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Weyl's and Weyl type theorems for $m-$ quasi k^*- paranormal operators and algebraically $m-$ quasi k^*- paranormal operators

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Abstract

In search of operators satisfying Weyl's theorem, a new class of operators $m-$ quasi k^*- paranormal operators for positive integers m and k are defined and studied. Also spectral properties of algebraically $m-$ quasi k^*- paranormal operators are discussed.

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1 Introduction and Preliminaries

Let $B(H)$ be the Banach algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator T, we mean an element in $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. The ascent of T denoted by $p(T)$, is the least non-negative integer n such that $\ker T^n = \ker T^{n+1}$. The descent of T denoted by $q(T)$, is the least non-negative integer n such that $ran(T^n) = ran(T^{n+1})$. T is said to be of finite ascent if $p(T-\lambda) < \infty$, for all $\lambda \in C$. If p(T) and q(T) are both finite, then p(T)=q(T)([11], Proposition 38.3). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T.

An operator T is called a Fredholm operator if the range of T denoted by $ran(T)$ is closed and both $\ker T$ and $\ker T^*$ are finite dimensional and is denoted by $T \in \Phi(H)$. An operator T is called upper semi-Fredholm operator, $T \in \Phi_+(H)$, if $ran(T)$ is closed and $\ker T$ is finite dimensional. An operator T is called lower semi- Fredholm operator, $T \in \Phi_-(H)$, if $\ker T^*$ is finite dimensional. The index of a semi-Fredholm operator is an integer defined as $ind(T) = \dim \ker T - \dim \ker T^*$. An upper semi-Fredholm operator, with index less than or equal to 0 is called upper semi-Weyl operator and is denoted by $T \in \Phi_+^-(H)$. A lower semi-Fredholm operator with index greater than or equal to 0 is called lower semi-Weyl operator and is denoted by $T \in \Phi_-^+(H)$. A Fredholm operator of index 0 is called Weyl operator.

The spectrum of T is denoted by $\sigma(T)$, where

$$\sigma(T) = \{\lambda \in C : T - \lambda I \text{ is not invertible}\}.$$

The Weyl spectrum of T is defined as

$$w(T) = \{\lambda \in C : T - \lambda I \text{ is not Weyl}\}.$$

The set of all isolated eigenvalues of finite multiplicity of T is denoted by $\pi_{00}(T)$.

Weyl's theorem holds for T [8] if T satisfies the equality

$$\sigma(T) - w(T) = \pi_{00}(T).$$

The approximate point spectrum of T is denoted by $\sigma_a(T)$, where

$$\sigma_a(T) = \{\lambda \in C : T - \lambda I \text{ is not bounded below}\}.$$

The essential spectrum of T is defined as

$$\sigma_e(T) = \{\lambda \in C : T - \lambda I \text{ is not Fredholm}\}.$$

An upper semi-Fredholm operator with finite ascent is called upper semi-Browder operator and is denoted by $T \in B_+(H)$ while a lower semi-Fredholm operator with finite descent is called lower semi-Browder operator and is denoted by $T \in B_-(H)$. A Fredholm operator with finite ascent and descent is called Browder operator. Clearly, the class of all Browder operators is contained in the class of all Weyl operators. Similarly the class of all upper semi-Browder operators is contained in the class of all upper semi-Weyl operators and the class of all lower semi- Browder operators is contained in the class of all lower semi-Weyl operators.

The Browder spectrum of T is defined as

$$\sigma_b(T) = \{\lambda \in C : T - \lambda I \text{ is not Browder}\}.$$

The set of all isolated eigenvalues of finite multiplicity of T in $\sigma_a(T)$ is denoted by $\pi_{00}^a(T)$ and $p_{00}(T)$ is defined as $p_{00}(T) = \sigma(T) - \sigma_b(T)$.

The essential approximate point spectrum of T is defined as

$$\sigma_{ea}(T) = \{\lambda \in C : T - \lambda I \notin \Phi_+^-(H)\}.$$

We say that a-Weyl's theorem holds for T [18], if T satisfies the equality

$$\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}^a(T).$$

We say that T satisfies property(w) if $\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}(T)$
and T satisfies property(b) if $\sigma_a(T) - \sigma_{ea}(T) = p_{00}(T)$.

