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A FEW MINUTES WITH A NEW TRIANGLE CENTRE COINED AS "VIVYA'S POINT"

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Abstract

We have nearly 6102 triangle centers in the literature of 'geometry of triangles', "vivya's point" is one of such triangle center, in this paper let us discuss very few and important properties of this point in a more analytical way.

Key Words:

Vivya's point, vivians,maneeal's identity, cevas theorem,eulers inequality, maneeals inequality, weill's point

INTRODUCTION

In our classical "*Euclidian Geometry*" and in "*Modern Geometry*" we come across with a lot of different classical centers of triangle. These are all defined by the concurrence of important lines in the triangle. These centres and the cevians that create them are

Centroid	*medians*
Incenter	*angle bisectors*
Circumcenter	*penpendicular bisectors*
Orthocenter	*altitudes*

Others discovered more recently (about 100 to 150 years ago) are

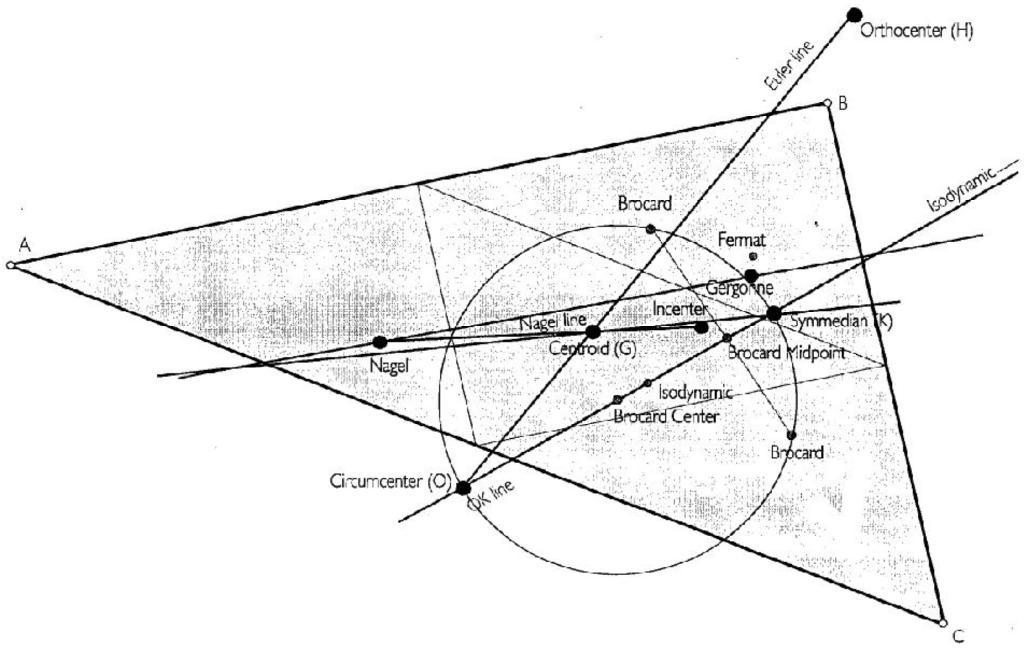
Associated with the incenter	
Gergonne point.	*cevians to contact points of incircle*
Nagel Point,	*cevians to contract points of excircles*
Spieker center	*incircle to medial triangle.*

Associated with the centroid	
symmedian point	*cevians which are reflections of medians respect to angular bisectors*
weills point	*centroid of intouch triangle*
9 pt center	*center of 9 pt circle*

Points that occur in pairs such as the isodynamic and isogonal points.

They play vital Role in Discussion and Improvement of our study in the Abstract Geometry.

Now in this paper Let us spend very few minutes with a new triangle centre coined as "Vivya's point" and let us discuss its properties, using this Discussion we will try to prove the famous Inequality named as ' Maneeal's Inequality" and also Famous classical Inequality "Euler's Inequality" and also many more curiosities.



Formal notations and some standard formulas :-

1) We know $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Rightarrow \Delta^2 = s(s-a)(s-b)(s-c)$

$\Rightarrow \Delta^2 = s(s^3 - s^2(a+b+c) + s(ab+bc+ca) - abc]$

$\Rightarrow (rs)^2 = s(s^3 - s^2(2s) + s(ab+bc+ca) - 4Rrs]$ Where $\Delta = rs \, and \, abc = 4R\Delta$

$\Rightarrow r^2 s^2 = s^2[s^2 - 2s^2 + (ab+bc+ca) - 4Rr]$

$\Rightarrow r^2 = s^2 + (ab+bc+ca) - 4Rr$

$\Rightarrow ab+bc+ca = r^2 + s^2 + 4Rr$

And

$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca)$

$= (2s)^2 - 2(r^2+s^2+4Rr) = 2s^2 - 2r^2 - 8Rr$

$\Rightarrow a^2+b^2+c^2 = 2\left(s^2 - r^2 - 4Rr\right)$

2) Let K = a(s-b) (s-c) + b(s-c) (s-a) +c(s-b) (s-a)

$= s^2 (a+b+c) - 2s (ab+ac+bc) + 3abc$

$= s^2 [2s] - 2s [r^2+s^2+4Rr] + 3 [4Rrs]$

$= 2s [s^2 - r^2 - s^2 - 4Rr + 6Rr]$

$= 2s [-r^2 + 2Rr]$

$K = 2rs\left(2R-r\right) = 2\Delta\left(2R-r\right) \Rightarrow 2R - r = \frac{K}{2\Delta}$

$\therefore K = a\left(s-b\right)\left(s-c\right) + b\left(s-c\right)\left(s-a\right) + c\left(s-b\right)\left(s-a\right) = 2rs\left(2R-r\right)$

3. Let P= c(s-b) +b(s-c)

$\Rightarrow$ P=s (b+c) -2bc

So, K = a(s-b) (s-c) + b(s-c) (s-a) +c(s-b) (s-a)

= a(s-b) (s-c) + (s-a) P

$= as^2 - as (b+c) + abc + P(s-a)$

$= as^2 - a(s (b+c) - 2bc) - abc + P(s-a)$

$= as^2 - aP - abc + P(s-a)$

$= as^2 + P(s-2a) - abc$

$= as^2 - abc + P(s-2a)$

$\therefore K = as^2 - abc + P\left(s-2a\right)$

4. $aK + a^2bc - 4(s-b)^2(s-c)^2$

$= a[as^2 - abc + P(s-2a)] + a^2bc - 4(s-b)(s-c)^2$

$= a^2s^2 - a^2bc + aP(s-2a) + a^2bc - (2(s-b)(s-c))^2$

$= (as)^2 - [2(s-b)(s-c)]^2 + P[as - 2a^2]$

$= [as + 2(s-b)(s-c)][as - 2(s-b)(s-c)] + P(as - 2a^2)$

$= [as + 2s^2 - 2s(b+c) + 2bc][as - 2s^2 + 2s(b+c) - 2bc] + P(2s - 2a^2)]$

$= [s(2s) - (b+c)] + 2s^2 - 2s(b+c) + 2bc]\ \ [s[(2s) - (b+c)] - 2s^2 + 2s(b+c) - 2bc] + P[as - 2a^2]$

$= [2s^2 - s(b+c) + 2s^2 - 2s(b+c) + 2bc][\ 2s^2 - s(b+c) - 2s^2 + 2s(b+c) - 2bc] + P(as - 2a^2)$

$= [4s^2 - 3s(b+c) + 2bc]\ [s(b+c) - 2bc] + P(as - 2a^2)$

$= [4s^2 - 3s(b+c) + 2bc]\ [P] + P(as - 2a^2)$

$= P[4s^2 - 3s(b+c) + 2bc + as - 2a^2]$

$= P[(a+b+c)^2 - 3(a+b+c)(s) + 4as + 2bc - 2a^2]$

$= P[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3(a+b+c)\left(\frac{a+b+c}{2}\right) + 4a\left(\frac{a+b+c}{2}\right) + 2bc - 2a^2]$

$= P[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - \frac{3}{2}(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) + 2a^2 + 2ab + 2ac + 2bc - 2a^2]$

$= \frac{P}{2}[2a^2 + 2b^2 + 2c^2 + 4ab + 4bc + 4ca - 3a^2 - 3b^2 - 3c^2 - 6ab - 6bc - 6ca + 4a^2 + 4ab + 4ac + 4bc - 4a^2]$

