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Syndetic Sequences and Hypercyclic Tuples

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Abstract. In this paper we give necessary and sufficient conditions under which a tuple of operators satisfying the conditions of Hypercyclicity Criterion.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbb{Z}_+, \quad i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic.

Definition 1.2. An strictly increasing sequence of positive integers $\{n_k\}$ is said to be syndetic if $\sup_n\{n_{k+1} - n_k\} < \infty$.

A nice criterion namely the Hypercyclicity Criterion has used to show that hypercyclic operators arise within the class of composition operators ([2]), weighted shifts ([8]), adjoints of multiplication operators ([3]), and hereditarily operators ([1]), and topologically mixing operators ([4]). The formulation of the Hypercyclicty Criterion in the next section was given by N. S. Feldman ([5]). Here, we want to extend some properties of Hypercyclicity Criterion from a single operator to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. The paper [7] has an important rule to prove our main result. For some other topics we refer to [1–14].

2. Main Results

In this section we characterize the equivalent conditions for a pair of operators, satisfying the Hypercyclicity Criterion.

Theorem 2.1. (The Hypercyclicity Criterion for a tuple) Suppose that X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n acting on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that :
1. $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$
2. There exist a sequence of functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} S_j z \to \; z$
then $\mathcal{T}$ is a hypercyclic tuple.

The proof of the following lemma, follows immediately by a method of the proof of Theorem 1.2 in [6].

Lemma 2.2. Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. Then the followings are equivalent:
(i) $\mathcal{T}$ is hypercyclic.
(ii) for all nonempty open subsets U, V in X, there exists a tuple of sequences $(\{m_{k(1)}\}_k, ..., \{m_{k(n)}\}_k)$ of integers such that

$$T_1^{m_{k(1)}} {T_2}^{m_{k(2)}} ... {T_n}^{m_{k(n)}} (U) \cap V \neq \emptyset$$

for all $k \geq 0$.

Following theorem states the Hypercyclicity Criterion in terms of open sets.

Theorem 2.3. Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. Then the followings conditions are equivalent:

(i) $\mathcal{T}$ satisfies the Hypercyclicity Criterion.

(ii) For each pair U and V of non-void open subsets of X, and each neighborhood W of zero,

$$T_1^{k(1)} {T_2}^{k(2)} ... {T_n}^{k(n)} U \cap W \neq \emptyset$$

and

$$T_1^{k(1)} {T_2}^{k(2)} ... {T_n}^{k(n)} W \cap V \neq \emptyset$$

for some positive integers $k(i)$ for $i = 1, ..., n$.
Proof. See [13, Theorem 2.2].

Note that if $T_1, T_2, ..., T_n$ are commutative bounded linear operators on a Banach space X, and $\{m_{k_{j(i)}}\}_j$ is a sequence of natural numbers for $i = 1, ..., n$, then we say

$$\{{T_1}^{m_{k_{j(1)}}} {T_2}^{m_{k_{j(2)}}} ... {T_n}^{m_{k_{j(n)}}} : j \geq 0\}$$

is hypercyclic if there exists $x \in X$ such that

$$T_1{}^{m_{k_{j(1)}}} {T_2}^{m_{k_{j(2)}}} ... {T_n}^{m_{k_{j(n)}}} x : j \geq 0\}$$

is dense in X.

Theorem 2.4. Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. Then the followings conditions are equivalent:

(i) $\mathcal{T}$ satisfies the Hypercyclicity Criterion.

(ii)For any increasing syndetic sequences $(\{p_{k(i)}\}_k$, the sequence

$$\{T_1{}^{p_{k(1)}} T_2{}^{p_{k(2)}} ... T_n{}^{p_{k(n)}} : k \geq 0\}$$

is hypercyclic.

Proof. (i) $\rightarrow$ (ii): Let $\mathcal{T}$ satisfies the Hypercyclicity Criterion, thus there exist a tuple of strictly increasing sequences $(\{m_{k(1)}\}_k, \{m_{k(2)}\}_k, ..., \{m_{k(n)}\}_k)$, dense subsets Y_1, Z_1 in X, and a sequence of function $\{S_k : Z \rightarrow X\}$ such that:

(1) $T_1^{m_{k(1)}} T_2^{m_{k(2)}} ... T_n^{m_{k(n)}} \rightarrow 0$ on Y_1,

(2) $S_k z \rightarrow 0$, and

$$T_1^{m_{k(1)}} T_2^{m_{k(2)}} ... T_n^{m_{k(n)}} S_k z \rightarrow \; z$$

for every $z \in Z_1$.