For an operator T and a non-negative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. In particular, $T_{[0]} = T$. If for some integer n, $R(T^n)$ is closed and $T_{[n]}$ is an upper(resp. a lower) semi-Fredholm operator, then T is called an upper(resp. lower) semi-B-Fredhom operator. Moreover if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. The index of a semi-B-Fredholm operator T is the index of semi-Fredholm operator $T_{[d]}$, where d is the degree of the stable iteration of T and defined as $d = inf\{n \in N;\ \text{for all } m \in N, m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}$. T is called a B-Weyl operator if it is B-Fredholm of index 0. An operator T is Drazin invertible, if it has finite ascent and descent. $E(T)$ denotes the isolated eigenvalues of T with no restriction on multiplicity.

The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in C : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

We say that T satisfies generalized Weyl's theorem[4] if

$$\sigma(T) - \sigma_{BW}(T) = E(T),$$

By [6], if Generalized Weyl's theorem holds for T, then Weyl's theorem holds for T.

An operator T is called normaloid if $r(T) = \|T\|$, where $r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}$. An operator T is called hereditarily normaloid, if every part of it is normaloid. An operator T is called polaroid if $iso\,\sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of poles of the resolvent of T and $iso\,\sigma(T)$ is the set of all isolated points of $\sigma(T)$. An operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator T is said to be reguloid if for every isolated point λ of $\sigma(T)$, $\lambda I - T$ is relatively regular. An operator T is known as relatively regular if and only if $\ker T$ and $T(X)$ are complemented. Also Polaroid $\Rightarrow$ reguloid $\Rightarrow$ isoloid.

An operator T is said to have the single valued extension property(SVEP) at $\lambda_0 \in C$, if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator T is said to have SVEP, if T has SVEP at every point $\lambda \in C$.

An operator T is said to have property(H) if $H_0(\lambda I - T) = \ker(\lambda I - T)$, where $H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}$.

An operator $T \in B(H)$ satisfies (Bishop's) property (β) if, for every open subset U of complex plane C and every sequence of analytic functions $f_n : U \to H$ with the property that $(T - \lambda)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U, then $f_n(\lambda) \to 0$ as $n \to \infty$ locally uniformly on U.

K.S.Ryoo and P.Y.Sik defined $k*-$paranormal operators in [19], k being a positive integer and showed that $*-$paranormal operators form a proper subclass of $k*-$paranormal operators for $k \geq 3$, and $k*-$paranormal operators are normaloid. In [7], Braha and Tanahasi have proved that $k*-$paranormal operators have Bishop's property(β). In [16], it is proved Weyl's theorem hold for $k*-$paranormal operators and other Weyl type theorems are discussed.

Definition 1.1. [19] An operator T is called $k*-$paranormal for a positive integer k, if for every unit vector x in H, $\|T^k x\| \geq \|T^* x\|^k$.

Theorem 1.1. *[7] $k*-$paranormal operators have Bishop's property(β)*

Theorem 1.2. *[16] An operator T is $k*-$paranormal for a positive integer k if and only if for any $\mu > 0$,*

$$T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0.$$

Theorem 1.3. *[16] If T is $k*-$paranormal operator for a positive integer k and for $\lambda \in C$, $\sigma(T) = \{\lambda\}$, then $T = \lambda$.*

Theorem 1.4. *[16] If T is $k*-$paranormal operator for some positive integer k, then T is polaroid.*

In this paper, we prove that if T is $m-$ quasi k^*- paranormal for positive integrs m and k, then T has property(H), Bishop's property(β,) T is not normaloid, Weyl's theorem hold for T and $f(T)$ for all positive integers m and k and $f \in H(\sigma(T))$ and if T^* has SVEP, then a-Weyl's theorem hold for T and $f(T)$ for all positive integers m and k and $f \in H(\sigma(T))$, where $H(\sigma(T))$ is the space of all analytic functions on an open neighborhood of spectrum of T. We further prove that if T is an algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then spectral mapping theorem and spectral mapping theorem for essential approximate point spectrum hold for T, T is polaroid, Generalised Weyl's theorem holds for T and other Weyl type theorems are discussed.

