$\mathfrak{A}' =(a''_{uk};0,\cdots,0,\alpha''^{(1)}_{uk},\alpha''^{(2)}_{uk},\cdots,\alpha''^{(u)}_{uk},0,\cdots,0) \tag{3.8}$$

$$\mathfrak{B} =(b'_{sk};\beta'^{(1)}_{sk},\beta'^{(2)}_{sk},\cdots,\beta'^{(s)}_{sk},0,\cdots,0,0,\cdots,0) \tag{3.9}$$

$$\mathfrak{B}' =(b''_{uk};0,\cdots,0,\beta''^{(1)}_{uk},\beta''^{(2)}_{uk},\cdots,\beta''^{(u)}_{uk},0,\cdots,0) \tag{3.10}$$

$$\mathfrak{A}_2 = (a'^{(1)}_k,\alpha'^{(1)}_k)_{1,p'^{(1)}};\cdots;(a'^{(s)}_k,\alpha'^{(s)}_k)_{1,p'^{(s)}};(a''^{(1)}_k,\alpha''^{(1)}_k)_{1,p''^{(1)}};\cdots;(a''^{(u)}_k,\alpha''^{(u)}_k)_{1,p''^{(u)}};$$
$$(1,0);\cdots;(1,0) \tag{3.11}$$

$$\mathfrak{B}_2 = (b'^{(1)}_k,\beta'^{(1)}_k)_{1,q'^{(1)}};\cdots;(b'^{(s)}_k,\beta'^{(s)}_k)_{1,q'^{(s)}};(b''^{(1)}_k,\beta''^{(1)}_k)_{1,q''^{(1)}};\cdots;(b''^{(u)}_k,\beta''^{(u)}_k)_{1,q''^{(u)}};$$
$$(0,1);\cdots;(0,1) \tag{3.12}$$

$$K_1 = [1-A_j;0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,P},$$

$$(1+\lambda+\sum_{j=1}^{k}\delta_j h_j - l_1;\sigma_1,\cdots,\sigma_s,\sigma'_1,\cdots,\sigma'_u,\epsilon'_1,\cdots,\epsilon'_t),$$

$$(1+\gamma+\sum_{j=1}^{k}\epsilon_j h_j - l_2;\mu_1,\cdots,\mu'_s,\mu'_1,\cdots,\mu_u,\lambda'_1,\cdots,\lambda'_t),$$

$$(Mh-\mu-N-l-vl_1-vl_2-\sum_{j=1}^{k}\lambda_j h_j;\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,\delta'_1,\cdots,\delta'_t) \tag{3.13}$$

$$K_2 = [1-B_j;0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,Q},$$

$$(1+\lambda+\sum_{j=1}^{k}\delta_j h_j;\sigma_1,\cdots,\sigma_s,\sigma'_1,\cdots,\sigma'_u,\epsilon'_1,\cdots,\epsilon'_t),$$

$$(1+\gamma+\sum_{j=1}^{k}\epsilon_j h_j;\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu_u,\lambda'_1,\cdots,\lambda'_t),$$

$$(Mh-\mu+\alpha-N-l-vl_1-vl_2-\sum_{j=1}^{k}\lambda_j h_j;\rho_1,\cdots,\rho_s,\cdots,\rho'_1,\cdots,\rho'_u,\delta'_1,\cdots,\delta'_t) \tag{3.14}$$

Theorem 1

$$D_x^{\alpha}\{x^{\mu}(x^{\upsilon}+\epsilon)^{\lambda}(x^{\upsilon}+\eta)^{\gamma}\,\phi_N(zx)\,S_{N_1',\cdots,N_k'}^{M_1',\cdots,M_k'}\left(\begin{array}{c}\mathrm{y}_1x^{\lambda_1}(x^{\upsilon}+\epsilon)^{\delta_1}(x^{\upsilon}+\eta)^{\epsilon_1}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{y}_kx^{\lambda_k}(x^{\upsilon}+\epsilon)^{\delta_k}(x^{\upsilon}+\eta)^{\epsilon_k}\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}_1x^{\rho_1}(x^{\upsilon}+\epsilon)^{-\sigma_1}(x^{\upsilon}+\eta)^{-\mu_1}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_sx^{\rho_s}(x^{\upsilon}+\epsilon)^{-\sigma_s}(x^{\upsilon}+\eta)^{-\mu_s}\end{array}\right)I\left(\begin{array}{c}\mathrm{z'}_1x^{\rho_1'}(x^{\upsilon}+\epsilon)^{-\sigma_1'}(x^{\upsilon}+\eta)^{-\mu_1'}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{z'}_ux^{\rho_u'}(x^{\upsilon}+\epsilon)^{-\sigma_u'}(x^{\upsilon}+\eta)^{-\mu_u'}\end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z_j''x^{\delta_j'}(x^{\upsilon}+\epsilon)^{-\epsilon_j'}(x^{\upsilon}+\eta)^{-\lambda_j'}\right]\Bigg\}$$