Let $(\{p_{k(1)}\}_k, \{p_{k(2)}\}_k, ..., \{p_{k(n)}\}_k)$ be a tuple of increasing syndetic sequences. Then, clearly there are some integers $M_1, ..., M_n \geq 0$ and subsequences $\{k_{v(i)}\}_v$, $i = 1, ..., n$ such that

$$p_{k_{v(i)}(i)} = m_{k_{v_i}(i)} + M_i$$

for all $v \geq 0$. Set $Y = Y_1$ and

$$Z = T_1^{M_1} T_2{}^{M_2} ... T_n{}^{M_n} Z_1,$$

and note that Y and Z are dense subsets of X. Now, by condition (1) of the Hypercyclicity Criterion, for all $y \in Y$ we have

$$T_1^{p_{k_{v(1)}(1)}} ... T_n^{p_{k_{v(n)}(n)}} y = T_1^{M_1} ... T_n{}^{M_n} (T_1^{m_{k_{v(1)}(1)}} ... T_n^{m_{k_{v(n)}(n)}} y) \rightarrow 0.$$

Also, if $z_1 \in Z_1$ and

$$z = T_1^{M_1} T_2{}^{M_2} ... T_n{}^{M_n} z_1,$$

then by condition (2) of the Hypercyclicity Criterion we get that

$$T_1^{p_{k_{v(1)}(1)}} T_2^{p_{k_{v(2)}(2)}} ... T_n^{p_{k_{v(n)}(n)}} S_v z$$

is equal to

$$T_1^{M_1} T_2{}^{M_2} ... T_n{}^{M_n} (T_1^{p_{k_{v(1)}(1)}} T_2^{p_{k_{v(2)}(2)}} ... T_n^{p_{k_{v(n)}(n)}} S_v z_1)$$

which tends to z and

$$S_v z = T_1^{M_1} T_2{}^{M_2} ... T_n{}^{M_n} S_v z_1 \rightarrow 0.$$

Hence, $\mathcal{T}$ satisfies the Hypercyclicity Criterion with respect to the pair of sequences $(\{p_{k_{v(1)}(1)}\}_v, \{\{p_{k_{v(2)}(2)}\}_v, ..., \{p_{k_{v(n)}(n)}$
This implies that

$$\{T_1^{p_{k_{v(1)}(1)}} T_2^{p_{k_{v(2)}(2)}} ... T_n^{p_{k_{v(n)}(n)}} : v \geq 0\}$$

and so

$$\{T_1^{p_{k(1)}} T_2{}^{p_{k(2)}} ... T_n{}^{p_{k(n)}} : k \geq 0\}$$

is hypercyclic. Thus the assertion (ii) holds.

(ii) $\rightarrow$ (i): It is sufficient to show that condition (ii) of Theorem 2.3 holds. For this let U and V

are nonempty open subsets of X and W be a neighborhood of zero in X. Choose nonempty open subsets U_1, V_1, W_1 of U, V, W respectively such that $U_1 - U_1 \subset W$ and $V_1 - W_1 \subset V$. Since

$$\{T_1^{p_{k(1)}} {T_2}^{p_{k(2)}} ... {T_n}^{p_{k(n)}} : k \geq 0\}$$

is hypercyclic, thus clearly $\mathcal{T}$ is also hypercyclic. Hence by Lemma 2.2, there exist a tuple of positive integers $(m_0(1), m_0(2), ..., m_0(n))$ such that

$$T_1^{m_0(1)} T_2^{m_0(2)} ... T_n^{m_0(n)}(W_1) \cap V_1 \neq \emptyset.$$

This implies that there is a nonempty open set $W_2 \subset W_1$ with

$$T_1^{m_0(1)} T_2^{m_0(2)} ... T_n^{m_0(n)} W_2 \subset V_1.$$

By condition (ii), there exist $j_i \in \mathbb{N}\ for\ i = 1, ..., n$, satisfying