2 $m-$quasi k^*- paranormal operators

In this section, $m-$quasi k^*- paranormal operators are defined and proved that the restriction of $m-$ quasi k^*- paranormal operators to an invariant subspace is k^*-paranormal, $m-$ quasi k^*- paranormal operators have property(H), Bishop's property(β) and if T is $m-$ quasi k^*- paranormal then Weyl's theorem hold for T, T^* and $f(T)$ for $f \in H(\sigma(T))$ and if T^* has SVEP, then a-Weyl's theorem hold for T, T^* and $f(T)$ for $f \in H(\sigma(T))$.

Definition 2.1. An operator T is called $m-$ quasi k^*- paranormal for positive integers m and k if for all $x \in H$,

$$\left\|T^{m+k}x\right\| \left\|T^m x\right\|^{k-1} \geq \left\|T^*T^m x\right\|^k$$

Example 1. *Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H, and let T be a weighted shift defined as $Te_0 = \frac{1}{2}e_1$, $Te_1 = \frac{1}{3}e_2$ and $Te_i = e_{i+1}$ for $i \geq 2$. Then T is $2-$quasi $*-$paranormal, but T is not quasi $*-$paranormal.*

Theorem 2.1. *T is $m-$quasi k^*-paranormal for positive integers m and k if and only if for any $\mu > 0$,*

$$T^{*m+k}T^{m+k} - k\mu^{k-1}T^{*m}TT^*T^m + (k-1)\mu^k T^{*m}T^m \geq 0.$$

Proof. Using Generalised arithmetic and geometric mean inequality, for $\mu > 0$ and $x \in H$,

$$\begin{aligned} \frac{1}{k}\left\langle \mu^{-k+1}\left|T^{m+k}\right|^2 x, \quad x\right\rangle \quad + \quad & \frac{k-1}{k}\left\langle \mu\left|T^m\right|^2 x, \quad x\right\rangle \\ \geq \quad & \left\langle \mu^{-k+1}\left|T^{m+k}\right|^2 x, \quad x\right\rangle^{\frac{1}{k}} \left\langle \mu\left|T^m\right|^2 x, \quad x\right\rangle^{\frac{k-1}{k}} \\ = \quad & \left\|T^{m+k}x\right\|^{\frac{2}{k}} \left\|T^m x\right\|^{\frac{2(k-1)}{k}} \\ \geq \quad & \left\|T^*T^m x\right\|^2 \\ = \quad & \left\langle \left|T^*T^m\right|^2 x, \quad x\right\rangle \end{aligned}$$

Hence $T^{*m+k}T^{m+k} - k\mu^{k-1}T^{*m}TT^*T^m + (k-1)\mu^k T^{*m}T^m \geq 0.$ ... (1)

To prove the sufficient part, if $\|T^m x\| = 0$, then the inequality(1) is trivial. On the other hand, if $x \in H$ with $\|T^m x\| > 0$, taking $\mu = \left(\frac{\|T^{m+k}x\|}{\|T^m x\|}\right)^{\frac{2}{k}}$ in (1), we get $\left\|T^{m+k}x\right\| \left\|T^m x\right\|^{k-1} \geq \left\|T^*T^m x\right\|^k$. Hence T is $m-$ quasi k^*- paranormal. $\square$

From Example 1 and theorem 1.2, it follows that for any fixed k

quasi k^*-paranormal $\subset$ 2$-$quasi k^*-paranormal $\subset$ 3$-$quasi k^*-paranormal $\subset \cdots$

From theorems 1.2 and 2.1, we get that the class of k^*-paranormal operators form a subclass of the class of $m-$ quasi k^*- paranormal operators for all positive integers m and k. The following example shows that the converse is not true and that $m-$ quasi k^*- paranormal operators need not be normaloids.

Example 2. *Let* $H = C^2$ *and* $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *Then for any* k, $\|Tx\| \geq \|T^*x\|$ *is not satisfied for* $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. *Hence* T *is not* $*-$*paranormal,* T *is not normaloid butT is* $m-$ *quasi* k^*- *paranormal for all positive integers* m *and* k.

