

On general multiple Eulerian integrals involving the multivariable I-function

of Prasad, a general class of polynomials and the

Aleph-function of one variable

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ABSTRACT

The object of this paper is first, to evaluate a general multiple Eulerian integrals with general integrand involving the product of a multivariable I-function defined by Prasad [4], a general class of polynomials, the Aleph-function of one variable and generalized hypergeometric function with general arguments. The second multiple Eulerian integral contain a general class of polynomials, the general polynomial set, the Aleph-function of one variable and multivariable I-function with general arguments. Our integral formulas are interesting and unified nature.

Keywords : Multivariable H-function, class of polynomial, general polynomials set, multiple Eulerian integral, Aleph-function of one variable, multivariable I-function.

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1. Introduction

In this paper, we evaluate two multiple Eulerian integrals involving the multivariable I-function defined by Prasad [4], the Aleph-function of one variable and multivariable class of polynomials with general arguments.

The multivariable I-function defined by Prasad [3] is an extension of the multivariable H-function defined by Srivastava et al [10]. We will use the contracted form.

The multivariable I-function of r -variables is defined in terms of multiple Mellin-Barnes type integral :

$$I(z_1, z_2, \dots, z_r) = I_{p_2, q_2, p_3, q_3, \dots; p_r, q_r; p^{(1)}, q^{(1)}, \dots; p^{(r)}, q^{(r)}}^{0, n_2; 0, n_3; \dots; 0, n_r; m^{(1)}, n^{(1)}, \dots; m^{(r)}, n^{(r)}} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \middle| \begin{matrix} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; \dots; \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; \dots; \end{matrix} \right)$$

$$(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)})_{1, p_r} : (a_j^{(1)}, \alpha_j^{(1)})_{1, p^{(1)}}; \dots; (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \left(\begin{matrix} (b_{rj}; \beta_{rj}^{(1)}, \dots, \beta_{rj}^{(r)})_{1, q_r} : (b_j^{(1)}, \beta_j^{(1)})_{1, q^{(1)}}; \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{matrix} \right) \quad (1.1)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi(s_1, \dots, s_r) \prod_{i=1}^r \theta_i(t_i) z_i^{t_i} dt_1 \dots dt_r \quad (1.2)$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [4]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_i| < \frac{1}{2}\Omega_i\pi, \text{ where}$$

$$\begin{aligned} \Omega_i = & \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \dots + \\ & \left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \dots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \end{aligned} \quad (1.3)$$

where $i = 1, \dots, r$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function. We may establish the asymptotic expansion in the following convenient form :

$$I(z_1, \dots, z_r) = 0(|z_1|^{\alpha_1}, \dots, |z_r|^{\alpha_r}), \max(|z_1|, \dots, |z_r|) \rightarrow 0$$

$$I(z_1, \dots, z_r) = 0(|z_1|^{\beta_1}, \dots, |z_r|^{\beta_r}), \min(|z_1|, \dots, |z_r|) \rightarrow \infty$$

where $k = 1, \dots, r : \alpha'_k = \min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \dots, m_k$ and

$$\beta'_k = \max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \dots, n_k$$

We will use these following notations in this section :

$$U_r = p_2, q_2; p_3, q_3; \dots; p_{r-1}, q_{r-1}; V_r = 0, n_2; 0, n_3; \dots; 0, n_{r-1} \quad (1.4)$$

$$W_r = (p^{(1)}, q^{(1)}); \dots; (p^{(r)}, q^{(r)}); X_r = (m^{(1)}, n^{(1)}); \dots; (m^{(r)}, n^{(r)}) \quad (1.5)$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \dots; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \dots, \alpha_{(r-1)k}^{(r-1)}) \quad (1.6)$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \dots; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \dots, \beta_{(r-1)k}^{(r-1)}) \quad (1.7)$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \dots, \alpha_{rk}^{(r)}) : \mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \dots, \beta_{rk}^{(r)}) \quad (1.8)$$

$$A_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \dots; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; B_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \dots; (b_k^{(r)}, \beta_k^{(r)})_{1,q^{(r)}} \quad (1.9)$$

The multivariable I-function of r-variables write :

$$I(z_1, \dots, z_r) = I_{U_r; p_r, q_r; W_r}^{V_r; 0, n_r; X_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \middle| \begin{matrix} A; \mathfrak{A}; A_1 \\ \\ \\ B; \mathfrak{B}; B_1 \end{matrix} \right) \quad (1.10)$$