$$=\frac{\epsilon^{\lambda}\eta^{\gamma}\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\frac{(-dh)_lc^{-\upsilon'-N+(M-1)h}(\upsilon')_{N+(1-M)h}(a)^{N-Mh}b^h(-)^{(d+1)h}2^lx^{N-Mh+1}}{l!h!(N-Mh)!}$$

$$\sum_{h_1=0}^{[N_1'/M_1']}\cdots\sum_{h_k=0}^{[N_k'/M_k']}A(h_1,\cdots,h_k)\prod_{j=1}^{k}\left[\frac{(-N_j')_{M_j'h_j}}{h_j!}(x^{\lambda_j}\epsilon^{\delta_j}\eta^{\epsilon_j}y_j)^{h_j}\right]\sum_{l_1,l_2=0}^{\infty}\frac{(-x)^{\upsilon l_1+\upsilon l_2}}{\epsilon^{l_1}l_1!\eta^{l_2}l_2!}$$

$$I_{U:n'+n''+P+3,q'+q''+Q+3;W}^{V;0,p'+p''+P+3;X}\left(\begin{array}{c|c}\mathrm{z}_1\epsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1} & \\ \cdot & \\ \cdot & \mathrm{A}\ ,\mathrm{K}_1,\mathfrak{A}:A'\\ \mathrm{z}_s\epsilon^{-\sigma_s}\eta^{-\mu_s}x^{\rho_s} & \\ \mathrm{z'}_1\epsilon^{-\sigma_1'}\eta^{-\mu_1'}x^{\rho_1'} & \\ \cdot & \\ \cdot & \\ \mathrm{z}_u\epsilon^{-\sigma_u'}\eta^{-\mu_u'}x^{\rho_u'} & \\ \mathrm{z''}_1\epsilon^{-\rho_1'}\eta^{-\lambda_1'}x^{\delta_1'} & \\ \cdot & \mathrm{B},\mathrm{K}_2,\mathfrak{B}:B'\\ \cdot & \\ \mathrm{z''}_t\epsilon^{-\rho_t'}\eta^{-\lambda_t'}x^{\delta_t'} & \end{array}\right)\tag{3.15}$$

The quantities $A, B, K_1, K_2, \mathfrak{A}, \mathfrak{B}, A'$ and B' are defined above, where the I-function on the right hand side is of $(s+u+t)$-variables.

Provided that

a) $\lambda_i, \delta_i, \epsilon_i, \rho_j, \sigma_j, \mu_j, \rho'_l, \sigma'_l, \mu'_l, \delta'_m, \epsilon'_m, , \lambda'_m > 0; i = 1, \cdots, k; j = 1, \cdots, s; l = 1, \cdots, u; m = 1, \cdots, t$

b) $Re[\mu + \sum_{i=1}^{s} \mu_i \min_{1 \leqslant j \leqslant m'^{(i)}} \frac{b_j'^{(i)}}{\beta_j'^{(i)}} + \sum_{i=1}^{u} \mu_i \min_{1 \leqslant j \leqslant m''^{(i)}} \frac{b_j''^{(i)}}{\beta_j''^{(i)}}] > 0$

Theorem 2

$$D_x^\alpha \Big\{ x^\mu (x^\upsilon + \epsilon)^\lambda (x^\upsilon + \eta)^\gamma \, \phi_N(zx) \, T_{N'}^{M'_1, \cdots, M'_k} \begin{pmatrix} \mathrm{y}_1 x^{\lambda_1} (x^\upsilon + \epsilon)^{\delta_1} (x^\upsilon + \eta)^{\epsilon_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{y}_k x^{\lambda_k} (x^\upsilon + \epsilon)^{\delta_k} (x^\upsilon + \eta)^{\epsilon_k} \end{pmatrix}$$