Theorem 2.2. *If* $T \in B(H)$ *is* $m-$ *quasi* k^*- *paranormal operator for positive integers* m *and* k, T *does not have dense range and* T *has the following representation:* $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *on* $H = \overline{ran(T^m)} \oplus \ker(T^{m*})$, *then* T_1 *is* k^*-*paranormal operator on* $\overline{ran(T^m)}$*and* $T_3^m = 0$. *Furthermore,* $\sigma(T) = \sigma(T_1) \bigcup \{0\}$, *where* $\sigma(T)$ *denotes the spectrum of* T.

Proof. Let P be the orthogonal projection onto $\overline{ran(T^m)}$. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$. Since T is $m-$ quasi k^*- paranormal operator, by theorem 2.1

$$P(T^{*m+k}T^{m+k} - k\mu^{k-1}T^{*m}TT^*T^m + (k-1)\mu^k T^{*m}T^m)P \geq 0.$$

That is, $P(T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k)P \geq 0$.
Hence $T_1^{*k}T_1^k - k\mu^{k-1}T_1T_1^* + (k-1)\mu^k I \geq k\mu^{k-1}T_2T_2^* \geq 0$.
Hence T_1 is k^*-paranormal operator on $\overline{ran(T^m)}$. Also for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\langle T_3^m x_2, x_2 \rangle = \langle T^m(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^{*m}(I-P)x \rangle = 0$. Hence $T_3^m = 0$. By ([10],corollary 7), $\sigma(T_1) \bigcup \sigma(T_3) = \sigma(T) \bigcup \tau$, where τ is the union of certain of the holes in $\sigma(T)$ which happen to be a subset of $\sigma(T_1) \bigcap \sigma(T_3)$ and $\sigma(T_1) \bigcap \sigma(T_3)$ has no interior points. Therefore $\sigma(T) = \sigma(T_1) \bigcup \sigma(T_3) = \sigma(T_1) \bigcup \{0\}$ □

Theorem 2.3. *If* T *is* $m-$ *quasi* k^*- *paranormal operator for positive integers* m *and* k, *and* M *is an invariant subspace of* T, *then the restriction* $T_{|M}$ *is* k^*-*paranormal.*

Proof. Let P be the orthogonal projection onto M and $T_1 = T_{|M}$. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$. By theorem 2.1,

$$P(T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k)P \geq 0.$$

$$\Longrightarrow T_1^{*k}T_1^k - k\mu^{k-1}T_1T_1^* + (k-1)\mu^k I \geq k\mu^{k-1}T_2T_2^* \geq 0.$$

Hence T_1 i.e $T_{|M}$ is k^*-paranormal operator on M. □

Theorem 2.4. *If* T *is* $m-$*quasi* k^**-paranormal operator for positive integers* m *and* k, *and for* $\lambda \in C$, $\sigma(T) = \{\lambda\}$ *then* $T = \lambda$ *if* $\lambda \neq 0$ *and* $T - \lambda I$ *is nilpotent, if* $\lambda = 0$.

Proof. If $\lambda = 0$, then by theorem 2.2, $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^m)} \oplus \ker(T^{m*})$, where T_1 is k^*-paranormal operator on $\overline{ran(T^m)}$and $T_3^m = 0$. Also $\sigma(T_1) = 0$. Since k^*-paranormal operators are normaloid [19], $T_1 = 0$. Hence $T^m = 0$. Assume that $\lambda \neq 0$. Then $T_1 = \frac{1}{\lambda}T$ is an invertible normaloid operator with $\sigma(T_1) = \{1\}$.

Hence T_1 is similar to an invertible isometry B(on an equivalent normed linear space) with $\sigma(B) = 1$(by theorem.2, [12]). T_1 and B being similar, 1 is an eigenvalue of $T_1 = \frac{1}{\lambda}T$(by theorem.5, [12]). Therefore by theorem 1.5.14 of [14], $T_1 = I$. Hence $T = \lambda$. □

If $\lambda \in iso\,\sigma(T)$, the spectral projection (or Riesz idempotent) E_λ of T with respect to λ is defined by $E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}\,dz$, where D is a closed disk with centre at λ and radius small enough such that $D\bigcap\sigma(T) = \{\lambda\}$. Then $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma(T_{|E_\lambda H}) = \{\lambda\}$ and $\ker(T-\lambda I) \subset E_\lambda H$.