The Aleph- function , introduced by Südländ [11] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i, Q_i, c_i; r'}^{M, N} \left(z \middle| \begin{matrix} (a_j, A_j)_{1, n}, [c_i(a_{ji}, A_{ji})]_{n+1, p_i; r} \\ (b_j, B_j)_{1, m}, [c_i(b_{ji}, B_{ji})]_{m+1, q_i; r} \end{matrix} \right) = \frac{1}{2\pi\omega} \int_L \Omega_{P_i, Q_i, c_i; r'}^{M, N}(s) z^{-s} ds \quad (1.11)$$

for all z different to 0 and

$$\Omega_{P_i, Q_i, c_i; r'}^{M, N}(s) = \frac{\prod_{j=1}^M \Gamma(b_j - B_j s) \prod_{j=1}^N \Gamma(1 - a_j + A_j s)}{\sum_{i=1}^{r'} c_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} - A_{ji} s) \prod_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} + B_{ji} s)} \quad (1.12)$$

$$\text{With : } |\arg z| < \frac{1}{2}\pi\Omega \quad \text{Where } \Omega = \sum_{j=1}^M \beta_j + \sum_{j=1}^N \alpha_j - c_i \left(\sum_{j=M+1}^{Q_i} \beta_{ji} + \sum_{j=N+1}^{P_i} \alpha_{ji} \right) > 0; i = 1, \dots, r'$$

For convergence conditions and other details of Aleph-function , see Südländ et al [11], the serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i, Q_i, c_i; r'}^{M, N}(z) = \sum_{G=1}^M \sum_{g=0}^{\infty} \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M, N}(s)}{B_G g!} z^s \quad (1.13)$$

$$\text{With } s = \eta_{G, g} = \frac{b_G + g}{B_G}, P_i < Q_i, |z| < 1 \text{ and } \Omega_{P_i, Q_i, c_i, r'}^{M, N}(s) \text{ is given in (1.2)} \quad (1.14)$$

The generalized polynomials defined by Srivastava [7], is given in the following manner :

$$S_{N_1, \dots, N_u}^{M_1, \dots, M_u}[y_1, \dots, y_u] = \sum_{K_1=0}^{[N_1/M_1]} \dots \sum_{K_u=0}^{[N_u/M_u]} \frac{(-N_1)_{M_1 K_1}}{K_1!} \dots \frac{(-N_u)_{M_u K_u}}{K_u!} A[N_1, K_1; \dots; N_u, K_u] y_1^{K_1} \dots y_u^{K_u} \quad (1.15)$$

Where M_1, \dots, M_u are arbitrary positive integers and the coefficients $A[N_1, K_1; \dots; N_u, K_u]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_u = \frac{(-N_1)_{M_1 K_1}}{K_1!} \dots \frac{(-N_u)_{M_u K_u}}{K_u!} A[N_1, K_1; \dots; N_u, K_u] \quad (1.16)$$

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

2. Sequence of functions

Agarwal and Chaubey [1], Salim [6] and several others have studied a general sequence of functions. In the present document we shall study the following useful series formula for a general sequence of functions.

$$R_n^{\alpha,\beta}[x; E, F, g, h; p, q; \gamma; \delta; e^{-sx^r}] = \sum_{w,v,u,t,e,k_1,k_2} \psi(w, v, u, t, e, k_1, k_2) x^R \quad (2.1)$$

$$\text{where } \sum_{w,v,u,t,e,k_1,k_2} = \sum_{w=0}^{\infty} \sum_{v=0}^n \sum_{u=0}^v \sum_{t=0}^n \sum_{e=0}^t \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \quad (2.2)$$

and the infinite series on the right side (2.1) is absolutely convergent, $R = ln + qv + pt + rw + k_1r + k_2q$ (2.3)

$$\text{and } \psi(w, v, u, t, e, k_1, k_2) = \frac{(-)^{t+w+k_2} (-v)_u (-t)_e (\alpha)_t l^n}{w!v!u!t!e!K_n k_1!k_2!} \frac{s^{w+k_1} F^{\gamma n-t}}{(1-\alpha-t)_e} (\alpha - \gamma n)_e$$

$$(-\beta - \delta n)_v g^{v+k_2} h^{\delta n-v-k_2} (v - \delta n)_{k_2} E^t \left(\frac{pe + rw + \lambda + qn}{l} \right)_n \quad (2.4)$$

where K_n is a sequence of constants. This function will note $R_n^{\alpha,\beta}[x]$

By suitably specializing the parameters involving in (2.1), a general sequence of function reduced to generalized polynomials set studied by Raizada [5], a class of polynomials introduced by Fujiwara [3] and several others authors.