$= \frac{P}{2}[2ab + 2bc + 2ca - a^2 - b^2 - c^2]$

$= \frac{P}{2}[2(ab + bc + ca) - (a^2 + b^2 + c^2)]$

$= \frac{P}{2}[2(r^2 + s^2 + 4Rr) - 2(s^2 - r^2 - 4Rr)]$

$= \frac{P}{2}(2)\ [r^2 + s^2 + 4Rr - s^2 + r^2 + 4Rr]$

$= P[2r^2 + 8Rr] = 2rp[r + 4R]$

$\therefore aK + a^2bc - 4(s-b)^2(s-c)^2 = 2rP(r + 4R)$

5. If W_e is a Weill's point and X be any point in the plant of Δ^{le} ABC, then

$$W_eX^2 = \frac{1}{3}\sum\left(\frac{s-b}{c}+\frac{s-c}{b}\right)AX^2 - \frac{r}{9R}(r+4R)(2r+3R)$$

Formal Definitions :

If ABC is a triangle and D, E, F are the points of contacts of the Incircle of Δ^{le} ABC on the sides BC, CA, AB. Let DP, EQ, FR are the perpendiculars Drawn from the vertices D,E,F to the sides EF, DF, DE of Δ^{le} DEF then the lines formed by joining vertices A,B,C of Δ^{le} ABC and the points P, Q, R are called as "*Vivian's*" .We have 3 *"Vivians"* in the triangle and If we extend the 3 Vivians in the triangle they meet the opposite sides of the Δ^{le} ABC at A^1, B^1, C^1 and these 3 Vivians Intersect at a point called as "*Vivya's point*" denoted by 'V_m'and the triangle formed by joining the 3 points A^1, B^1, C^1 is called as "*Vivians Triangle*" and for any orbitary triangle the *Circumcentre (S), Vivya's point (V_m) and Incentre (I)* are *collinear*. The line through these 3 points is called as *Vivya's line*

The following figures demonstrate about the vivians.

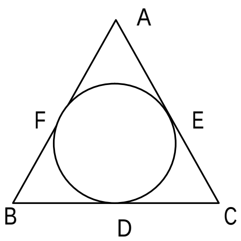


Fig (1)

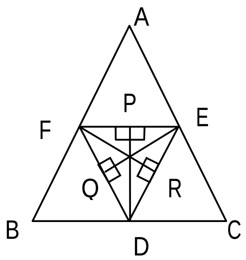


Fig (2)

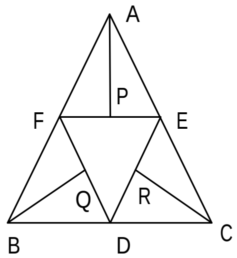


Fig (3)

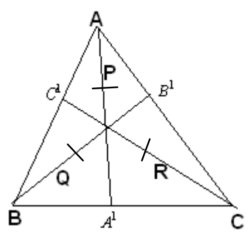


Fig (4)

In the above 4 figures we can see how the vivians can be constructed and also we can identify how the new triangle centre "*Vivya's point*" denoted by "V_m" formed. Now let us prove some theorems related to this point.

Theorem 1 :

The three Vivians of triangle are concurrent and the point of concurrence is called as "*Vivya's point*" (V_m).

Proof:

Step - I

"From Figure 1"

We can observe that Δ^{le} DEF is a contact triangle (or) In touch triangle, so we can find the lengths of AE, AF, BF, BD, CE, CD as follows.

Let us consider formal notations Say BC= a units, CA= b units, AB= c units. $\overline{BDC}$, $\overline{BFA}$ are Tangents to the Incircle.

We know that "*Tangents Drawn from an External Point are equal in length*".

So BD=BF=x, CD=CE=y, AF=AE=z (let)

Hence BC = a = BD + BC = *xy*

CA = b = CE + EA = y +z

AB = c = AF + BF = x + z

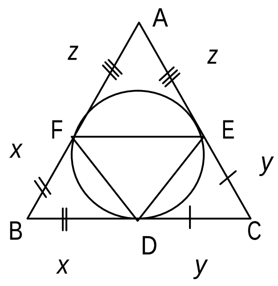
$\therefore$ 2s = a + b +c = 2(x+y+z)

$\Rightarrow$ s= x + y + z where s is semi perimeter

So BD=BF= x = (x+y+z) - (y+z) = s-b

CD= CE = y =(x+y+z) -(x+z) = s-c

AE = AF = z= (x+y+z) -(x+y) = s-a



Now from Δ^{le}BDF, BD=BF and $\angle$FBD =B

So $\angle$BED=$\angle$BDE = $90-\dfrac{B}{2}$

Similarly for Δ^{le}CDE, $\angle$CDE= $\angle$CED = $90-\dfrac{C}{2}$ ($\because$ $\angle$OCE =C)

and for Δ^{le}AEF, $\angle$AEF= $\angle$AFE= $90-\dfrac{A}{2}$ ($\because$ $\angle$EAF =A)

$\therefore$ $\angle$DFE = 180^o - ($\angle$BFD + $\angle$ EFA)

$$= 180^o - \left(90-\frac{B}{2}+90-\frac{A}{2}\right)$$

= $\frac{A}{2}+\frac{B}{2}=90-\frac{C}{2}$ ($\therefore \angle$A+$\angle$B+$\angle$C =180^o)

$\underline{|DEF} = 90 - C/2$

Similarly,

$\underline{|FDE} = 90 - A/2$

$\underline{|DEF} = 90 - B/2$

Now from Δ^{le} BDF, By Consine Rule

$DF^2 = BD^2 + BF^2 - 2BD\ BF \cos B$

$\Rightarrow$ $DF^2 = (s-b)^2 + (s-b)^2 - 2(s-b)(s-b)\cos B$

$\Rightarrow$ $DF^2 = 2(s-b)^2 - 2(s-b)^2 \cos B$

$\Rightarrow$ $DF^2 = 2(s-b)^2 [1-\cos B] = 2(s-b)^2 2\sin^2 B/2 = 4(s-b)^2.\sin^2 B/2 = [2(s-b)(\sin B/2)]^2$

$\Rightarrow$ $DF = [2(s-b)(\sin B/2)]$

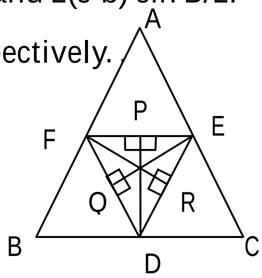
Similarly,

$DE = 2(s-c) \sin C/2$

$EF = 2(s-a) \sin A/2$

$\therefore$ The sides of Δ^{le} DEF are 2(s-c) sin C/2, 2(s-a) Sin A/2 and 2(s-b) sin B/2.

The angles of Δ^{le} DEF are 90-C/2, 90 -A/2, 90 -B/2 respectively.



Step - 2:

" From Fig 2 "

Since EQ, DP, FR are $\perp^r$ to the sides FD, EF, DE

For Δ^{le}FED,

EQ is cevian (the line from the vertex to the opposite side

Now For Δ^{le}EQF, $\angle$EQF =90^o and $\angle$QFE = 90-C/2

So $\angle$FEQ= 180^o - ($\angle$EQF + $\angle$QFE) =C/2

Similarly

$\angle$QED = A/2

So since EQ is cevian of the Δ^{le} FED meets the opposite side DF in Q then the Ratio it divides the opposite side equal to the ratio of Areas of two triangles formed by the cevian.

i.e., $\frac{FQ}{QB} = \frac{[\Delta^{le} FEQ]}{[\Delta^{le} EQD]}$ ($\because$ [ ] represents area)

and we know that Area of triangle Δ^{le} = ½ ab sinC

So [Δ^{le}FEQ] $= \frac{1}{2}$ EF .EQ. Sin $\angle$PEQ

$= \frac{1}{2}$ 2 (s-a) Sin $\frac{A}{2}$ .EQ sin $\frac{C}{2}$ = (s-a) Sin $\frac{A}{2}$ .EQ sin $\frac{C}{2}$

Similarly

$$\left[\Delta^{le} EQR\right] = \frac{1}{2} ED.EQ \sin\angle QED = \frac{1}{2}.2(s-c)\sin C/2 .EQ.\sin A/2 = (s-c) \sin A/2 .EQ \sin C/2$$

$$\therefore \quad \frac{FQ}{QD} = \frac{(s-a)\sin A/2 \sin C/2\, EQ}{(s-c)\sin A/2 \sin C/2\, EQ} = \frac{s-a}{s-c} \Rightarrow \frac{FQ}{QD} = \frac{s-a}{s-c}$$