$$I \begin{pmatrix} \mathrm{z}_1 x^{\rho_1} (x^\upsilon + \epsilon)^{-\sigma_1} (x^\upsilon + \eta)^{-\mu_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{z}_s x^{\rho_s} (x^\upsilon + \epsilon)^{-\sigma_s} (x^\upsilon + \eta)^{-\mu_s} \end{pmatrix} I \begin{pmatrix} \mathrm{z'}_1 x^{\rho'_1} (x^\upsilon + \epsilon)^{-\sigma'_1} (x^\upsilon + \eta)^{-\mu'_1} \\ \cdot \ \cdot \ \cdot \\ \cdot \ \cdot \ \cdot \\ \mathrm{z'}_u x^{\rho'_u} (x^\upsilon + \epsilon)^{-\sigma'_u} (x^\upsilon + \eta)^{-\mu'_u} \end{pmatrix}$$

$${}_P F_Q \left[(A_P); (B_Q); -\sum_{j=1}^{t} z''_j x^{\delta'_j} (x^\upsilon + \epsilon)^{-\epsilon'_j} (x^\upsilon + \eta)^{-\lambda'_j} \right] \Big\}$$

$$= \frac{\epsilon^\lambda \eta^\gamma \prod_{j=1}^{Q} \Gamma(B_j)}{\prod_{j=1}^{P} \Gamma(A_j)} \sum_{h=0}^{[N/M]} \sum_{l=0}^{[dh]} \frac{(-dh)_l c^{-\upsilon' - N + (M-1)h} (\upsilon')_{N+(1-M)h} (a)^{N-Mh} b^h (-)^{(d+1)h} 2^l x^{N-Mh+1}}{l! h! (N-Mh)!}$$

$$\sum_{h_1, \cdots, h_k = 0}^{M' \leqslant N'} [(-N')_{M'} B(N', h_1, \cdots, h_k)] \prod_{j=1}^{k} \left[\frac{(-N'_j)_{M'_j h_j}}{h_j!} (x^{\lambda_j} \epsilon^{\delta_j} \eta^{\epsilon_j} y_j)^{h_j} \right] \sum_{l_1, l_2 = 0}^{\infty} \frac{(-x)^{\upsilon l_1 + \upsilon l_2}}{\epsilon^{l_1} l_1! \eta^{l_2} l_2!}$$

$$I_{U:n'+n''+P+3,q'+q''+Q+3;W}^{V;0,p'+p''+P+3;X}\left(\begin{array}{c|c} \mathrm{z}_1\epsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1} & \\ \cdot & \\ \cdot & \mathrm{A}\ ,\ \mathrm{K}_1,\mathfrak{A}:A' \\ \mathrm{z}_s\epsilon^{-\sigma_s}\eta^{-\mu_s}x^{\rho_s} & \\ \mathrm{z'}_1\epsilon^{-\sigma'_1}\eta^{-\mu'_1}x^{\rho'_1} & \\ \cdot & \\ \cdot & \\ \mathrm{z}_u\epsilon^{-\sigma'_u}\eta^{-\mu'_u}x^{\rho'_u} & \\ \mathrm{z''}_1\epsilon^{-\rho'_1}\eta^{-\lambda'_1}x^{\delta'_1} & \\ \cdot & \mathrm{B},\ \mathrm{K}_2,\mathfrak{B}:B' \\ \cdot & \\ \mathrm{z''}_t\epsilon^{-\rho'_t}\eta^{-\lambda'_t}x^{\delta'_t} & \end{array}\right) \tag{3.16}$$

under the same conditions that (3.15)

Concerning the two following theorems, we will use the following notations

$$U_1 = p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}; V_1 = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1} \tag{3.17}$$

$$X_1 = m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)}; Y_1 = p^{(1)}, q^{(1)}; \cdots ; p^{(r)}, q^{(r)} \tag{3.18}$$

$$A_1 = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \cdots ; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots , \alpha_{(r-1)k}^{(r-1)}) \tag{3.19}$$

$$B_1 = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \cdots ; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots , \beta_{(r-1)k}^{(r-1)}) \tag{3.20}$$

$$\mathfrak{A}_1 = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots , \alpha_{rk}^{(r)});\ \mathfrak{B}_1 = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots , \beta_{rk}^{(r)}) \tag{3.21}$$

$$\mathfrak{A}_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; \mathfrak{B}_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (b_k^{(r)}, \beta_k^{(r)})_{1,q^{(r)}} \tag{3.22}$$

$$K_3 = (-\mu; \lambda_1, \cdots , \lambda_s), K_4 = (-\mu + \alpha - en' - \epsilon; \lambda_1, \cdots , \lambda_s) \tag{3.23}$$