Theorem 2.5. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, and $\lambda \in \sigma(T)$ is an isolated point, then the Riesz idempotent operator E_λ with respect to λ satisfies $E_\lambda H = \ker(T-\lambda I)$. Hence λ is an eigenvalue of T.*

Proof. It is sufficient to show that $E_\lambda H \subseteq \ker(T-\lambda)$. Now $\sigma(T_{|E_\lambda H}) = \{\lambda\}$ and $T_{|E_\lambda H}$ is k^*-paranormal. Hence by theorem 2.4 $T_{|E_\lambda H} = \lambda$. Hence $E_\lambda H = \ker(T-\lambda I)$. □

Theorem 2.6. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, then T has property(H).*

Proof. By [13], $E_\lambda H = H_0(\lambda I - T)$. Hence by theorem 2.5, $m-$ quasi k^*- paranormal operators have (H) property. □

Hence by theorems 2.5, 2.6 and 2.8 of [2], we get the following results.

Theorem 2.7. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, then T has SVEP, $p(\lambda I - T) \leq 1$ for all $\lambda \in C$, Furthermore both T and T^* are reguloid.*

Corollary 2.8. *If T is m-quasi k-paranormal operator for positive integers m and k, then both T and T^* are isoloid.*

Theorem 2.9. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, then Weyl's theorem hold for T and T^*. If in addition, T^* has SVEP, then a-Weyl's theorem holds for both T and T^*.*

Theorem 2.10. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k and T^* has SVEP, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Theorem 2.11. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, then $w(f(T)) = f(w(T))$.*

Proof. By theorem 2.2, if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^m)} \oplus kerT^{m*}$, then T_1 is k^*-paranormal operator on $\overline{ran(T^m)}$. By theorem 4.2 of [16], $w(f(T_1)) = f(w(T_1))$. Hence by theorem 2.2, $w(f(T)) = w(f(T_1)) \cup w(f(T_3)) = f(w(T_1)) \cup f(w(T_3)) = f(w(T_1) \cup w(T_3)) = f(w(T))$ □

Using the Lemma of [15] and theorems 2.9 and 2.11, we getthe following result.

Theorem 2.12. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Theorem 2.13. *If T is $m-$ quasi k^*- paranormal operator for positive integers m and k, then T satisfies property (β).*

Proof. Let $\{f_n(\lambda)\}$ be a sequence of analytic functions $f_n : U \to H$ on an open subset U of C such that

$$(T - \lambda I)f_n(\lambda) \to 0 \text{ as } n \to \infty$$

uniformly on all compact subsets of U. Then taking $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ as in theorem 2.2, we get

$$\begin{pmatrix} (T_1 - \lambda I)f_{n_1}(\lambda) + T_2 f_{n_2}(\lambda) \\ (T_3 - \lambda I)f_{n_2}(\lambda) \end{pmatrix} \to 0$$

Hence $(T_1 - \lambda I)f_{n_1}(\lambda) + T_2 f_{n_2}(\lambda) \to 0$ and $(T_3 - \lambda I)f_{n_2}(\lambda) \to 0$ as $n \to \infty$. Since T_3 is nilpotent, the second of the above equations implies $f_{n_2}(\lambda) \to 0$ locally uniformly on U. Substituting in the first equation we get $(T_1 - \lambda I)f_{n_1}(\lambda) \to 0$. Then by theorem 1.1, $f_{n_1}(\lambda) \to 0$ locally uniformly on U. Hence $f_n(\lambda) \to 0$ locally uniformly on U. Hence $m-$ quasi k^*- paranormal operators satisfy property(β). □

3 Algebraically $m-$ quasi k^*- paranormal operators

In this section, we prove spectral mapping theorem and essential approximate point spectral theorem for algebraically $m-$quasi k^*-paranormal operators and show that they are polaroids. we also prove Generalised Weyl's theorem for algebraically $m-$quasi k^*-paranormal operators and discuss other Weyl type theorems.