3. Integral representation of generalized hypergeometric function

The following generalized hypergeometric function in terms of multiple contour integrals is also required [8 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_P F_Q [(A_P); (B_Q); -(x_1 + \dots + x_r)]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \dots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \dots + s_r)} \Gamma(-s_1) \dots \Gamma(-s_r) x_1^{s_1} \dots x_r^{s_r} ds_1 \dots ds_r \quad (3.1)$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \dots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \dots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \dots, r$

The equivalent form of Eulerian beta integral is :

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} dt = (b-a)^{\alpha+\beta} B(\alpha, \beta) (Re(\alpha) > 0, Re(\beta) > 0, a \neq b) \quad (3.2)$$

4. First integral

We note :

$$U = p_2, q_2; p_3, q_3; \dots; p_{r-1}, q_{r-1}; 0, 0; 0, 0; \dots; 0, 0; 0, 0; \dots; 0, 0 \quad (4.1)$$

$$V = 0, n_2; 0, n_3; \dots; 0, n_{r-1}; 0, 0; 0, 0; \dots; 0, 0; 0, 0; \dots; 0, 0 \quad (4.2)$$

$$X = m^{(1)}, n^{(1)}; \dots; m^{(r)}, n^{(r)}; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0 \quad (4.3)$$

$$Y = p^{(1)}, q^{(1)}; \dots; p^{(r)}, q^{(r)}; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1 \quad (4.4)$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \dots; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \dots, \alpha_{(r-1)k}^{(r-1)}) \quad (4.5)$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \dots; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \dots, \beta_{(r-1)k}^{(r-1)}) \quad (4.6)$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \dots, \alpha_{rk}^{(r)}, 0, \dots, 0, 0, \dots, 0) \quad (4.7)$$

$$\mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \dots, \beta_{rk}^{(r)}, 0, \dots, 0, 0, \dots, 0) \quad (4.8)$$

$$\mathfrak{A}_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \dots; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; (1, 0); \dots; (1, 0); (1, 0); \dots; (1, 0) \quad (4.9)$$

$$\mathfrak{B}_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \dots; (b_k^{(r)}, \beta_k^{(r)})_{1,q^{(r)}}; (0, 1); \dots; (0, 1); (0, 1); \dots; (0, 1) \quad (4.10)$$

$$A^* = [1 + \sigma'_i - \eta_{G,g}c'_i - \sum_{j=1}^u K_j \xi_j^{(1,j)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, \tau_i^{(1,1)}, \dots, \tau_i^{(1,l)}, 1, 0, \dots, 0]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - \eta_{G,g}c_i^{(T)} - \sum_{j=1}^u K_j \xi_j^{(T,j)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, \tau_i^{(T,1)}, \dots, \tau_i^{(T,l)}, 0, \dots, 0, 1]_{1,s},$$

$$[1 - A_j; 0, \dots, 0, 1, \dots, 1, 0, \dots, 0]_{1,P},$$

$$[1 - \alpha_i - \eta_{G,g}a_i - \sum_{j=1}^u K_j \alpha_i^{(j)}; \delta'_i, \dots, \delta_i^{(r)}, \mu'_i, \dots, \mu_i^{(l)}, 1, \dots, 1, 0, \dots, 0]_{1,s},$$

W-items (T-W)-items

$$[1 - \beta_i - \eta_{G,g}b_i - \sum_{j=1}^u K_j \beta_i^{(j)}; \eta'_i, \dots, \eta_i^{(r)}, \theta'_i, \dots, \theta_i^{(l)}, 0, \dots, 0, 1, \dots, 1]_{1,s} \quad (4.11)$$

W-items (T-W)-items

$$B^* = [1 + \sigma'_i - \eta_{G,g}c'_i - \sum_{j=1}^u K_j \xi_i^{(1,j)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, \tau_i^{(1,1)}, \dots, \tau_i^{(1,l)}, 0, \dots, 0]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - \eta_{G,g}c_i^{(T)} - \sum_{j=1}^u K_j \xi_i^{(T,j)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, \tau_i^{(T,1)}, \dots, \tau_i^{(T,l)}, 0, \dots, 0]_{1,s},$$

$$[1 - B_j; 0, \dots, 0, 1, \dots, 1, 0, \dots, 0]_{1,Q},$$

$$[1 - \alpha_i - \beta_i - \eta_{G,g}(a_i + b_i) - \sum_{j=1}^u K_j(\alpha_i^{(j)} + \beta_i^{(j)}); (\delta'_i + \eta'_i), \dots, (\delta_i^{(r)} + \eta_i^{(r)})$$

$$(\mu'_i + \theta'_i), \dots, (\mu_i^{(l)} + \theta_i^{(l)}), 1, \dots, 1]_{1,s} \quad (4.12)$$