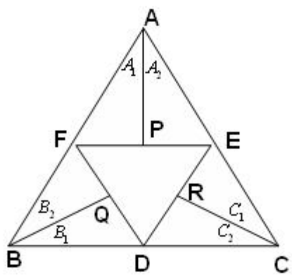
Similarly we can find

$$\frac{DR}{RE} = \frac{s-b}{s-c}$$

And $$\frac{EP}{PF} = \frac{s-c}{s-b}$$

Step - 3

From figure 3



Let $\angle FAP = A_1, \angle EAP = A_2$

$\angle$ECR=C_1, $\angle$DCR=C_2, $\angle$DBQ=B_1, $\angle$FBQ=B_2

Now consider Δ^{le} BFD,

Since BQ is cevian

BQ divides the opposite side DF in the ratio of Area of Δ^{le} BQD, Δ^{le} BQF

i.e. $$\frac{DQ}{FQ} = \frac{[\Delta^{le} BQD]}{[\Delta^{le} BQF]} = \frac{\frac{1}{2}.BD.BQ\sin B_1}{\frac{1}{2}.BQ.BF\sin B_2}$$

$$\Rightarrow \frac{FQ}{DQ} = \frac{BF \sin B_2}{BD \sin B_1}$$

$$\Rightarrow \frac{s-a}{s-c} = \frac{(s-b)\sin B_2}{(s-b)\sin B_1}$$ (Using step -2)

$$\Rightarrow \frac{\sin B_2}{\sin B_1} = \frac{s-a}{s-c} \Rightarrow \frac{\sin B_1}{\sin B_2} = \frac{s-c}{s-a}$$

similarly we can prove that

$$\Rightarrow \frac{\sin A_1}{\sin A_2} = \frac{s-b}{s-c} \; and \; \frac{\sin C_1}{\sin C_2} = \frac{s-a}{s-b}$$

Step -4

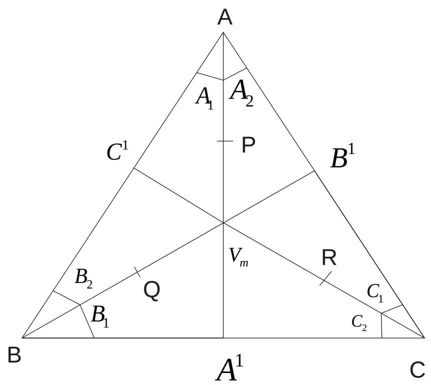
"From Fig 4"

Now AA^1, BB^1, CC^1 are required *Vivian's* and also they are *Cevians*.

We know that Cevian of triangle divides the opposite side in the Ratio of Areas of triangles thus formed.

So, $$\frac{BA^1}{A^1C} = \frac{\left[\Delta^{le} B\, AA^1\right]}{\left[\Delta^{le} C\, AA^1\right]}$$

$$\Rightarrow \frac{BA^1}{A^1C} = \frac{\frac{1}{2} BA.AA^1.Sin\, A_1}{\frac{1}{2} CA.AA^1.Sin\, A_2}$$



$$\Rightarrow \frac{BA^1}{A^1C} = \frac{c.Sin\, A_1}{b.Sin\, A_2}$$

$$\Rightarrow \frac{BA^1}{A^1C} = \left(\frac{c}{b}\right)\left(\frac{s-b}{s-c}\right)$$ (using step-3)

$$\therefore \frac{BA^1}{A^1C} = \frac{c(s-b)}{b(s-c)}$$

Similarly can prove that $\frac{CB^1}{B^1A} = \frac{a(s-c)}{c(s-a)}, \frac{AC^1}{C^1B} = \frac{b(s-a)}{a(s-b)}$

Final step

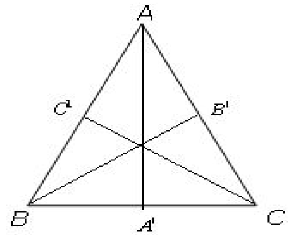
Now $$\frac{AC^1}{C^1B}.\frac{BA^1}{A^1C}\frac{CB^1}{B^1A} = \frac{b(s-a)}{a(s-b)}.\frac{c(s-b)}{b(s-c)}.\frac{a(s-b)}{c(s-a)}$$

$$= \frac{abc(s-a)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)} = 1$$

Hence by the converse of ceva's theorem we can conclude that the three *Vivian's* AA^1, BB^1, CC^1 are Concurrent and the point of concurrence is called as "*Vivya's point* (V_m)".

COROLLARIES:

I. *Let AA^1, BB^1, CC^1 are Vivians then*



$$\frac{BA^1}{A^1C} = \frac{c(s-b)}{b(s-a)}$$

Similarly $\frac{CB^1}{B^1A} = \frac{a(s-c)}{c(s-a)}$ and $\frac{AC^1}{C^1B} = \frac{b(s-a)}{a(s-b)}$

II. *The point of concurrency of 3 Vivian's is called as "Vivya's point" and is denoted as V_m.*

III. *If V_m is the Vivya's point then the length of the each Vivian in the triangle ABC is given by*

$$\left(AA^{1}\right)^2 = \frac{a^2b^2c^2 - 4abc(s-b)^2(s-c)^2}{[c(s-b)+b(s-c)]^2}$$

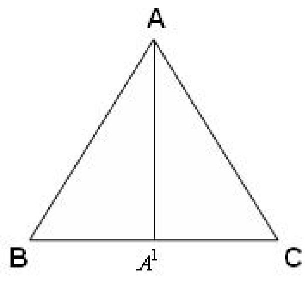
Hence $\quad AA^1 = \frac{1}{c(s-b)+b(s-c)}\sqrt{a^2b^2c^2 - 4bc(s-b)^2(s-c)^2}$

Similarly $\quad BB^1 = \frac{1}{a(s-c)+c(s-a)}\sqrt{a^2b^2c^2 - 4ac(s-a)^2(s-c)^2}$

$$CC^1 = \frac{1}{a(s-b)+b(s-c)}\sqrt{a^2b^2c^2 - 4ab(s-a)^2(s-b)^2}$$

<u>Proof:-</u>

Let AA^1 is *Vivian*



We know by using previous Discussion

$$\frac{BA^1}{A^1C} = \frac{c(s-b)}{c(s-c)}$$

So let $BA^1 = c(s-b)\,t$ for some constant t

$A^1C = b(s-c)\,t$

$BA^{|} + A^{|}C$ = t [c(s-b) +b(s-c)]

BC = t[c(s-b)+b(s-c)]

$\Rightarrow$ a = t[c(s-b)+b(s-c)]

$\Rightarrow$ t = $\dfrac{a}{c(s-b)+b(s-c)}$

$\therefore$ $BA^{|}$ = c(s-b) t = $\dfrac{ca(s-b)}{c(s-b)+b(s-c)}$

$AC^{|}$ = b(s-c) t = $\dfrac{ba(s-c)}{c(s-b)+b(s-c)}$

Since $AA^{|}$ is a vivian as well as Cevian So By STEWART ' S THEOREM

$$(AA^{|})^2 = \frac{BA^{|}.AC^2}{BC} + \frac{CA^{|}.BA^2}{BC} - BA^{|}.CA^{|}$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{ac(s-b)b^2}{[c(s-b)+b(s-c)](a)} + \frac{ab(s-c)c^2}{[c(s-b)+b(s-c)](a)} - \frac{a^2bc(s-b)(s-c)}{[c(s-b)+b(s-c)]^2}$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]}[b(s-b)+c(s-c)] - \frac{a^2bc(s-b)(s-c)}{[c(s-b)+b(s-c)]^2}$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[[b(s-b)+c(s-c)][c(s-b)+b(s-c)] - [a^2(s-b)(s-c)\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[bc(s-b)^2 + b^2(s-b)(s-c) + c^2(s-c)(s-b) + bc(s-c)^2 - a^2(s-b)(s-c)\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[bc\left[(s-b)^2 + (s-c)^2\right] + (s-b)(s-c)(b^2+c^2-a^2)\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[bc(s-b+s-c)^2 - 2bc(s-b)(s-c) + (s-b)(s-c)(b^2+c^2-a^2)\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[bc\left[2s-(b+c)\right]^2 + (s-b)(s-c)[b^2+c^2-a^2-2bc]\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[bc(a)^2 + (s-b)(s-c)\left[(b-c)^2 - a^2\right]\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[a^2bc + (s-b)(s-c)(b-c+a)(b-c-a)\right]$$

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{bc}{[c(s-b)+b(s-c)]^2}\left[a^2bc - (s-b)(s-c)(2s-2c)(2s-2b)\right]$$ where 2s= a+b+c

$$\Rightarrow \left(AA^{|}\right)^2 = \frac{a^2b^2c^2 - 4bc(s-b)^2(s-c)^2}{[c(s-b)+b(s-c)]^2}$$