Definition 3.1. An operator T is defined to be of algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, if there exists a non-constant complex polynomial $p(t)$ such that $p(T)$ is of $m-$ quasi k^*- paranormal operator for positive integers m and k.

If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then there exists a non-contant polynomial $p(t)$ such that $p(T)$ is $m-$ quasi k^*- paranormal operator for positive integers m and k. By the theorem 2.7, $p(T)$ is of finite ascent. Hence $p(T)$ has SVEP and hence T has SVEP([14], Theorem 3.3.6).

Theorem 3.1. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k and $\sigma(T) = \{\mu_0\}$, then $T - \mu_0 I$ is nilpotent.*

Proof. Since T is algebraically $m-$ quasi k^*- paranormal operator there is a non-constant polynomial $p(t)$ such that $p(T)$ is $m-$ quasi k^*- paranormal operator for positive integers m and k, then applying theorem 2.4,

$$\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\} \text{ implies } p(T) - p(\mu_0)I \text{ is nilpotent.}$$

Let $p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1} \cdots (z - \mu_t)^{k_t}$ where $\mu_j \neq \mu_s$ for $j \neq s$. Then for some positive integer l,

$$0 = (p(T) - p(\mu_0)I)^l = a(T - \mu_0 I)^{lk_0}(T - \mu_1 I)^{lk_1} \cdots (T - \mu_t I)^{lk_t}.$$

Since $T - \mu_1 I$, $T - \mu_2 I$, $\cdots$ $T - \mu_t I$ are invertible, $(T - \mu_0 I)^{lk_0} = 0$. Hence $T - \mu_0 I$ is nilpotent. □

Theorem 3.2. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then $w(f(T)) = f(w(T))$ for every $f \in Hol(\sigma(T))$.*

Proof. Suppose that T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then T has SVEP. Hence by ([11],Proposition 38.5) $ind(T-\lambda I) \leq 0$ for all complex numbers λ. Now to prove the result it is sufficient to show that $f(w(T)) \subseteq w(f(T))$. Let $\lambda \in f(w(T))$. Suppose if $\lambda \notin w(f(T))$, then $f(T) - \lambda I$ is Weyl and hence $ind(f(T) - \lambda I) = 0$. Let $f(z) - \lambda = (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_n)g(z)$. Then $f(T) - \lambda I = (T - \lambda_1 I)(T - \lambda_2 I) \ldots (T - \lambda_n I)g(T)$ where $g(T)$ is invertible and $ind(f(T)-\lambda I) = 0 = ind(T-\lambda_1 I)+ind(T-\lambda_2 I)+\cdots+ind(T-\lambda_n I)+ind\, g(T)$. Since each of $ind(T-\lambda_i I) \leq 0$, we get that $ind(T-\lambda_i I) = 0$, for all $i = 1, 2, \ldots n$. Therefore $T - \lambda_i I$ is weyl for each $i = 1, 2, \ldots n$. Hence $\lambda_i \notin w(T)$ and hence $\lambda \notin f(w(T))$, which is a contradiction. Hence the theorem. □

Theorem 3.3. *If T or T^* is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.*

Proof. For $T \in B(H)$, by [17] the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in H(\sigma(T))$ with no restrictions on T. Therefore, it is enough to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose if $\lambda \notin \sigma_{ea}(f(T))$ then $f(T) - \lambda I \in \Phi_+^-(H)$, that is $f(T) - \lambda I$ is upper semi-Fredholm operator with index less than or equal to zero. Also $f(T) - \lambda I = c(T - \alpha_1 I)(T - \alpha_2 I) \ldots (T - \alpha_n I)g(T)$ where $g(T)$ is invertible and $c, \alpha_1\, \alpha_2, \ldots \alpha_n \in C$.

