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Hypercyclicity of the Direct Sum of Tuples

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Abstract. In this paper we give conditions under which a set of direct sum of operators to be hypercyclic.

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1. Introduction

Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbf{Z}_+, \quad i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d$ we will refer to the set

$$\mathcal{T}_d = \{S \oplus S : S \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d$ is hypercyclic provided there exist $x_1, x_2 \in X$ such that

$$\{W(x_1 \oplus x_2) : W \in \mathcal{T}_d\}$$

is dense in $X \oplus X$. By a polynomial p we will mean

$$p(z_1, z_2, ..., z_n) = \sum_{j=1}^{n} \sum_{i_j=1}^{m_j} c_{i_1 \ldots i_n} z^{i_1} z^{i_2} ... z^{i_n}.$$

We will denote the generalized kernel of the pair $\mathcal{T} = (T_1, T_2, ..., T_n)$ by $GK(\mathcal{T})$ that is defined as:

$$GK(\mathcal{T}) = \bigcup \{ker(p(T_1, T_2, ..., T_n)) : p \; is \; a \; poly.\}.$$

Hypercyclic operators arise within the class of composition operators ([4]), weighted shifts ([12]), adjoints of multiplication operators ([5]), and adjoints of subnormal and hyponormal operators ([2]), and hereditarily operators ([1]), and topologically mixing operators ([7, 11]). Here, we want to extend some properties of hypercyclic operators to a tuple of commuting operators. For some other topics we refer to [1–18].

2. Main Results

In this section we characterize conditions for the direct sum of a pair of operators to be hypercyclic. We will denote the collection of hypercyclic vectors of a pair $\mathcal{T}$ by $HC(\mathcal{T})$.

Theorem 2.1. Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators T_1, $T_2, ..., t_n$, and T_i^* has no eigenvalues for $i = 1, 2, ..., n$. If $GK(\mathcal{T})$ is dense in X, then for every nonempty open subsets U, V of X and every neighborhood W of 0, there exists a polynomial q such that

$$q(T_1, T_2, ..., T_n)(U) \cap W \neq \emptyset$$

and

$$q(T_1, T_2, ..., T_n)(W) \cap V \neq \emptyset.$$

Proof. Let U, V be any nonempty open subsets of X and W be a neighborhood of 0. Since $GK(\mathcal{T})$ is dense in X, thus

$$GK(\mathcal{T}) \cap U \neq \emptyset$$

and so there exists a vector $u \in U$ and a nonzero polynomial q such that $q(T_1, T_2, ..., T_n)u = 0$. Thus

$$0 \in rq(T_1, T_2, ..., T_n)(U) \cap W$$

for every $r > 0$, and so

$$rq(T_1, T_2, ..., T_n)(U) \cap W \neq \emptyset$$

for all $r > 0$. Since T_i^* has no any eigenvalue for $i = 1, 2, ..., n$, $ran(q(T_1, T_2, ..., T_n))$ is dense in X. But

$$ran(q(T_1, T_2, ..., T_n)) = \bigcup_{r>0} q(T_1, T_2, ..., T_n)(rW),$$

thus there exists $r > 0$ such that

$$rq(T_1, T_2, ..., T_n)(W) \cap V \neq \emptyset.$$

Also, as we saw earlier that

$$rqq(T_1, T_2, ..., T_n)(U) \cap W \neq \emptyset,$$

so the proof is complete.

Corollary 2.2. Under the conditions of Theorem 2.1, if $HC(\mathcal{T}) \neq \emptyset$, then $\mathcal{T}_d$ is hypercyclic.

Proof. By Theorem 2.1, for any nonempty open subsets U, V of X and every neighborhood W of 0, there exists a polynomial q such that

$$q(T_1, T_2, ..., T_n)(U) \cap W \neq \emptyset$$

and

$$q(T_1, T_2, ..., T_n)(W) \cap V \neq \emptyset.$$

Since $HC(\mathcal{T}$ is dense in X, there exists $x \in U$ such that $q(T_1, T_2, ..., T_n)x \in W$. Also, since $Orb(\mathcal{T}, x)$ is dense, thus there exist integers $m_1, ..., m_n$ such that

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n} x \in q(T_1, T_2, ..., T_n)^{-1} V \cap W.$$

So

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}(U)\cap W \neq \emptyset.$$

But

$$q(T_1,T_2,...,T_n)T_1^{m_1}T_2^{m_2}...T_n^{m_n}x \in V$$

and

$$q(T_1,T_2,...,T_n)x \in W,$$

thus

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}(W)\cap V \neq \emptyset.$$

Hence $\mathcal{T}_d$ is hypercyclic and the proof is complete.