$\Rightarrow AA^1 = \frac{1}{c(s-b)+b(s-c)}\sqrt{a^2b^2c^2 - 4bc(s-b)^2(s-c)^2}$

Similarly we can prove that

$BB^1 = \frac{1}{a(s-c)+c(s-a)}\sqrt{a^2b^2c^2 - 4bc(s-a)^2(s-c)^2}$

$CC^1 = \frac{1}{a(s-b)+b(s-a)}\sqrt{a^2b^2c^2 - 4ab(s-a)^2(s-b)^2}$

IV. If V_m is the Vivya's point of a triangle then the Vivya's point divides each Vivian in the ratio is given by

$$\frac{AV_m}{V_mA^1} = \frac{c(s-a)(s-b)+b(s-a)(s-c)}{a(s-b)(s-c)}$$

Similarly $$\frac{BV_m}{V_mB^1} = \frac{c(s-b)(s-a)+a(s-b)(s-c)}{b(s-a)(s-c)}$$

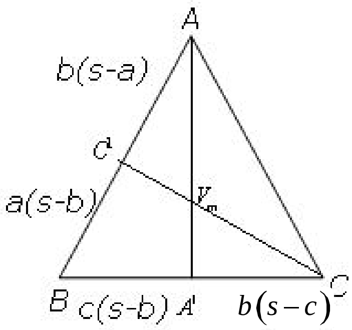
and $$\frac{CV_m}{V_mC^1} = \frac{a(s-c)(s-b)+b(s-c)(s-a)}{c(s-a)(s-b)}$$

Proof :-

Let AA^1, CC^1 are vivians of Δ^{le} ABC and V_m is "Vivya's point"

Since C^1, V_m, C are collinear

For Δ^{le} ABA^1 and $\overleftrightarrow{C^1V_mC}$ is a transversal



Hence By Menelau's theorem

We have $\frac{AC^1}{C^1B}\cdot\frac{BC}{CA^1}\cdot\frac{A^1V_m}{V_mA} = -1$

So $\frac{A^1V_m}{V_mA} = \frac{BC^1}{C^1A}\frac{CA^1}{BC}$

("-" is removed since it represents only direction but not magnitude)

we know that

$$\frac{BC^1}{C^1A} = \frac{a(s-b)}{b(s-a)}$$

and $\frac{BA^1}{A^1C} = \frac{c(s-b)}{b(s-c)}$ (By adding 1 on both sides)

$$\Rightarrow \frac{BA^1}{A^1C} + 1 = \frac{c(s-b)}{b(s-c)} + 1$$

$$\Rightarrow \frac{BA^1 + A^1C}{A^1C} = \frac{c(s-b)+b(s-c)}{b(s-c)}$$

$$\Rightarrow \frac{BC}{A^1C} = \frac{c(s-b)+b(s-c)}{b(s-c)}$$

$$\Rightarrow \frac{CA^1}{BC} = \frac{b(s-c)}{c(s-b)+b(s-c)}$$

since $\frac{A^1V_m}{V_mA} = \frac{BC^1}{C^1A}.\frac{CA}{BC}$

$$\frac{A^1V_m}{V_mA} = \frac{a(s-b)}{(s-a)}.\frac{b(s-c)}{c(s-b)+b(s-c)} = \frac{a(s-b)(s-c)}{c(s-a)(s-b)+(s-a)(s-c)}$$

$$\Rightarrow \frac{AV_m}{V_mA^1} = \frac{c(s-a)(s-b)+b(s-a)(s-c)}{a(s-a)(s-c)}.$$

Similarly we can prove

$$\frac{BV_m}{V_mB^1} = \frac{c(s-b)(s-a)+a(s-b)(s-c)}{b(s-a)(s-c)}$$

And $\frac{CV_m}{V_mC^1} = \frac{a(s-c)(s-b)+b(s-c)(s-a)}{c(s-a)(s-b)}$

V. *If V_m is Vivya’s points of Δ^{le} ABC and AA^1, BB^1, CC^1 are Vivian’s then*

$$AV_m^2 + BV_m^2 + CV_m^2 = S^2\left[\frac{2R+r}{2R-r}\right] - \frac{(r^2+4Rr)(8R^2+r^2)}{(2R-r)^2}$$

Proof: We have by previous theorem :

$$\frac{AV_m}{V_mA^1} = \frac{[(s-a)(s-b)+b(s-a)(s-c)]}{a(s-b)(s-c)} = \frac{K-a(s-b)(s-c)}{a(s-b)(s-c)}$$

$$= \frac{(s-a)[c(s-b)+b(s-c)]}{a(s-b)(s-c)} = \frac{(s-a)p}{a(s-b)(s-c)}$$

So $AV_m = [K-a(s-b)(s-c)]t$ for some real constant ‘ ’t’ ’

$V_mA^1 = [a(s-b)(s-c)]t$

Now $AV_m + V_mA^1 = t[K]$

$\Rightarrow AA^1 = tK$

$$\Rightarrow t = \frac{AA^1}{K}$$

So $AV_m = \left[K - a(s-b)(s-c)\right]\frac{AA^1}{K} = \frac{p(s-a)AA^1}{K}$

$$V_m A^1 = \frac{a(s-b)(s-c)AA^1}{K}$$

Siimilarly $BV_m = \frac{\left[k - b(s-a)(s-b)\right]BB^1}{K}$, $V_m B^| = \frac{b(s-a)(s-c)}{K}BB^|$

and $CV_m = \frac{\left[k - c(s-a)(s-b)\right]CC^1}{k}$, $V_m C^1 = \frac{c(s-a)(s-b)CC^1}{k}$

Now

$$AV_m^2 + BV_m^2 + CV_m^2 = \Sigma AV_m^2$$

$$= \Sigma \frac{p^2(s-a)^2}{k^2} AA^1$$

$$= \Sigma \frac{p^2(s-a)^2}{k^2} \frac{1}{\left[c(s-b)+b(s-c)\right]^2}\left[a^2b^2c^2 - 4bc(s-b)^2(s-c)^2\right]$$

$$= \Sigma \frac{p^2(s-a)^2}{k^2}\left[\frac{a^2b^2c^2 - 4bc(s-b)^2(s-c)^2}{p^2}\right]$$

$$= \Sigma\left[\frac{a^2b^2c^2(s-a)^2}{k^2} - \frac{4bc(s-a)^2(s-b)^2(s-c)^2}{k^2}\right]$$

$$= \frac{a^2b^2c^2}{k^2}\left[(s-a)^2 + (s-b)^2 + (s-c)^2\right] - \frac{4}{k^2}(s-a)^2(s-b)^2(s-c)^2\left[ab+bc+ca\right]$$

$$= \frac{a^2b^2c^2}{k^2}\left[3s^2 - 2s(a+b+c) + a^2 + b^2 + c^2\right] - \frac{4}{k^2}\frac{s^2(s-a)^2(s-b)^2(s-c)^2}{s^2}\left[ab+bc+ca\right]$$

$$= \frac{16R^2\Delta^2}{k^2}\left[3s^2 - 4s^2 + a^2 + b^2 + c^2\right] - \frac{4}{k^2}\frac{\Delta^4}{s^2}(ab+bc+ca) \quad (\therefore abc = 4R\Delta \ and\, 2s = a+b+c)$$

$$= \frac{16R^2\Delta^2}{k^2}\left[-s^2 + 2s^2 - 2r^2 - 8Rr\right] - \frac{4\Delta^4}{k^2s^2}\left[r^2 + s^2 + 4Rr\right]$$

$$= \frac{4\Delta^2}{k^2}\left[4R^2\left(s^2 - 2r^2 - 8Rr\right) - \frac{\Delta^2}{s^2}\left(r^2 + s^2 + 4Rr\right)\right]\left[\because \Delta = rs\right]$$

$$= \frac{4\Delta^2}{k^2}\left[4R^2s^2 - 8R^2r^2 - 32R^3r - r^2\left(r^2rs^2 + 4Rr\right)\right]$$

$$= \frac{4\Delta^2}{k^2}\left[4R^2s^2 - 8R^2r^2 - 32R^3r - r^4 - r^2s^2 - 4R4r^3\right]$$

$$= \frac{4\Delta^2}{k^2}\left[s^2\left(4R^2 - r^2\right) - 8r^2\left(8R^2 + r^2\right) - 4Rr\left(8R^2 + r^2\right)\right]$$

$$= \frac{4\Delta^2}{k^2}\left[s^2\left(4R^2 - r^2\right) - \left(r^2 + 4Rr\right)\left(8R^2 + r^2\right)\right]$$

$$= \frac{4\Delta^2}{4\Delta^2\left(2R - r\right)^2}\left[s^2\left(4R^2 - r^2\right) - \left(r^2 + 4Rr\right)\left(8R^2 + r^2\right)\right]$$

$$= S^2\left(\frac{2R + r}{2R - r}\right) - \frac{\left(r^2 + 4Rr\right)\left(8R^2 + r^2\right)}{\left(2R - r\right)^2}$$

$$\therefore \quad AV_m^2 + BV_m^2 + CV_m^2 = S^2\left[\frac{2R + r}{2R - r}\right] - \frac{\left(r^2 + 4Rr\right)\left(8R^2 + r^2\right)}{\left(2R - r\right)^2}$$