$$K_1 = [1 - A_j; 0, \cdots , 0, 0, \cdots , 0, 1, \cdots , 1]_{1,P},$$

$$(1 + \lambda + \sum_{j=1}^{k} \delta_j h_j; \sigma_1, \cdots , \sigma_s, \sigma'_1, \cdots , \sigma'_u, \epsilon'_1, \cdots , \epsilon'_t),$$

$$(1+\gamma+\sum_{j=1}^{k}\epsilon_j h_j - l_2;\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu_u,\lambda'_1,\cdots,\lambda'_t),$$

$$(Mh-\mu-N-l-\upsilon l_1-\upsilon l_2-\sum_{j=1}^{k}\lambda_j h_j;\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,\delta'_1,\cdots,\delta'_t) \tag{3.24}$$

$$K_2=[1-B_j;0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,Q},$$

$$(1+\lambda+\sum_{j=1}^{k}\delta_j h_j;\sigma_1,\cdots,\sigma_s,\sigma'_1,\cdots,\sigma'_u,\epsilon'_1,\cdots,\epsilon'_t),$$

$$(1+\gamma+\sum_{j=1}^{k}\epsilon_j h_j;\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu_u,\lambda'_1,\cdots,\lambda'_t),$$

$$(Mh-en'+\epsilon-N-l-\upsilon l_1-\upsilon l_2-\sum_{j=1}^{k}\lambda_j h_j;\rho_1,\cdots,\rho_s,\cdots,\rho'_1,\cdots,\rho'_u,\delta'_1,\cdots,\delta'_t) \tag{3.25}$$

Theorem 3

$$D_x^{\alpha}\Big\{x^{\mu}(x^{\upsilon}+\epsilon)^{\lambda}(x^{\upsilon}+\eta)^{\gamma}\,\phi_N(zx)\,S_{N'_1,\cdots,N'_k}^{M'_1,\cdots,M'_k}\begin{pmatrix} \mathrm{y}_1 x^{\lambda_1}(x^{\upsilon}+\epsilon)^{\delta_1}(x^{\upsilon}+\eta)^{\epsilon_1}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{y}_k x^{\lambda_k}(x^{\upsilon}+\epsilon)^{\delta_k}(x^{\upsilon}+\eta)^{\epsilon_k}\end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z}_1 x^{\rho_1}(x^{\upsilon}+\epsilon)^{-\sigma_1}(x^{\upsilon}+\eta)^{-\mu_1}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_s x^{\rho_s}(x^{\upsilon}+\epsilon)^{-\sigma_s}(x^{\upsilon}+\eta)^{-\mu_s}\end{pmatrix} I\begin{pmatrix} \mathrm{z'}_1 x^{\rho'_1}(x^{\upsilon}+\epsilon)^{-\sigma'_1}(x^{\upsilon}+\eta)^{-\mu'_1}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{z'}_u x^{\rho'_u}(x^{\upsilon}+\epsilon)^{-\sigma'_u}(x^{\upsilon}+\eta)^{-\mu'_u}\end{pmatrix}$$

$$I_{U_1:p,q;W_1}^{V_1;0,n;X_1}\left(\begin{array}{c|c} \omega x^{\alpha_1} & \mathrm{A}\ ;\mathfrak{A}_1;\mathrm{A}_1\\ \cdot & \\ \cdot & \\ \cdot & \\ \omega x^{\alpha_r} & \mathrm{B};\ \mathfrak{B}_1;\ \mathrm{B}_1\end{array}\right)\ {}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z''_j x^{\delta'_j}(x^{\upsilon}+\epsilon)^{-\epsilon'_j}(x^{\upsilon}+\eta)^{-\lambda'_j}\right]\Big\}$$

$$=\frac{\epsilon^{\lambda}\eta^{\gamma}x^{\mu-\alpha}\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\frac{(-dh)_l x^{N-Mh+l}c^{-\upsilon'-N+(M-1)h}(\upsilon')_{N+(1-M)h}(a)^{N-Mh}b^h(-)^{(d+1)h}2^l x^{N-Mh+1}}{l!h!(N-Mh)!}$$

$$\sum_{h_1=0}^{[N'_1/M'_1]}\cdots\sum_{h_k=0}^{[N'_k/M'_k]}A(h_1,\cdots,h_k)\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_jh_j}}{h_j!}(x^{\lambda_j}\epsilon^{\delta_j}\eta^{\epsilon_j}y_j)^{h_j}\right]\sum_{n'=-\infty}^{\infty}\sum_{l_1,l_2=0}^{\infty}e\binom{\alpha}{\epsilon n'+e}\frac{(-x)^{\upsilon l_1+\upsilon l_2}}{\epsilon^{l_1}l_1!\eta^{l_2}l_2!}$$