If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is $m-$ quasi k^*- paranormal operator. Then $p(T)$ has SVEP and hence T has SVEP. Therefore $ind(T - \alpha_i I) \leq 0$ and hence $T - \alpha_i I \in \Phi_+^-(H)$ for each $i = 1, 2, \ldots, n$. Therefore $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.
If T^* is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then there exists a non-constant polynomial $p(t)$ such that $p(T^*)$ is $m-$ quasi k^*- paranormal operator. Then $p(T^*)$ has SVEP and hence T^* has SVEP. Therefore $ind(T-\alpha_i I) \geq 0$ for each $i = 1, 2, \ldots, n$. Therefore $0 \leq \Sigma_{i=1}^n ind(T-\alpha_i I) = ind(f(T)-\lambda I) \leq 0$. Therefore $ind(T - \alpha_i I) = 0$ for each $i = 1, 2, \ldots, n$. Therefore $T - \alpha_i I$ is Weyl for each $i = 1, 2, \ldots, n$. $(T - \alpha_i I) \in \Phi_+^-(H)$ and hence $\alpha_i \notin \sigma_{ea}(T)$. Therefore $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. □

Theorem 3.4. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then T is polaroid.*

Proof. If $\lambda \in iso\, \sigma(T)$ using the spectral projection of T with respect to λ, we can write $T = T_1 \oplus T_2$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Since T_1 is algebraically $m-$ quasi k^*- paranormal operator and $\sigma(T_1) = \{\lambda\}$, by theorem 3.1, $T_1 - \lambda I$ is nilpotent. Since $\lambda \notin \sigma(T_2)$, $T_2 - \lambda I$ is invertible. Hence both $T_1 - \lambda I$ and $T_2 - \lambda I$ and hence $T - \lambda I$ have finite ascent and descent. Hence λ is a pole of the resolvent of T. Hence T is polaroid. □

Corollary 3.5. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then T is reguloid.*

Corollary 3.6. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then T is isoloid.*

Theorem 3.7. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then generalized Weyl's theorem holds for T.*

Proof. Assume that $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Then $T - \lambda I$ is B-Weyl and not invertible. Claim: $\lambda \in \partial\sigma(T)$ Assume the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim\ker(T-\mu I)>0$ for all μ in U. Hence by ([[9],theorem.10) T does not have SVEP which is a contradiction. Hence $\lambda\in\partial\sigma(T)-\sigma_{BW}(T)$. Therefore by punctured neighborhood theorem, $\lambda\in E(T)$.

Conversely suppose that $\lambda\in E(T)$. Then λ is isolated in $\sigma(T)$. Using the Riesz idempotent E_λ with respect to λ, we can represent T as the direct sum $T=\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1)=\{\lambda\}$ and $\sigma(T_2)=\sigma(T)-\{\lambda\}$. Then by theorem 3.1, $T_1-\lambda I$ is nilpotent. Since $\lambda\notin\sigma(T_2)$, $T_2-\lambda I$ is invertible. Hence both $T_1-\lambda I$ and $T_2-\lambda I$ have both finite ascent and descent. Hence $T-\lambda I$ has both finite ascent and descent. Hence $T-\lambda I$ is Drazin invertible. Therefore, by ([5], Lemma 4.1) $T-\lambda$ is B-Fredholm of index 0. Hence $\lambda\in\sigma(T)-\sigma_{BW}(T)$. Therefore $\sigma(T)-\sigma_{BW}(T)=E(T)$. □

Corollary 3.8. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then Weyl's theorem holds for T.*

By ([3], Theorem 2.16) we get the following result.

Corollary 3.9. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, and T^* has SVEP then a-Weyl's theorem and property(w) hold for T.*

By the Lemma of [15], theorem 3.8 and theorem 3.2, we get the following result.

Theorem 3.10. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then Weyl's theorem holds for $f(T)$, for every $f\in Hol(\sigma(T))$.*

If T^* has SVEP, then by ([1], Lemma 2.15) $\sigma_{ea}(T)=w(T)$ and $\sigma(T)=\sigma_a(T)$. Hence we get the following results.

Corollary 3.11. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, and if in addition T^* has SVEP, then a-Weyl's theorem holds for $f(T)$, for every $f\in H(\sigma(T))$.*

Corollary 3.12. *If T^* is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, then $w\left(f(T)\right)=f\left(w(T)\right)$.*

By ([1], Theorem 2.17) , we get the following results.

Corollary 3.13. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, and T^* has SVEP then property (b) hold for T.*

Corollary 3.14. *If T is algebraically $m-$ quasi k^*- paranormal operator for positive integers m and k, Weyl's theorem, a-Weyl's theorem, property(w) and property(b) hold for T^*.*

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