Theorem 2 :

MANEEAL 's Identity:

Let ABC is a triangle and V_m is it 's 'Vivya 's point' and X be any point in the plane of Δ^{le}ABC

Then

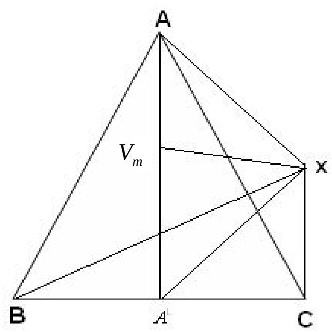
$$V_mX^2 = \frac{a(s-b)(s-c)}{K}AX^2 + \frac{b(s-a)(s-c)}{K}BX^2 + \frac{c(s-a)(s-b)}{K}CX^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

where $K = 2rs\left(2R - r\right)$

The other form of maneeal 's Identity as

$$V_mX^2 = \frac{2RSin^2A/2}{(2R-r)}AX^2 + \frac{2RSin^2B/2}{2R-r}BX^2 + \frac{2RSin^2C/2}{2R-r}CX^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

Proof:-



Let R= circum radius of ΔABC

r = In radius of ΔABC

s =semi perimeter

V_m = Vivya 's point

X = Any point in the plane of ΔABC

AB = c, BC=a, CA=b

P = c(s-b) +b(s-c)

K = a(s-b) (s-c) +b(s-a) (s-c) +(s-a) (s-b)

$\therefore$ K-a(s-b) (s-c) = (s-a) [b(s-c) +c(s-b)] =(s-a) P

We know by previous calculations

$$\frac{BA^1}{A^1C} = \frac{c(s-b)}{b(s-c)}$$

And $BA^1 = \dfrac{ac(s-b)}{c(s-b)+b(s-c)}$

$= \dfrac{ac(s-b)}{p}$

Similarly $A^1C = \dfrac{ab(s-c)}{p}$

and V_m is *Vivya's point* and we have

$$\frac{AV_m}{V_mA^1} = \frac{c(s-a)(s-b)+b(s-a)(s-c)}{a(s-b)(s-c)}$$

by previous theorems we have

$$\therefore \quad AV_m = \frac{\left[c(s-a)(s-b)+b(s-a)(s-c)\right]AA^{|}}{K} = \frac{\left[K-a(s-b)(s-c)\right]AA^{|}}{K}$$

and $V_mA^1 = \dfrac{a(s-b)(s-c)AA^1}{k}$

<u>Step - 1 :</u>

Now from Δ^{le} BXC,

Since $XA^{|}$ is a cevian so by STEWARTS THEOREM, we have

$$\left(XA^{|}\right)^2 = \frac{BA^{|}.CX^2}{BC} + \frac{CA^{|}.BX^2}{BC} - BA^{|}.A^{|}C$$

$$\Rightarrow \left(XA^{|}\right)^2 = \frac{ac(s-b)}{Pa}CX^2 + \frac{ab(s-c)}{Pa}BX^2 - \frac{a^2bc(s-b)(s-c)}{P^2}$$

$$\Rightarrow \left(XA^{|}\right)^2 = \frac{c(s-b)}{P}CX^2 + \frac{b(s-c)}{P}BX^2 - \frac{a^2bc(s-b)(s-c)}{P^2}$$

<u>Step - 2 :</u>

Now From Δ^{le} $AXA^{|}$,

Since V_mX is a cevian so by Stewart 's theorem, we have

$$V_mX^2 = \frac{AV_m\left(XA^{|}\right)^2}{AA^{|}} + \frac{V_mA^{|}.AX^2}{AA^{|}} - AV_m.V_mA^{|}$$

$$\Rightarrow V_mX^2 = \frac{\left[K - a(s-b)(s-c)\right]AA^{|}}{K.AA^{|}}\left(XA^{|}\right)^2 + \frac{a(s-b)(s-c)AA^{|}}{K.AA^{|}}AX^2 - \frac{\left[K - a(s-b)(s-c)\right](a)(s-b)(s-c)\left(AA^{|}\right)^2}{K^2}$$

$$\Rightarrow V_mX^2 = \left[\frac{K - a(s-b)(s-c)}{K}\right]\left[\frac{c(s-b)CX^2}{P} + \frac{b(s-c)BX^2}{P} - \frac{a^2bc(s-b)(s-c)}{P^2}\right]$$

$$+\frac{a(s-b)(s-c)}{K}AX^2 - \frac{\left[K - a(s-b)(s-c)\right](a)(s-b)(s-c)\left(AA^{|}\right)^2}{K^2}$$

$$\Rightarrow V_mX^2 = \left[\frac{(s-a)P}{K}\right]\left[\frac{c(s-b)CX^2}{P} + \frac{b(s-c)BX^2}{P} - \frac{a^2bc(s-b)(s-c)}{P^2}\right]$$

$$+\frac{a(s-b)(s-c)}{K}AX^2 - \frac{(s-a)P(a)(s-b)(s-c)\left(AA^{|}\right)^2}{K^2}$$

$$\Rightarrow V_mX^2 = \frac{c(s-b)(s-a)CX^2}{K} + \frac{b(s-c)(s-a)BX^2}{K} + \frac{a(s-b)(s-c)AX^2}{K}$$

$$-\frac{a^2bc(s-a)(s-b)(s-c)p}{KP^2} - \frac{p(s-a)(s-b)(s-c)}{K^2}AA^{1^2}$$

$$\Rightarrow V_mX^2 = \frac{\sum a(s-b)(s-c)}{K}AX^2 - \frac{a(s-a)(s-b)(s-c)}{K^2P}\left[Kabc + P^2AA^{1^2}\right]$$

<u>Step - 3 :</u>

Now

Kabc + $P^2\left(AA^{|}\right)^2$

= Kabc+$P^2\left[\frac{a^2b^2c^2 - 4bc(s-b)^2(s-c)^2}{P^2}\right]$

Where $\left(AA^{|}\right)^2 = \frac{a^2b^2c^2 - 4bc(s-b)^2(s-c)^2}{P^2}$

= K abc+$a^2b^2c^2$ -4bc$(s-b)^2$ $(s-c)^2$

= bc[Ka+a^2bc-4$(s-b)^2(s-c)^2$]

= bc[2rp(r+4R)] [By notation 4]

$$\therefore V_m X^2 = \sum \frac{a(s-b)(s-c)}{\mathrm{K}} AX^2 - \frac{a(s-a)(s-b)(s-c)}{\mathrm{K}^2 P}\left[Kabc + P^2 AA^{1^2}\right]$$

$$\Rightarrow V_m X^2 = \sum \frac{a(s-b)(s-c)}{\mathrm{K}} AX^2 - \frac{a(s-a)(s-b)(s-c)}{\mathrm{K}^2 P}\left[2rpbc + (r+4R)\right]$$

$$\Rightarrow V_m X^2 = \sum \frac{a(s-b)(s-c)}{\mathrm{K}} AX^2 - \frac{abc(s-a)(s-b)(s-c)(r+4R)(2r)}{\mathrm{K}^2}$$

$$\Rightarrow V_m X^2 = \sum \frac{a(s-b)(s-c)}{2\mathrm{rs}(2\mathrm{R}-\mathrm{r})} AX^2 - \frac{abc(s-a)(s-b)(s-c)(2r)(r+4R)}{(2\mathrm{rs})^2(2R-r)^2}$$

$$\Rightarrow V_m X^2 = \sum \frac{a(s-b)(s-c)}{2\mathrm{rs}(2\mathrm{R}-\mathrm{r})} AX^2 - \frac{abc(s-a)(s-b)(s-c)(2r)(r+4R)}{4r^2 s^2(2R-r)^2}$$

Step - 4 :