$$I^{V;0,p'+p''+P+3;X}_{U:n'+n''+P+3,q'+q''+Q+3;W}\left(\begin{array}{c|c} \mathrm{z}_1\epsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1} & \\ \cdot & \\ \cdot & \mathrm{A}\ ,\ \mathrm{K}_5,\mathfrak{A}:A' \\ \mathrm{z}_s\epsilon^{-\sigma_s}\eta^{-\mu_s}x^{\rho_s} & \\ \mathrm{z'}_1\epsilon^{-\sigma'_1}\eta^{-\mu'_1}x^{\rho'_1} & \\ \cdot & \\ \cdot & \\ \mathrm{z}_u\epsilon^{-\sigma'_u}\eta^{-\mu'_u}x^{\rho'_u} & \\ \mathrm{z''}_1\epsilon^{-\rho'_1}\eta^{-\lambda'_1}x^{\delta'_1} & \\ \cdot & \mathrm{B},\ \mathrm{K}_6,\mathfrak{B}:B' \\ \cdot & \\ \mathrm{z''}_t\epsilon^{-\rho'_t}\eta^{-\lambda'_t}x^{\delta'_t} & \end{array}\right)$$

$$I^{V_1;0,n+1;X_1}_{U_1:p+1,q;W_1}\left(\begin{array}{c|c} \omega x^{\alpha_1} & \mathrm{A}\ ;\mathfrak{A}_1,\mathrm{K}_3;A_1 \\ \cdot & \\ \cdot & \\ \omega x^{\alpha_r} & \mathrm{B};\ \mathfrak{B}_1,\mathrm{K}_4;B_1 \end{array}\right) \quad \textbf{(3.26)}$$

Provided that

a) $\lambda_i,\delta_i,\epsilon_i,\rho_j,\sigma_j,\mu_j,\rho'_l,\sigma'_l,\mu'_l,\delta'_m,\epsilon'_m,,\lambda'_m,\alpha_p>0;i=1,\cdots,k;j=1,\cdots,s;l=1,\cdots,u;m=1,\cdots,t;p=1,\cdots,r$

b) $Re\big[1+\sum_{i=1}^{s}\mu_i\min_{1\leqslant j\leqslant m'^{(i)}}\frac{b'^{(i)}_j}{\beta'^{(i)}_j}+\sum_{i=1}^{u}\mu_i\min_{1\leqslant j\leqslant m''^{(i)}}\frac{b''^{(i)}_j}{\beta''^{(i)}_j}\big]>0$ and

$$Re\big[1+\mu+\sum_{i=1}^{r}\alpha_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b^{(i)}_j}{\beta^{(i)}_j}\big]>0$$

Theorem 4

$$D^{\alpha}_x\big\{x^{\mu}(x^{\upsilon}+\epsilon)^{\lambda}(x^{\upsilon}+\eta)^{\gamma}\,\phi_N(zx)\,T^{M'_1,\cdots,M'_k}_{N'}\left(\begin{array}{c} \mathrm{y}_1x^{\lambda_1}(x^{\upsilon}+\epsilon)^{\delta_1}(x^{\upsilon}+\eta)^{\epsilon_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_kx^{\lambda_k}(x^{\upsilon}+\epsilon)^{\delta_k}(x^{\upsilon}+\eta)^{\epsilon_k} \end{array}\right)$$

$$I\begin{pmatrix} \mathrm{z}_1x^{\rho_1}(x^\upsilon+\epsilon)^{-\sigma_1}(x^\upsilon+\eta)^{-\mu_1} \\ \cdots \\ \cdots \\ \mathrm{z}_sx^{\rho_s}(x^\upsilon+\epsilon)^{-\sigma_s}(x^\upsilon+\eta)^{-\mu_s} \end{pmatrix} I\begin{pmatrix} \mathrm{z'}_1x^{\rho'_1}(x^\upsilon+\epsilon)^{-\sigma'_1}(x^\upsilon+\eta)^{-\mu'_1} \\ \cdots \\ \cdots \\ \mathrm{z'}_ux^{\rho'_u}(x^\upsilon+\epsilon)^{-\sigma'_u}(x^\upsilon+\eta)^{-\mu'_u} \end{pmatrix}$$