We have the following multiple Eulerian integral and we obtain the I-function of $(r + l + T)$ -variables

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\mathfrak{N}_{P_i, Q_i, c_i; r'}^{M, N} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{c_i^{(j)}}} \right] \right]$$

$$S_{N_1, \dots, N_u}^{M_1, \dots, M_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\xi_i^{(j,u)}}} \right] \end{array} \right)$$

$$I_{U_r; p_r, q_r; W_r}^{V_r; 0, n_r; X_r} \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\rho_i^{(j,r)}}} \right] \end{array} \right)$$

$${}_PF_Q \left[(A_P); (B_Q); - \sum_{k=1}^l g_k \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\mu_i^{(k)}} (v_i - x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\tau_i^{(j,k)}}} \right] \right] dx_1 \cdots dx_s$$

$$= \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \right]$$

$$\sum_{G=1}^M \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} a_u \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M, N}(\eta_{G, g})}{B_G g!} y_1^{K_1} \cdots y_u^{K_u} \prod_{i=1}^s \left[(v_i - u_i)^{\eta_{G, g}(a_i + b_i) + \sum_{j=1}^u K_j(\alpha_i^{(j)} + \beta_i^{(j)})} \right]$$

$$z^{\eta_{G, g}} \prod_{i=1}^s \left[\prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G, g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j, l)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G, g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j, l)}} \right]$$

$$I_{U;p_r+sT+P+2s;X}^{V;p_r+sT+P+2s;q_r+sT+Q+s;Y} \left(\begin{array}{c|c} \begin{matrix} z_1 w_1 \\ \vdots \\ \vdots \\ z_r w_r \\ g_1 W_1 \\ \vdots \\ \vdots \\ g_l W_l \\ G_1 \\ \vdots \\ \vdots \\ G_T \end{matrix} & \begin{matrix} A ; A^*, \mathfrak{A}; \mathfrak{A}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ B ; B^*, \mathfrak{B}; \mathfrak{B}_1 \end{matrix} \end{array} \right) \quad (4.13)$$

Where

$$w_m = \prod_{i=1}^s \left[(v_i - u_i)^{\delta_i^{(m)} + \eta_i^{(m)}} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\rho_i^{(j, m)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\rho_i^{(j, m)}} \right], m = 1, \dots, r \quad (4.14)$$

$$W_k = \prod_{i=1}^s \left[(v_i - u_i)^{\mu_i^{(k)} + \theta_i^{(k)}} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\tau_i^{(j, k)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\tau_i^{(j, k)}} \right], k = 1, \dots, l \quad (4.15)$$

$$G_j = \prod_{i=1}^s \left[\frac{(v_i - u_i) U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = 1, \dots, W \quad (4.16)$$

$$G_j = - \prod_{i=1}^s \left[\frac{(v_i - u_i) U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = W + 1, \dots, T \quad (4.17)$$

Provided that :

$$\textbf{(A)} \ 0 \leq W \leq T; u_i, v_i \in \mathbb{R}; \min\{a_i, b_i, c_i^{(j)}, \alpha_i^{(k)}, \beta_i^{(k)}, \xi_i^{(j, k)}\} > 0, i = 1, \dots, s; k = 1, \dots, u; j = 1, \dots, T$$

$$\textbf{(B)} \ \min\{\delta_i^{(t)}, \eta_i^{(t)}, \rho_i^{(j, t)}, \mu_i^{(k)}, \theta_i^{(k)}, \tau_i^{(j, k)}\} \geq 0; j = 1, \dots, T; i = 1, \dots, s; k = 1, \dots, l; t = 1, \dots, r$$

$$\textbf{(C)} \ \sigma_i^{(j)} \in \mathbb{R}, U_i^{(j)}, V_i^{(j)} \in \mathbb{C}, z_t, g_k \in \mathbb{C}; j = 1, \dots, s; t = 1, \dots, r; k = 1, \dots, l; t = 1, \dots, r$$

$$(D) \max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = 1, \dots, W \text{ and}$$

$$\max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = W + 1, \dots, T$$

$$(F) \operatorname{Re} \left[\alpha_i + a_i \min_{1 \leq j \leq M} \frac{b_j}{B_j} + \sum_{t=1}^r \delta_i^{(t)} \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(t)}}{\beta_j^{(t)}} \right] > 0;$$

$$\operatorname{Re} \left[\beta_i + b_i \min_{1 \leq j \leq M} \frac{b_j}{B_j} + \sum_{t=1}^r \eta_i^{(t)} \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(t)}}{\beta_j^{(t)}} \right] > 0; i = 1, \dots, s$$