Now

$$\frac{abc(s-a)(s-b)(s-c)(2r)(r+4R)}{4\mathrm{r}^2 s^2(2R-r)^2}$$

$$= \frac{4Rrs(s-a)(s-b)(s-c)(2r)(r+4R)}{4\mathrm{r}^2 s^2(2R-r)^2} \quad [\because abc = 4Rrs]$$

$$= \frac{8Rr^2 s(s-a)(s-b)(s-c)(r+4R)}{4(\mathrm{rs})^2(2R-r)^2}$$

$$= \frac{8Rr^2\Delta^2(r+4R)}{4\Delta^2(2R-r)^2}[\because \Delta = rs]$$

$$= \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$\therefore \quad V_m X^2 = \sum \frac{a(s-b)(s-c)}{2rs(2R-r)} AX^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$\therefore \quad V_m X^2 = \frac{a(s-b)(s-c)}{K} AX^2 + \frac{b(s-a)(s-c)}{K} BX^2 + \frac{c(s-a)(s-b)}{K} CX^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

where $K = 2rs(2R-r)$

This is our MANEEAL 's Identity

Now

$$\frac{a(s-b)(s-c)}{2rs(2R-r)} = \frac{abc\sin^2 A/2}{2rs(2R-r)} = \frac{4R\sin^2 A/2}{2\Delta(2R-r)} = \frac{2R\sin^2 A/2}{2R-r}$$

Hence the other form of MANEEAL 's identity is

$$V_mX^2 = \frac{2R\sin^2 A/2}{2R-r}AX^2 + \frac{2R\sin^2 B/2}{2R-r}BX^2 + \frac{2R\sin^2 C/2}{2R-r}CX^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

Corollaries :

I. *If Vivya's point (V_m) and circumcentre (S) of a triangle then*

$$V_mS^2 = R(R-2r)\left[\frac{2R+r}{2R-r}\right]^2$$

Proof

1. If we put instead of X as S (Circumcentre) in V_mX using the previous theorem then we can find the distance between our ' Vivya's point and Circumcentre (S).

$$\therefore V_mS^2 = \frac{a(s-b)(s-c)}{2rs(2R-r)}AS^2 + \frac{b(s-a)(s-c)}{2rs(2R-r)}BS^2 + \frac{c(s-a)(s-b)}{2rs(2R-r)}CS^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

and we have AS=BS=CS=R= Circum Radius

Hence $\therefore V_mS^2 = \frac{a(s-b)(s-c)}{K}R^2 + \frac{b(s-a)(s-c)}{K}R^2 + \frac{c(s-a)(s-b)}{K}Rr^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$

$$V_mS^2 = \frac{R^2}{K}\left[a(s-a)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)\right] - \left[\frac{2Rr^2(r+4R)}{(2R-r)^2}\right]$$

$$\Rightarrow V_mS^2 = \frac{R^2}{K}(k) - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$\Rightarrow V_mS^2 = R^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$\Rightarrow V_mS^2 = R\left[\frac{R(2R-r)^2 - 2r^2(r+4R)}{(2R-r)^2}\right]$$

$$\Rightarrow V_mS^2 = \frac{R}{(2R-r)^2}\left[R[4R^2 + r^2 - 4Rr] - 2r^2(r+4R)\right]$$

$$\Rightarrow V_mS^2 = \frac{R}{(2R-r)^2}\left[4R^3 + Rr^2 - 4R^2r - 2r^3 - 8Rr^2\right]$$

$$\Rightarrow V_mS^2 = \frac{R}{(2R-r)^2}\left[4R^3 - 4R^2r - 7Rr^2 - 2r^3\right]$$

$$\therefore \quad V_mS^2 = \frac{R}{(2R-r)^2}\left[4R^3 - 4R^2r - 7Rr^2 - 2r^3\right]$$

$$= \frac{R}{(2R-r)^2}\left[(R-2r)(2R+r)^2\right] = R(R-2r)\left[\frac{2R+r}{2R-r}\right]^2$$

$\therefore \quad V_mS^2 = R(R-2r)\left[\frac{2R+r}{2R-r}\right]^2$

II. If Vivya's point (V_m) and incentre (I) of a triangle then $V_mI^2 = \frac{4Rr^2(R-2r)}{(2R-r)^2}$

Proof :

If we put instead of X in V_mX^2 as I (In centre) we can get a relation to find the distance between our "Vivya's point and Incentre.

$$V_mI^2 = \frac{2R\sin^2(A/2)}{2R-r}AI^2 + \frac{2R\sin^2(B/2)}{2R-r}BI^2 + \frac{2R\sin^2(C/2)}{2R-r} - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

And we know that

$$AI = \frac{r}{\sin A/2}, BI = \frac{r}{\sin B/2}, CI = \frac{r}{\sin C/2} \Rightarrow r^2 = AI^2\sin^2 A/2 = BI^2 \sin B/2 = CI^2\sin^2 C/2$$

$$\Rightarrow V_mI^2 = \frac{2Rr^2}{2R-r} + \frac{2Rr^2}{2R-r} + \frac{2Rr^2}{2R-r} - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$\Rightarrow V_mI^2 = \frac{6Rr^2}{2R-r} - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$\Rightarrow V_mI^2 = \frac{2Rr^2}{2R-r}[3(2R-r)-(r+4R)]$$

$$\Rightarrow V_mI^2 = \frac{2Rr^2}{2(R-r)^2}[2R-4r]$$

$$\Rightarrow V_mI^2 = \frac{4Rr^2(R-2r)}{(2R-r)^2}$$

III. The relation between the V_mS, V_mI, IS is $\frac{V_mS^2 - IS^2}{V_mI^2} = \frac{2R}{r}$

Proof : We know that $SI^2=R^2-2Rr=R[R-2r]$ where S, I are circum centre, In centre.

By previous theorems we have

$$V_mS^2 = R(R-2r)\left[\frac{2R+r}{2R-r}\right]^2 \;\&\; V_mI^2 = \frac{4Rr^2(R-2r)}{(2R-r)^2}$$

Now $\frac{(2R-r)^2}{R}[V_mS^2 - SI^2] = \frac{(2R-r)^2}{R}\left[\frac{R}{(2R-r)^2}\left(4R^3-4R^2r-7Rr^2-2r^3\right) - R(R-2r)\right]$

$= [4R^3-4R^2r-7R^2-2r^3-(R-2r)(2R-r)^2]$

$= 4R^3-4R^2r-7Rr^2-2r^3-(R-2r)(4R^2+r^2-4Rr)$

$= 4R^3-4R^2r-7Rr^2-2r^3-[4R^3+Rr^2-4R^2r-8R^2r-2r^3+8Rr^2]$

$= 8R^2r-16Rr^2$

$= 8Rr(R-2r)$

$$= \frac{8Rr^2(R-2r)(2R-r)^2}{(2R-r)^2(r)}$$

$$= \left[\frac{4Rr^2(R-2r)}{(2R-r)}\right]\left[\frac{2(2R-r)^2}{r}\right] = (V_mI^2)\left(\frac{2}{r}\right)(2R-r)^2$$

$$\therefore \frac{(2R-r)^2}{R}[V_mS^2-IS^2] = \frac{2}{r}(2R-r)^2(V_mI^2)$$

$\Rightarrow r(V_mS^2 - IS^2) = 2R(V_mI^2)$

$$\Rightarrow r\left[V_mS^2 - IS^2\right] = 2R\left(V_mI^2\right)$$

$$\Rightarrow \frac{V_mS^2 - IS^2}{V_mI^2} = \frac{2R}{r}$$

IV. If V_mS, V_mI are the distances from Vivya's point (V_m) to circumcentre (S), Incentre (I) then $\frac{V_mS}{V_mI} = \frac{2R+r}{2r}$

Proof :

We know by previous theorem

$V_mS^2 = R(R-2r)\left[\frac{2R+r}{2R-r}\right]^2$, $V_mI^2 = \frac{4Rr^2(R-2r)}{(2R-r)^2}$ & $SI^2=R^2-2Rr=R[R-2r]$

So clearly we have $\frac{V_mS^2}{R(2R+r)^2} = \frac{V_mI^2}{4Rr^2}$

$$\Rightarrow \quad \frac{V_mS}{2R+r} = \frac{V_mI}{2r}$$

$$\Rightarrow \quad \frac{V_mS}{V_mI} = \frac{2R+r}{2r}$$

V. If W_e is the Weill's point then $W_eI^2 = \left(1+\frac{r}{3R}\right)^2\left(R^2 - 2Rr\right)$

Proof :

If W_e is a Weill's point and X is any point in the plane of Δ^{le}, then we have

$$W_eX^2 = \frac{1}{3}\sum\left(\frac{s-b}{c}+\frac{s-c}{b}\right)AX^2 - \frac{r}{9R}\left(r+4R\right)\left(2r+3R\right)$$