$$I_{U_1:p,q;W_1}^{V_1;0,n;X_1}\left(\begin{array}{c|c} \omega x^{\alpha_1} & \mathrm{A}\ ;\mathfrak{A}_1;\mathrm{A}_1 \\ . & \\ . & \\ . & \\ \omega x^{\alpha_r} & \mathrm{B};\ \mathfrak{B}_1;\ \mathrm{B}_1 \end{array}\right) {}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z''_j x^{\delta'_j}(x^\upsilon+\epsilon)^{-\epsilon'_j}(x^\upsilon+\eta)^{-\lambda'_j}\right]\Bigg\}$$

$$=\frac{\epsilon^\lambda\eta^\gamma x^{\mu-\alpha}\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\frac{(-dh)_l x^{N-Mh+l}c^{-\upsilon'-N+(M-1)h}(\upsilon')_{N+(1-M)h}(a)^{N-Mh}b^h(-)^{(d+1)h}2^l x^{N-Mh+1}}{l!h!(N-Mh)!}$$

$$\sum_{h_1,\cdots,h_k=0}^{M'\leqslant N'}[(-N')_{M'}B(N',h_1,\cdots,h_k)]\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_jh_j}}{h_j!}(x^{\lambda_j}\epsilon^{\delta_j}\eta^{\epsilon_j}y_j)^{h_j}\right]\sum_{n'=-\infty}^{\infty}\sum_{l_1,l_2=0}^{\infty}e\binom{\alpha}{\epsilon n'+e}\frac{(-x)^{\upsilon l_1+\upsilon l_2}}{\epsilon^{l_1}l_1!\eta^{l_2}l_2!}$$

$$I_{U:n'+n''+P+3,q'+q''+Q+3;W}^{V;0,p'+p''+P+3;X}\left(\begin{array}{c|c} \mathrm{z}_1\epsilon^{-\sigma_1}\eta^{-\mu_1}x^{\rho_1} & \\ . & \\ . & \mathrm{A}\ ,\ \mathrm{K}_5,\mathfrak{A}:A' \\ \mathrm{z}_s\epsilon^{-\sigma_s}\eta^{-\mu_s}x^{\rho_s} & \\ \mathrm{z'}_1\epsilon^{-\sigma'_1}\eta^{-\mu'_1}x^{\rho'_1} & \\ . & \\ . & \\ \mathrm{z}_u\epsilon^{-\sigma'_u}\eta^{-\mu'_u}x^{\rho'_u} & \\ \mathrm{z''}_1\epsilon^{-\rho'_1}\eta^{-\lambda'_1}x^{\delta'_1} & \\ . & \mathrm{B},\ \mathrm{K}_6,\mathfrak{B}:B' \\ . & \\ \mathrm{z''}_t\epsilon^{-\rho'_t}\eta^{-\lambda'_t}x^{\delta'_t} & \end{array}\right)$$

$$I_{U_1:p+1,q;W_1}^{V_1;0,n+1;X_1}\left(\begin{array}{c|c} \omega x^{\alpha_1} & \text{A };\mathfrak{A}_1,\text{K}_3;A_1 \\ \cdot & \\ \cdot & \\ \cdot & \\ \omega x^{\alpha_r} & \text{B}; \mathfrak{B}_1,\text{K}_4;B_1 \end{array}\right) \tag{3.27}$$

where $M' = \sum_{i=1}^{k} M'_i h_i$, under the same conditions and notations that (3.10)
under the same conditions that (3.26)

Proofs

To prove the theorem 1 ,we have

$$\text{L.H.S} = D_x^{\alpha}\Big\{x^{\mu}(x^{\upsilon}+\epsilon)^{\lambda}(x^{\upsilon}+\eta)^{\gamma}\,\phi_N(zx)\,S_{N'_1,\cdots,N'_k}^{M'_1,\cdots,M'_k}\left(\begin{array}{c} \text{y}_1 x^{\lambda_1}(x^{\upsilon}+\epsilon)^{\delta_1}(x^{\upsilon}+\eta)^{\epsilon_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{y}_k x^{\lambda_k}(x^{\upsilon}+\epsilon)^{\delta_k}(x^{\upsilon}+\eta)^{\epsilon_k} \end{array}\right)$$

$$I\left(\begin{array}{c} \text{z}_1 x^{\rho_1}(x^{\upsilon}+\epsilon)^{-\sigma_1}(x^{\upsilon}+\eta)^{-\mu_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{z}_s x^{\rho_s}(x^{\upsilon}+\epsilon)^{-\sigma_s}(x^{\upsilon}+\eta)^{-\mu_s} \end{array}\right) I\left(\begin{array}{c} \text{z'}_1 x^{\rho'_1}(x^{\upsilon}+\epsilon)^{-\sigma'_1}(x^{\upsilon}+\eta)^{-\mu'_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \text{z'}_u x^{\rho'_u}(x^{\upsilon}+\epsilon)^{-\sigma'_u}(x^{\upsilon}+\eta)^{-\mu'_u} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z''_j x^{\delta'_j}(x^{\upsilon}+\epsilon)^{-\epsilon'_j}(x^{\upsilon}+\eta)^{-\lambda'_j}\right]\Bigg\}$$