$$(G) \Omega_i = \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} \beta_j^{(i)} + \left(\sum_{j=1}^{n_2} \alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2} \alpha_{2j}^{(i)} \right) + \dots +$$

$$\left(\sum_{j=1}^{n_s} \alpha_{sj}^{(i)} - \sum_{j=n_s+1}^{p_s} \alpha_{sj}^{(i)} \right) - \left(\sum_{j=1}^{q_2} \beta_{2j}^{(i)} + \sum_{j=1}^{q_3} \beta_{3j}^{(i)} + \dots + \sum_{k=1}^{q_s} \beta_{sj}^{(i)} \right) - \delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^T \rho_i^{(j,k)} > 0$$

$(i = 1, \dots, s; k = 1, \dots, r)$

$$(H) \left| \arg \left(z_i \prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{-\rho_i^{(j,k)}} \right) \right| < \frac{1}{2} \Omega_i \pi$$

$$(I) \quad a_{ij}, b_{ik}, \in \mathbb{C} \ (i = 1, \dots, r; j = 1, \dots, p_i; k = 1, \dots, q_i); a_j^{(i)}, b_j^{(k)} \in \mathbb{C}$$

$$(i = 1, \dots, r; j = 1, \dots, p^{(i)}; k = 1, \dots, q^{(i)})$$

$$\alpha_{ij}^{(k)}, \beta_{ij}^{(k)} \in \mathbb{R}_+ \ ((i = 1, \dots, r, j = 1, \dots, p_i, k = 1, \dots, r); \alpha_j^{(i)}, \beta_i^{(i)} \in \mathbb{R}_+ \ (i = 1, \dots, r; j = 1, \dots, p_i))$$

$$(J) \ P \leq Q + 1. \text{ The equality holds, when , in addition,}$$

$$\text{either } P > Q \text{ and } \sum_{k=1}^l \left| g_k \left(\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{-\tau_i^{(j,k)}} \right) \right|^{\frac{1}{Q-P}} < 1 \quad (u_i \leq x_i \leq v_i; i = 1, \dots, s)$$

$$\text{or } P \leq Q \text{ and } \max_{1 \leq k \leq l} \left[\left| \left(g_k \prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{-\tau_i^{(j,k)}} \right) \right| \right] < 1 \quad (u_i \leq x_i \leq v_i; i = 1, \dots, s)$$

$$(K) \text{ The series occuring on the right-hand side of (4.5) are absolutely and uniformly convergent}$$

Proof

To establish the formula (4.13), we first use series representation (1.13) and (1.15) for $\aleph_{P_i, Q_i, c_i; r}^{M, N}(\cdot)$ and $S_{N_1, \dots, N_u}^{M_1, \dots, M_u}[\cdot]$ respectively, we use contour integral representation with the help of (1.1) for the multivariable I-function defined by Prasad [4] occurring in its left-hand side and use the contour integral representation with the help of (3.1) for the generalized hypergeometric function ${}_P F_Q(\cdot)$. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process).

Now we write :

$$\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}}$$

$$\text{where } K_i^{(j)} = \eta_i^{(j)} - \eta_{G,g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j,l)} - \sum_{t=1}^r \rho_i^{(j,t)} \zeta_t - \sum_{k=1}^l \tau_i^{(j,k)} \zeta_k \quad i = 1, \dots, s; j = 1, \dots, T \quad (4.18)$$

and express the factor occuring in R.H.S. Of (4.13) in terms of following Mellin-Barnes contour integral with the help of the result [9, page 18, eq.(2.6.4)]

$$\frac{1}{(2\pi\omega)^W} \int_{L'_1} \cdots \int_{L'_W} \prod_{j=1}^W \left[\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j) \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \right] = \prod_{j=1}^W \left[\frac{(U_i^{(j)} u_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right]$$

$$\prod_{j=1}^W \left[\frac{(U_i^{(j)} (x_i - u_i))}{(u_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta'_j} d\zeta'_1 \cdots d\zeta'_W \quad (4.11)$$

and

$$\frac{1}{(2\pi\omega)^{T-W}} \int_{L'_{W+1}} \cdots \int_{L'_{Tj=W+1}} \prod_{j=W+1}^T \left[\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j) \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \right] = \prod_{j=W+1}^T \left[\frac{(U_i^{(j)} v_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right]$$

$$\prod_{j=W+1}^T \left[-\frac{(U_i^{(j)} (v_i - x_i))}{(v_i U_i^{(j)} + V_i^{(j)})} \right]^{\zeta'_j} d\zeta'_{W+1} \cdots d\zeta'_T \quad (4.12)$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost \mathbf{x} -integral with the help of (3.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r + l + T)$ -variables, we obtain the formula (4.13).