Put X = S where $AS = BS = CS = R$

$$\Rightarrow W_eS^2 = \frac{1}{3}\sum\left(\frac{s-b}{c}+\frac{s-c}{b}\right)AS^2 - \frac{r}{4R}\left(r+4R\right)\left(2r+3R\right)$$

$$= \frac{1}{3}\sum\left(\frac{s-b}{c}+\frac{s-c}{b}\right)R^2 - \frac{r}{9R}\left(r+4R\right)\left(2r+3R\right)$$

$$= \frac{1}{3}R^2\left[\frac{s-b}{c}+\frac{s-c}{b}+\frac{s-a}{b}+\frac{s-b}{a}+\frac{s-c}{a}+\frac{s-a}{c}\right] - \frac{r}{9R}\left(r+4R\right)\left(2r+3R\right)$$

$$= \frac{1}{3}R^2\left[1+1+1\right] - \frac{r}{9R}\left(r+4R\right)\left(2r+3R\right)$$

$$= R^2 - \frac{r}{9R}\left(r+4R\right)\left(2r+3R\right)$$

$$= \frac{9R^3 - r\left(2r^2+3Rr+8Rr+12R^2\right)}{9R}$$

$$= \frac{9R^3 - 2r^3 - 11Rr^2 - 12R^2r}{9R}$$

$$= \frac{\left(R^2-2Rr\right)\left(9R^2+r^2+6Rr\right)}{9R^2}$$

$$= \left(\frac{3R+r}{3R}\right)^2\left(R^2-2Rr\right)$$

$$= \left(\frac{3R+r}{3R}\right)^2\left(R^2-2Rr\right)$$

$$= \left(\frac{3R+r}{3R}\right)^2 SI^2$$

$$= \left(\left(\frac{3R+r}{3R}\right)SI\right)^2 = \left(\left(1+\frac{r}{3R}\right)SI\right)^2$$

$$\therefore W_eS^2 = \left[\left(1+\frac{r}{3R}\right)SI\right]^2$$

$$\therefore W_eS = \left(1+\frac{r}{3R}\right)SI$$

VI. *If W_e is the Weill's point then* $W_eI^2 = \dfrac{r^2(R-2r)}{9R}$

Proof :

If W_e is a Weill 's point and X is any point in the plane of Δ^{le}, then we have

$$W_eX^2 = \frac{1}{3}\sum\left(\frac{s-b}{c}+\frac{s-c}{b}\right)AX^2 - \frac{r}{9R}(r+4R)(2r+3R)$$

Put X=I and $AI = \dfrac{r}{Sin\,A/2} = \sqrt{r^2+(s-a)^2}$

$$BI = \frac{r}{Sin\,B/2} = \sqrt{r^2+(s-b)^2}$$

$$CI = \frac{r}{Sin\,C/2} = \sqrt{r^2+(s-c)^2}$$

So $W_eI^2 = \dfrac{1}{3}\left[\sum\left(\dfrac{s-b}{c}+\dfrac{s-c}{b}\right)AI^2\right] - \dfrac{r}{9R}(r+4R)(2r+3R)$

$$= \frac{1}{3}\left[\sum\left(\frac{s-b}{c}+\frac{s-c}{b}\right)\left(r^2+(s-a)^2\right)\right] - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= \frac{1}{3}\left[r^2\left(\frac{s-b}{c}+\frac{s-c}{b}+\frac{s-a}{b}+\frac{s-b}{a}+\frac{s-a}{c}+\frac{s-c}{a}\right)\right.$$

$$+\frac{(s-a)^2(s-b)}{c}+\frac{(s-a)^2(s-c)}{b}+\frac{(s-b)^2(s-a)}{c}+\frac{(s-b)^2(s-c)}{a}$$

$$\left.+\frac{(s-c)^2(s-a)}{b}+\frac{(s-c)^2(s-b)}{a}\right] - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= \frac{1}{3}r^2[3] + \frac{1}{3}\frac{(s-a)(s-b)}{c}[s-a+s-b] + \frac{1}{3}\frac{(s-b)(s-c)}{a}[s-b+s-c]$$

$$+\frac{1}{3}\frac{(s-c)(s-a)}{b}[s-c+s-a] - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= r^2 + \frac{1}{3}\left[(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\right] - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= r^2 + \frac{1}{3}\left[3S^2 - 2S(a+b+c)+ab+bc+ca\right] - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= r^2 + \frac{1}{3}\left[3s^2 - 2s(a+b+c) + s^2 + r^2 + 4Rr\right] - \frac{r}{9R}\left[r+4R\right]\left[2r+3R\right]$$

$$= r^2 + \frac{1}{3}\left[r^2 + 4Rr\right] - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= \frac{4r^2+4Rr}{3} - \frac{r}{9R}(r+4R)(2r+3R)$$

$$= \frac{12r^2R + 12R^2r - 2r^3 - 11r^2R - 12R^2r}{9R}$$

$$= \frac{r^2R - 2r^3}{9R} = \frac{r^2(R-2r)}{9R} = \frac{r^2(R^2-2Rr)}{9R^2}$$

$$\therefore W_eI^2 = \frac{r^2(R^2-2Rr)}{9R^2} = \left(\frac{r}{3R}\right)^2 SI^2 = \left(\frac{r}{3R}SI\right)^2$$

$$\therefore W_eI = \frac{r}{3R}SI$$

VII. *If W_e is the Weill's point and V_m is vivya's point then* $V_mW_e^2 = \frac{r^2(r+4R)^2(R^2-2Rr)}{9R^2(2R-r)^2}$

Proof :

we have by theorem 2

$$V_mX^2 = \sum \frac{a(s-b)(s-c)}{K}AX^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

Put X= W_e and

$$AW_e^2 = \frac{1}{3}\left(\frac{s-a}{c} + \frac{s-c}{a}\right)b^2 + \frac{1}{3}\left(\frac{s-a}{b} + \frac{s-b}{a}\right)c^2 - \frac{r}{9R}(r+4R)(2r+3R)$$

$$BW_e^2 = \frac{1}{3}\left(\frac{s-b}{c} + \frac{s-c}{b}\right)a^2 + \frac{1}{3}\left(\frac{s-b}{a} + \frac{s-a}{b}\right)c^2 - \frac{r}{9R}(r+4R)(2r+3R)$$

$$CW_e^2 = \frac{1}{3}\left(\frac{s-c}{a} + \frac{s-a}{c}\right)b^2 + \frac{1}{3}\left(\frac{s-c}{b} + \frac{s-b}{c}\right)a^2 - \frac{r}{9R}(r+4R)(2r+3R)$$

so $\quad V_mW_e^2 = \sum \frac{a(s-b)(s-c)}{K}AW_e^2 - \frac{2Rr^2(r+4R)}{(2R-r)^2}$

$$= \sum \frac{a(s-b)(s-c)}{K}\left(\frac{1}{3}\left(\frac{s-c}{a} + \frac{s-a}{c}\right)b^2 + \frac{1}{3}\left(\frac{s-c}{b} + \frac{s-b}{c}\right)a^2 - \frac{r}{9R}(r+4R)(2r+3R)\right) - \frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$= \sum \frac{1}{3K}\left[(s-b)(s-c)^2c^2 + (s-c)(s-b)^2b^2 + ac(s-a)(s-b)(s-c) + ab(s-a)(s-b)(s-c)\right]$$

$$-\frac{r}{9RK}(r+4R)(2r+3R)\left[\sum a(s-b)(s-c)\right]-\frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$=\frac{1}{3K}\sum(s-b)(s-c)\left[c^2(s-c)+b^2(s-b)\right]+\frac{1}{3K}\sum(s-a)(s-b)(s-c)\left[ab+ac\right]$$

$$-\frac{r}{9RK}(r+4R)(2r+3R)\left[s^2(a+b+c)-2s(ab+bc+ca)-3abc\right]-\frac{2Rr^2(r+4R)}{(2R-r)^2}$$

$$=\frac{1}{3K}\sum(s-b)(s-c)\left[c^2(s-c)+b^2(s-b)\right]+\frac{2}{3K}(s-a)(s-b)(s-c)\left[ab+bc+ac\right]$$

$$-\frac{r}{9RK}(r+4R)(2r+3R)\left[s^2(2s)-2s(r^2+s^2+4Rr)-34Rrs\right]-\frac{2Rr^2(r+4R)}{(2R-r)^2}$$

Where K = 2rs (2R-r) by brute force algebraic manipulation we get

$$V_mW_e^{\,2}=\frac{r^2(r+4R)^2(R^2-2Rr)}{9R^2(2R-r)^2}=\frac{r^2(r+4R)^2\,SI^2}{9R^2(2R-r)^2}=\left(\frac{r\;(r+4R)SI}{3R(2R-r)}\right)^2$$

(where SI is the distance between the circumcenter and incenter)

hence $V_mW_e=\left(\frac{r\;(r+4R)SI}{3R(2R-r)}\right)$

Theorem 3 :

Prove that Vivya's point (V_m), circumcentre (S), Incentre (I) are collinear. The line through these points are called as Vivya's line.