$$T_1^{j_1+r_1} {T_2}^{j_2+r_2} ... {T_n}^{j_n+r_n}(U_1) \cap W_2 \neq \emptyset$$

for $r_i = 0, ..., m_0(i)\ i = 1, ..., n$. Then for the cases $r_i = 0$ and $r_i = m_0(i)$ for $i = 1, ..., n$, we get

$$T_1^{j_1} {T_2}^{j_2} ... {T_n}^{j_n}(U_1) \cap W_2 \neq \emptyset$$

and

$$T_1^{j_1+m_0(1)} ... {T_n}^{j_n+m_0(n)}(U_1) \cap W_2 \neq \emptyset$$

respectively. Set

$$E = (T_1^{j_1+m_0(1)} ... {T_n}^{j_n+m_0(n)}(U_1) \cap V_1) - (T_1^{j_1+m_0(1)} ... {T_n}^{j_n+m_0(n)}(U_1) \cap W_2)$$

and note that $E \neq \emptyset$. Define $r_i = j_i + m_0(i)$, $i = 1, ..., n$. Then, we have

$${T_1}^{r_1} {T_2}^{r_2} ... {T_n}^{r_n}(U_1) \cap W_2 \neq \emptyset$$

and so

$${T_1}^{r_1} {T_2}^{r_2} ... {T_n}^{r_n}(U_1) \cap W \neq \emptyset.$$

Note that

$$\begin{aligned} E &\subset {T_1}^{j_1+m_0(1)} {T_2}^{j_2+m_0(2)} ... {T_n}^{j_n+m_0(n)}(U_1 - U_1) \cap (V_1 - W_2) \\ &\subset {T_1}^{r_1} {T_2}^{r_2} ... {T_n}^{r_n}(W) \cap V. \end{aligned}$$

Hence, ${T_1}^{r_1} {T_2}^{r_2} ... {T_n}^{r_n}(W) \cap V$ is also nonempty and so (i) holds.

References

[1] J. Bes and A. Peris, Hereditarily hypercyclic operators, *J. Func. Anal.*, **167** (1) (1999), 94-112.

[2] P. S. Bourdon and J. H. Shapiro, Cyclic phenomena for composition operators, *Memoirs of the Amer. Math. Soc.,* **125***, Amer. Math. Soc. Providence*, RI, 1997.

[3] P. S. Bourdon and J. H. Shapiro, Hypercyclic operators that commute with the Bergman backward shift, *Trans. Amer. Math. Soc.*, **352** (2000), 5293-5316.

[4] G. Costakis and M. Sambarino, Topologically mixing hypercyclic operators, *Proc. Amer. Math. Soc.*, **132** (2003), 385-389.

[5] N. S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, *J. Math. Appl.*, **346** (2008), 82-98.

[6] G. Godefroy and J. H. Shapiro, operators with dense invariant cyclic manifolds, *J. Func. Anal.*, **98** (1991), 229-269.

[7] K. G. Grosse-Erdmann, Recent developments in hypercyclicity, *Rev. R. Acad. Cien. Serie A. Mat.,* **97** (2) (2003), 273-286.

[8] H. N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.*, **347** (1995), 993-1004.

[9] B. Yousefi, and H. Rezaei, Hypercyclicity on the algebra of Hilbert-Schmidt operators, *Results in Mathematics*, **46** (2004), 174-180.

[10] B. Yousefi and A. Farrokhinia, On the hereditarily hypercyclic vectors, *Journal of the Korean Mathematical Society*, **43** (6) (2006), 1219-1229.

[11 B. Yousefi and H. Rezaei, Hypercyclic property of weighted composition operators, *Proc. Amer. Math. Soc.*, **135** (10) (2007), 3263-3271.

[12] B. Yousefi and H. Rezaei, On the supercyclicity and hypercyclicity of the operator algebra, *Acta Mathematica Sinica*, **24** (7) (2008), 1221-1232.

[13] B. Yousefi and J. Izadi, Weighted composition operators and supercyclicity criterion, *International Journal of Mathematics and Mathematical Sciences*, Volume **2011**, DOI 10.1155/2011/514370 (2011).

[14] B. Yousefi, Hereditarily transitive tuples, *Rend. Circ. Mat. Palermo*, Volume **2011**, DOI 10.1007/S12215-011-0066-y (2011).