First, using the definitions of general class of polynomials $\phi_n(.)$, $S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[.]$ given in (1.19) and (1.20) respectively and expressing the I-functions of s variables and u-variables and generalized hypergeometric function in Mellin-Barnes contour integral with the help of equations (1.9), (1.12) and and (1.25) respectively, changing the order of summations and integrations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process), we have :

$$\text{L.H.S } = \frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h=0}^{[N/M]}\sum_{l=0}^{[dh]}\frac{(-dh)_l x^{N-Mh+l}c^{-\upsilon'-N+(M-1)h}(\upsilon')_{N+(1-M)h}(a)^{N-Mh}b^h(-)^{(d+1)h}2^l x^{N-Mh+1}}{l!h!(N-Mh)!}$$

$$\sum_{h_1=0}^{[N'_1/M'_1]}\cdots\sum_{h_k=0}^{[N'_k/M'_k]}A(h_1,\cdots,h_k)\prod_{j=1}^{k}\left[\frac{(-N'_j)_{M'_jh_j}}{h_j!}y_j^{h_j}\right]\frac{1}{(2\pi\omega)^{s+u+t}}\int_{L'_1}\cdots\int_{L'_s}\int_{L''_1}\cdots\int_{L''_u}\int_{L_1}\cdots\int_{L_t}$$

$$\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z_i'^{t_i}\,\psi(x_1,\cdots,x_u)\prod_{i=1}^{u}\xi_i(x_i)z_i''^{x_i}\;\frac{\prod_{j=1}^{P}\Gamma(A_j+\sum_{i=1}^{k}s_i)}{\prod_{j=1}^{Q}\Gamma(B_j+\sum_{i=1}^{k}s_i)}\prod_{j=1}^{t}\left[\Gamma(-s_j)(z_j')^{s_j}\right]$$

$$D_x^{\alpha}\Big\{x^{\mu+N-Mh+1-\sum_{j=1}^{k}\lambda_jh_j+\sum_{j=1}^{s}\rho_jt_j+\sum_{j=1}^{t}\delta_j's_j+\sum_{i=1}^{u}\rho_i'x_i}$$

$$(x^{\upsilon}+\epsilon)^{\lambda+\sum_{j=1}^{k}\delta_jh_j-\sum_{j=1}^{s}\sigma_jt_j-\sum_{j=1}^{u}\sigma_j'x_j-\sum_{j=1}^{t}\rho_i's_j}\,(x^{\upsilon}+\eta)^{\gamma+\sum_{j=1}^{k}\epsilon_jh_j-\sum_{j=1}^{s}\mu_jt_j-\sum_{j=1}^{u}\mu_j'x_j-\sum_{j=1}^{t}\lambda_i's_j}\Big\}$$

$$\mathrm{d}t_1\cdots\mathrm{d}t_s\mathrm{d}x_1\cdots\mathrm{d}x_u\,\mathrm{d}s_1\cdots\mathrm{d}s_t \tag{3.28}$$

Now use the binomial formula : $(x^{\upsilon}+a)^{\lambda}=a^{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(\frac{x^{\upsilon}}{a}\right)^{m};\left|\frac{x^{\upsilon}}{a}\right|<1$ (3.29)

twice and then the result (1.23) therein. On interpreting the contour integrals into I-function of $(s+u+t)$- variables, we obtain the desired result.
To prove the theorem 2, we use the similar method using the definition of general class of polynomials $T_n^{m_1,\cdots,m_k}[.]$.