5. Second formula

We note :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; 0, 0; \cdots; 0, 0 \quad (5.1)$$

$$V = 0, n_2; 0, n_3; \cdots; 0, n_{r-1}; 0, 0; \cdots; 0, 0 \quad (5.2)$$

$$X = m^{(1)}, n^{(1)}; \cdots; m^{(r)}, n^{(r)}; 1, 0; \cdots; 1, 0 \quad (5.3)$$

$$Y = p^{(1)}, q^{(1)}; \cdots; p^{(r)}, q^{(r)}; 0, 1; \cdots; 0, 1 \quad (5.4)$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \cdots; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots, \alpha_{(r-1)k}^{(r-1)}) \quad (5.5)$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \cdots; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots, \beta_{(r-1)k}^{(r-1)}) \quad (5.6)$$

$$\mathfrak{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)}, 0, \cdots, 0) \quad (5.7)$$

$$\mathfrak{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots, \beta_{rk}^{(r)}, 0, \cdots, 0) \quad (5.8)$$

$$\mathfrak{A}_1 = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; (1, 0); \cdots; (1, 0) \quad (5.9)$$

$$\mathfrak{B}_1 = (b_k^{(1)}, \beta_k^{(1)})_{1,p^{(1)}}; \dots; (b_k^{(r)}, \beta_k^{(r)})_{1,p^{(r)}}; (0, 1); \dots; (0, 1) \quad (5.10)$$

$$A^* = [1 + \sigma'_i - \theta'_i R - \eta_{G,g} c'_i - \sum_{j=1}^u K_j \xi_i^{(1,j)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, 1, 0, \dots, 0]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - \theta_i^{(T)} R - \eta_{G,g} c_i^{(T)} - \sum_{j=1}^u K_j \xi_i^{(T,j)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, 0, \dots, 0, 1]_{1,s},$$

$$[1 - \alpha_i - \zeta_i R - \eta_{G,g} a_i - \sum_{j=1}^u K_j \alpha_i^{(j)}; \delta'_i, \dots, \delta_i^{(r)}, 1, \dots, 1, 0, \dots, 0]_{1,s},$$

W-items (T-W)-items

$$[1 - \beta_i - \lambda_i R - \eta_{G,g} b_i - \sum_{j=1}^u K_j \beta_i^{(j)}; \eta'_i, \dots, \eta_i^{(r)}, 0, \dots, 0, 1, \dots, 1]_{1,s} \quad (5.11)$$

W-items (T-W)-items

$$B^* = [1 + \sigma'_i - \theta'_i R - \eta_{G,g} c'_i - \sum_{j=1}^u K_j \xi_i^{(1,j)}; \rho_i^{(1,1)}, \dots, \rho_i^{(1,r)}, 0, \dots, 0]_{1,s}, \dots,$$

$$[1 + \sigma_i^{(T)} - \theta_i^{(T)} R - \eta_{G,g} c_i^{(T)} - \sum_{j=1}^u K_j \xi_i^{(T,j)}; \rho_i^{(T,1)}, \dots, \rho_i^{(T,r)}, 0, \dots, 0]_{1,s},$$

$$[1 - \alpha_i - \beta_i (\zeta_i + \lambda_i) R - \eta_{G,g} (a_i + b_i) - \sum_{j=1}^u K_j (\alpha_i^{(j)} + \beta_i^{(j)}); (\delta'_i + \eta'_i), \dots, (\delta_i^{(r)} + \eta_i^{(r)})$$

$$(\mu'_i + \theta'_i), \dots, (\mu_i^{(l)} + \theta_i^{(l)}), 1, \dots, 1]_{1,s} \quad (5.12)$$

We have the following multiple Eulerian integral

$$\int_{u_1}^{v_1} \dots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\mathfrak{N}_{P_i, Q_i, c_i; r'}^{M, N} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right] \right] R_n^{\alpha, \beta} \left[Z \prod_{j=1}^s \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\theta_i^{(j)}}} \right] \right]$$

$$S_{N_1, \dots, N_u}^{M_1, \dots, M_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\xi_i^{(j,u)}}} \right] \end{array} \right)$$

$$I_{U_r;p_r,q_r;W_r}^{V_r;0,n_r;X_r} \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{\rho_i^{(j,r)}}} \right] \end{array} \right) dx_1 \cdots dx_s$$