Theorem 2.3. Let X be a separable infinite dimensional Banach space and $\mathcal{T}=(T_1,T_2,...,T_n)$ be a tuple of operators T_1, $T_2,...,T_n$. Also, let $HC(\mathcal{T})\neq\emptyset$ and

$$D=\{x : Orb(\mathcal{T},x) \textit{ is bounded}\}$$

be dense in X. Then for every nonempty open subsets U, V of X and every neighborhood W of 0, there exists a polynomial q such that

$$q(T_1,T_2,...,T_n)(U)\cap W \neq \emptyset$$

and

$$q(T_1,T_2,...,T_n)(W)\cap V \neq \emptyset.$$

Proof. Let U, V be a pair of nonempty open subsets of X and W be a neighborhood of 0. Since D is dense in X, thus $D\cap U\neq\emptyset$ and so there exists a vector $x\in U$ such that $Orb(\mathcal{T},x)$ is bounded. Hence for some $c>0$, $Orb(\mathcal{T},x)\in cW$ and so for all $m,n\in\mathbb{N}$ we have

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}(U)\cap cW \neq \emptyset.$$

On the other hand, since $HC(\mathcal{T})\neq\emptyset$, thus $\mathcal{T}$ is hypercyclic and so there exist $m_1,...,m_n\in\mathbb{N}$ such that

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}(c^{-1}W)\cap V \neq \emptyset.$$

Define the polynomial q by

$$q(z_1,z_2,...,z_n)=c^{-1}{z_1}^{m_1}{z_2}^{m_2}...{z_n}^{m_n},$$

then

$$q(T_1,T_2,...,T_n)(U)\cap W \neq \emptyset$$

and

$$q(T_1,T_2)(W)\cap V \neq \emptyset.$$

Thus the proof is complete.

Corollary 2.4. Under the conditions of Theorem 2.3, if T_i^* has no eigenvalues for $i=1,...,n$, then $\mathcal{T}_d$ is hypercyclic.

Theorem 2.5. Let X be a separable infinite dimensional Banach space and $\mathcal{T}=(T_1,T_2,...,T_n)$ be a hypercyclic tuple of operators T_1, $T_2,...,T_n$. Also, let

$$E=\{x\in X : \exists \textit{ bounded sequence } \{x_{m_1...m_n}\}_{m_i,i=1,...,n} \textit{ such}$$

$$that\ T_1^{m_1}T_2^{m_2}...T_n^{m_n}x_{m_1...m_n} = x;\ \forall m_i \geq 0\ ,\ i = 1, ..., n\}$$

be dense in X. Then for every nonempty open subsets U, V of X and every neighborhood W of 0, there exists a polynomial q such that

$$q(T_1, T_2, ..., T_n)(U) \cap W \neq \emptyset$$

and

$$q(T_1, T_2, ..., T_n)(W) \cap V \neq \emptyset.$$

Proof. Let U, V be a any nonempty open subsets of X and W be a neighborhood of 0. Since E is dense in X, thus $E \cap V \neq \emptyset$ and so there exists a vector $x \in V$ and a bounded sequence $\{x_{m_1...m_n}\}$ such that $T_1^{m_1}T_2^{m_2}...T_n^{m_n}x_{m_1...m_n} = x$ for all $m, n \geq 0$. Since $\{x_{mn}\}$ is bounded, there exists $c > 0$ such that $x_{mn} \in cW$ for all $m_i \geq 0$ for $i = 1, ..., n$. But, $\mathcal{T}$ is hypercyclic and so there exists $m_0(i) \in \mathbb{N}$ for $i = 1, ..., n$ such that

$$T_1^{m_0(1)}T_2^{m_0(2)}...T_n^{m_0(n)}(cU) \cap W \neq \emptyset.$$

Define the polynomial q by

$$q(z_1, ..., z_n) = cz_1{}^{m_0(1)}z_2{}^{m_0(2)}...z_n{}^{m_0(n)}.$$

Then

$$q(T_1, T_2, ..., T_n)(W) \cap V \neq \emptyset$$

and

$$q(T_1, T_2, ..., T_n)(U) \cap W \neq \emptyset.$$

This completes the proof.

Corollary 2.6. Under the conditions of Theorem 2.5, if T_i^* has no any eigenvalue for $i = 1, ..., n$, then $\mathcal{T}_d$ is hypercyclic.

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