To prove the theorem 3, we set $f(x),g(x)$ in equation (1.24) as the following

$$f(x)=\quad I^{V_1;0,n;X_1}_{U_1:p,q;W_1}\left(\begin{array}{c|c}\omega x^{\alpha_1} & \mathrm{A};\mathfrak{A}_1;\mathrm{A}_1\\ \cdot & \\ \cdot & \\ \cdot & \\ \omega x^{\alpha_r} & \mathrm{B};\mathfrak{B}_1;\mathrm{B}_1\end{array}\right)\text{ and}$$

$$g(x)=x^{\mu}(x^{\upsilon}+\epsilon)^{\lambda}(x^{\upsilon}+\eta)^{\gamma}\,\phi_N(zx)\,S^{M_1',\cdots,M_k'}_{N_1',\cdots,N_k'}\left(\begin{array}{c}\mathrm{y}_1x^{\lambda_1}(x^{\upsilon}+\epsilon)^{\delta_1}(x^{\upsilon}+\eta)^{\epsilon_1}\\ \cdots\\ \cdots\\ \mathrm{y}_kx^{\lambda_k}(x^{\upsilon}+\epsilon)^{\delta_k}(x^{\upsilon}+\eta)^{\epsilon_k}\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}_1x^{\rho_1}(x^{\upsilon}+\epsilon)^{-\sigma_1}(x^{\upsilon}+\eta)^{-\mu_1}\\ \cdots\\ \cdots\\ \mathrm{z}_sx^{\rho_s}(x^{\upsilon}+\epsilon)^{-\sigma_s}(x^{\upsilon}+\eta)^{-\mu_s}\end{array}\right)I\left(\begin{array}{c}\mathrm{z'}_1x^{\rho_1'}(x^{\upsilon}+\epsilon)^{-\sigma_1'}(x^{\upsilon}+\eta)^{-\mu_1'}\\ \cdots\\ \cdots\\ \mathrm{z'}_ux^{\rho_u'}(x^{\upsilon}+\epsilon)^{-\sigma_u'}(x^{\upsilon}+\eta)^{-\mu_u'}\end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t}z_j''x^{\delta_j'}(x^{\upsilon}+\epsilon)^{-\epsilon_j'}(x^{\upsilon}+\eta)^{-\lambda_j'}\right] \tag{3.30}$$

We have L.H.S $=\sum_{n'=-\infty}^{\infty}\epsilon\binom{\alpha}{en'+\epsilon}D_x^{\alpha-en'-\epsilon}\left[x^{\mu}I(w_1x^{\alpha_1},\cdots,w_sx^{\alpha_r}\right]D_x^{an+\epsilon}\Big[\Big\{x^{\mu}(x^{\upsilon}+\epsilon)^{\lambda}(x^{\upsilon}+\eta)^{\gamma}$

$$\phi_N(zx)S_{N_1',\cdots,N_k'}^{M_1',\cdots,M_k'}\begin{pmatrix} \mathrm{y}_1x^{\lambda_1}(x^\upsilon+\epsilon)^{\delta_1}(x^\upsilon+\eta)^{\epsilon_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_kx^{\lambda_k}(x^\upsilon+\epsilon)^{\delta_k}(x^\upsilon+\eta)^{\epsilon_k} \end{pmatrix} I\begin{pmatrix} \mathrm{z}_1x^{\rho_1}(x^\upsilon+\epsilon)^{-\sigma_1}(x^\upsilon+\eta)^{-\mu_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_sx^{\rho_s}(x^\upsilon+\epsilon)^{-\sigma_s}(x^\upsilon+\eta)^{-\mu_s} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z'}_1x^{\rho_1'}(x^\upsilon+\epsilon)^{-\sigma_1'}(x^\upsilon+\eta)^{-\mu_1'} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z'}_ux^{\rho_u'}(x^\upsilon+\epsilon)^{-\sigma_u'}(x^\upsilon+\eta)^{-\mu_u'} \end{pmatrix} {}_PF_Q\left[(A_P);(B_Q);-\sum_{j=1}^{t} z_j''x^{\delta_j'}(x^\upsilon+\epsilon)^{-\epsilon_j'}(x^\upsilon+\eta)^{-\lambda_j'}\right]$$

Now use the Lemme and the theorem 1, we obtain the theorem 3

To prove the theorem 4, we use the similar method using the definition of general class of polynomials $T_n^{m_1,\cdots,m_k}[.]$.

Remarks :

If $U=V=A=B=A_1=B_1=U_1=V_1=0$, the multivariable I-functions defined by Prasad [4] reduce to multivariable H-functions defined by Srivastava and al [9], we obtain the similar relations.

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by Prasad [4].

4.Conclusion

In this paper we have evaluated a generalized contour integral involving the multivariable I-functions defined by Prasad [4], two classes of polynomials and the generalized hypergeometric function. The four formulaes established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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