$$= \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \right]$$

$$\sum_{G=1}^M \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} a_u \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M, N}(\eta_{G, g})}{B_G g!} y_1^{K_1} \cdots y_u^{K_u}$$

$$\sum_{w,v,u,t,e,k_1,k_2} \psi'(w,v,u,t,e,k_1,k_2) Z^R \prod_{i=1}^s \left[(v_i - u_i)^{\eta_{G,g}(a_i+b_i) + \sum_{j=1}^u K_j(\alpha_i^{(j)} + \beta_i^{(j)})} \right]$$

$$\prod_{i=1}^s \left[\prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G,g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j,l)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G,g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j,l)}} \right]$$

$$I_{U:p_r+sT+2s;X}^{V;p_r+sT+2s;q_r+sT+s;Y} \left(\begin{array}{c|c} \begin{matrix} z_1 w_1 \\ \vdots \\ z_r w_r \\ \vdots \\ G_1 \\ \vdots \\ G_T \end{matrix} & \begin{matrix} A^* ; A^*, \mathfrak{A}; \mathfrak{A}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ B^* ; B^*, \mathfrak{B}; \mathfrak{B}_1 \end{matrix} \end{array} \right) \quad (5.13)$$

where

$$\psi'(w,v,u,t,e,k_1,k_2) = \frac{\psi(w,v,u,t,e,k_1,k_2) \prod_{i=1}^s (v_i - u_i)^{(\zeta_i + \lambda_i)R}}{\prod_{i=1}^s \left[\prod_{j=1}^W (u_i U_i^{(j)} + V_i^{(j)})^{\theta_i^{(j)} R} \prod_{j=W+1}^T (u_i U_i^{(j)} + V_i^{(j)})^{\theta_i^{(j)} R} \right]} \quad (5.14)$$

$\psi(w,v,u,t,e,k_1,k_2)$ and R are given by (2.4) and (2.3) respectively.

$$w_l = \prod_{i=1}^s \left[(v_i - u_i)^{\delta_i^{(l)} + \eta_i^{(l)}} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\rho_i^{(j,l)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\rho_i^{(j,l)}} \right], l = 1, \dots, r \quad (5.15)$$

$$G_j = \prod_{i=1}^s \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = 1, \dots, W \quad (5.16)$$

$$G_j = - \prod_{i=1}^s \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = W + 1, \dots, T \quad (5.17)$$

Provided that :

$$\textbf{(A)} \quad 0 \leq W \leq T; u_i, v_i \in \mathbb{R}; \min\{a_i, b_i, c_i^{(j)}, \alpha_i^{(k)}, \beta_i^{(k)}, \xi_i^{(j,k)}\} > 0, i = 1, \dots, s; k = 1, \dots, u; j = 1, \dots, T$$

$$\textbf{(B)} \quad \min\{\zeta_i, \lambda_i, \theta_i^{(j)}, \delta_i^{(l)}, \eta_i^{(l)}, \rho_i^{(j,l)}\} \geq 0; j = 1, \dots, T; i = 1, \dots, s; l = 1, \dots, r$$

$$\textbf{(C)} \quad \operatorname{Re}(\alpha_i, \beta_i, v_i^{(j)}) > 0 (i = 1, \dots, s; j = 1, \dots, T); |\tau| < 1$$

$$\textbf{(D)} \quad \max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = 1, \dots, W \text{ and}$$

$$\max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \dots, s; j = W + 1, \dots, T$$

$$\begin{aligned} \textbf{(E)} \quad \Omega_i &= \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} \beta_j^{(i)} + \left(\sum_{j=1}^{n_2} \alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2} \alpha_{2j}^{(i)} \right) + \dots + \\ &\left(\sum_{j=1}^{n_s} \alpha_{sj}^{(i)} - \sum_{j=n_s+1}^{p_s} \alpha_{sj}^{(i)} \right) - \left(\sum_{j=1}^{q_2} \beta_{2j}^{(i)} + \sum_{j=1}^{q_3} \beta_{3j}^{(i)} + \dots + \sum_{k=1}^{q_s} \beta_{sj}^{(i)} \right) - \delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^T \rho_i^{(j,k)} > 0 \end{aligned}$$

$$(i = 1, \dots, s; k = 1, \dots, r)$$

$$\textbf{(G)} \quad \operatorname{Re} \left[\alpha_i + R\zeta_i + a_i \min_{1 \leq j \leq M} \frac{b_j}{B_j} + \sum_{t=1}^r \delta_i^{(t)} \min_{1 \leq j \leq m_i} \frac{d_j^{(t)}}{\delta_j'^{(t)}} \right] > 0;$$