Proof:

<u>Method - I</u>

By previous theorems

We have $V_ms^2=\frac{R(R-2r)}{(2R-r)^2}(2R+r)^2$ and $V_mI^2=\frac{R(R-2r)}{(2R-r)^2}(2r)^2$

So clearly $V_ms^2>,V_mI^2$

We have

$\frac{V_mS^2-IS^2}{V_mI^2}=\frac{2R}{r}$ and $\frac{V_mS}{V_mI}=\frac{2R+r}{2r}$

$\Rightarrow\frac{V_mS^2-IS^2}{V_mI^2}=\frac{2R}{r}$ and $\frac{V_mS}{V_mI}=\frac{1}{2}\left(\frac{2R}{r}\right)+\frac{1}{2}$

$$\Rightarrow \frac{V_mS}{V_mI} = \frac{1}{2}\left[\frac{V_mS^2 - IS^2}{V_mI^2}\right] + \frac{1}{2}$$

$$\Rightarrow 2V_mI.V_mS = V_mS^2 - IS^2 + V_mI^2$$

$$\Rightarrow IS^2 = \left[V_mS - V_mI\right]^2$$

$$V_mS = V_mI + IS$$

<u>Method - II</u>

We know that $V_mS = \sqrt{R(R-2r)}\,\frac{2R+r}{2R-r}$

$$V_mI = \sqrt{R(R-2r)}\,\frac{2r}{2R-r}$$

$$IS = \sqrt{R(R-2r)}$$

$$V_mI + IS = \sqrt{R(R-2r)}\left[\frac{2R-r+2r}{2R-r}\right] = \sqrt{R(R-2r)}\left[\frac{2R+r}{2R-r}\right] = V_mS$$

$\therefore$ The Vivya 's point (V_m), circumcentre (S), Incentre (I) are collinear.

and $\frac{V_mI}{IS} = \frac{2r}{2R-r}$

$\therefore$ Incentre (I) Divides the line joining of V_m, S in the ratio 2r : 2R-r

Theorem : 4

Prove that weill's point also lies on the vivya's line.

Proof :

we have

$$V_mW_e = \left(\frac{r\ (r+4R)SI}{3R(2R-r)}\right)$$

$$W_eS = \left(1 + \frac{r}{3R}\right)SI$$

$$V_mI = \frac{2r}{2R-r}SI$$

$$V_mS = \frac{2R+r}{2R-r}SI$$

$SI = (R^2 - 2Rr)$

consider

$$V_mW_e + W_eI = \frac{r}{3R}\left[\frac{r+4R+2R-r}{2R-r}\right](SI)$$

$$= \frac{2r}{2R-r}(SI)$$

$$= V_mI$$

$$V_mW_e + W_eI = V_mI \quad \ldots\ldots\ldots\ldots\ldots(1)$$

and similarly consider

$$W_eI + IS = \frac{r}{3R}SI + SI = \left(\frac{r}{3R}+1\right)SI$$

$$= W_eS$$

$$W_eI + IS = W_eS \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1), V_m, W_e *and* I are collinear and W_e lies in between V_m *and* I

From (2), W_e, I, S are collinear and I lies in between W_e and S

i.e., V_m, W_e, I, S are collinear and they lies on line Join of V_m, I, S.

So The point We is also lies on the Vivya's line

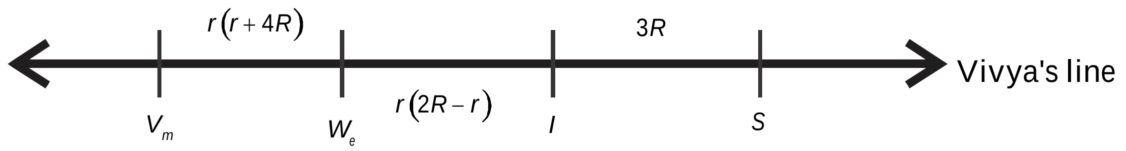
Now.

$V_mW_e : W_eI : IS$

$$= \frac{r(r+4R)}{3R(2R-r)}SI : \frac{r}{3R}SI : SI$$

$$= r(r+4R) : r(2R-r) : 3R$$

i.e., the points V_m, W_e, I, S are collinear and $V_mW_e : W_eI : IS = r(r+4R) : r(2R-r) : 3R$



In the similar argument what ever we adopted by using 'Maneeal's Identity', we can also prove that the Vivya's point lies on the line join of different triangle centres. Such as Vivya's point also lies on the line join of spieker centre and schiffler point. (I am omitting proof of this statement) etc., Proving of famous Inequalities using Vivya's point :

I. If R, r are circumradii, inradii then prove that $\frac{R}{r} \geq 2$ [Euler's inequality]

Proof :

We already proved that $V_m I^2 = \frac{4Rr^2(R-2r)}{(2R-r)^2}$

Since $V_m I^2 \geq o \quad \Rightarrow \frac{4Rr^2(R-2r)}{(2R-r)^2} \geq o$

$$\Rightarrow R - 2r \geq o \left[\because \frac{4Rr^2}{2(R-r)^2} > o \right]$$

$$\Rightarrow R \geq 2r$$

$$\Rightarrow \frac{R}{r} \geq 2$$

Which is a famous Euler's Inequality.

II. If R, r are circumradii, inradii then prove that $4R^3 \geq 4R^2 r - 7Rr^2 - 2r^3$

[Maneeals inequality]

Proof :

We already proved that

$$V_m S^2 = \frac{R}{(2R-r)^2}\left[4R^3 - 4R^2 r - 7Rr^2 - 2r^3\right]$$

Since $V_m S^2 \geq o \Rightarrow \frac{R}{(2R-r)^2}\left[4R^3 - 4R^2 r - 7Rr^2 - 2r^3\right] \geq o$

$$\Rightarrow 4R^3 - 4R^2 r - 7Rr^2 - 2r^3 \geq o \left[\because \frac{R}{(2R-r)^2} > o \right]$$

$$\Rightarrow 4R^3 \geq 4R^2 r - 7Rr^2 - 2r^3$$

Which is a famous maneeal's inequality

III. Prove that Euler's inequality using Maneeals inequality.

Proof :

Consider $\Rightarrow 4R^3 \geq 4R^2 r - 7Rr^2 - 2r^3$

$\Rightarrow 4R^3 \geq 4R^2r - 7Rr^2 - 2r^3 \geq o$

$\Rightarrow 4R^3 + 4R^2r + Rr^2 - 8R^2r - 8Rr^2 - 2r^3 \geq o$

$\Rightarrow R[4R^2 + 4Rr + r^2] - 2r[4R^2 + 4Rr + r^2] \geq o$

$\Rightarrow [R - 2r]\left[4R^2 + 4Rr + r^2\right] \geq o$

$\Rightarrow [R - 2r][2R + r]^2 \geq o$

$\Rightarrow R - 2r \geq o \quad \left(\because (2R + r)^2 > o\right)$

$\Rightarrow R \geq 2r$

$\Rightarrow \dfrac{R}{r} \geq 2$ Which is a famous Euler 's inequality

IV. Prove that if V_mS, V_mI, IS are the distances from Vivya 's point to Circumcentre, Incentre and Circumcentre to Incentre respectively. Then prove that $V_mS^2 \geq 4V_mI^2 + IS^2$.

Proof :

We already proved that

$$r(V_ms^2 - Is^2) = 2R\left[V_mI^v\right]$$

$$\Rightarrow \frac{V_ms^2 - IS^2}{V_mI^2} = \frac{2R}{r}$$

But $\dfrac{R}{r} \geq 2 \quad \Rightarrow \dfrac{2R}{r} \geq 4$

So $\dfrac{V_mS^2 - IS^2}{V_mI^2} \geq 4$

$\Rightarrow V_mS^2 - IS^2 \geq 4V_mI^2$

$\Rightarrow V_mS^2 \geq 4V_mI^2 + IS^2$

These are the very few properties for the Vivya 's point.

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