$$\operatorname{Re} \left[\beta_i + R\lambda_i + b_i \min_{1 \leq j \leq M} \frac{b_j}{B_j} + \sum_{t=1}^r \eta_i^{(t)} \min_{1 \leq j \leq m_i} \frac{d_j^{(t)}}{\delta_j'^{(t)}} \right] > 0; i = 1, \dots, s$$

$$\textbf{(H)} \quad \left| \arg \left(z_i \prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{-\rho_i^{(j,k)}} \right) \right| < \frac{1}{2} \Omega_i \pi$$

$$\textbf{(I)} \quad a_{ij}, b_{ik} \in \mathbb{C} (i = 1, \dots, r; j = 1, \dots, p_i; k = 1, \dots, q_i); a_j^{(i)}, b_j^{(k)} \in \mathbb{C}$$

$$(i = 1, \dots, r; j = 1, \dots, p^{(i)}; k = 1, \dots, q^{(i)})$$

$$\alpha_{ij}^{(k)}, \beta_{ij}^{(k)} \in \mathbb{R}_+ ((i = 1, \dots, r, j = 1, \dots, p_i, k = 1, \dots, r); \alpha_j^{(i)}, \beta_i^{(i)} \in \mathbb{R}_+ (i = 1, \dots, r; j = 1, \dots, p_i))$$

(J) The series occuring on the right-hand side of (5.5) are absolutely and uniformly convergent

Proof

To establish the formula (5.13), we first use series representation (1.15), (1.17) and (3.1) for $\mathfrak{N}_{P_i, Q_i, c_i; r}^{M, N}(\cdot)$, $S_{N_1, \dots, N_u}^{M_1, \dots, M_u}[\cdot]$ and $R_n^{\alpha, \beta}[\cdot]$ respectively and the contour integral representation with the help of (1.2) for the multivariable I-function defined by Prasad [4] occurring in its left-hand side. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now, we write:

$$\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} \quad (5.18)$$

$$\text{where } K_i^{(j)} = \eta_i^{(j)} - \eta_{G, g} c_i^{(j)} - R \theta_i^{(j)} - \sum_{l=1}^u L_l \xi_i^{(j, l)} - \sum_{t=1}^r \rho_i^{(j, t)} \zeta_t; i = 1, \dots, s; j = 1, \dots, T \quad (5.19)$$

and express the factors occurring in R.H.S. Of (5.13) in terms of following Mellin-Barnes contour integral, we obtain :

$$\begin{aligned} \prod_{j=1}^W (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} &= \prod_{j=1}^W \left[\frac{(U_i^{(j)} u_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right] \frac{1}{(2\pi\omega)^W} \int_{L'_1} \cdots \int_{L'_W} \prod_{j=1}^W [\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j)] \\ &\quad \prod_{j=1}^W \left[\frac{(U_i^{(j)} (x_i - u_i))^{K_i^{(j)}}}{(u_i U_i^{(j)} + V_i^{(j)})^{K_i^{(j)}}} \right]^{\zeta'_j} d\zeta'_1 \cdots d\zeta'_W \end{aligned} \quad (5.20)$$

and

$$\begin{aligned} \prod_{j=W+1}^T (U_i^{(j)} x_i + V_i^{(j)})^{K_i^{(j)}} &= \prod_{j=W+1}^T \left[\frac{(U_i^{(j)} v_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})} \right] \frac{1}{(2\pi\omega)^{T-W}} \int_{L'_{W+1}} \cdots \int_{L'_T} \prod_{j=W+1}^T [\Gamma(-\zeta'_j) \Gamma(-K_i^{(j)} + \zeta'_j)] \\ &\quad \prod_{j=W+1}^T \left[\frac{(U_i^{(j)} (x_i - v_i))^{K_i^{(j)}}}{(v_i U_i^{(j)} + V_i^{(j)})^{K_i^{(j)}}} \right]^{\zeta'_j} d\zeta'_{W+1} \cdots d\zeta'_T \end{aligned} \quad (5.21)$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost \mathbf{x} -integral with the help of (3.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r+T)$ -variables, we obtain the formula (5.13).

6. Multivariable H-function

If $U_r = V_r = A = B = 0$, the multivariable I-function reduces to the multivariable H-function defined by Srivastava et al [10] and we obtain the two following formulae.

Formula 1

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\mathfrak{N}_{P_i, Q_i, c_i; r'}^{M, N} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T (U_i^{(j)} x_i + V_i^{(j)})^{c_i^{(j)}}} \right] \right]$$

$$S_{N_1, \dots, N_u}^{M_1, \dots, M_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right) \xi_i^{(j,1)}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right) \xi_i^{(j,u)}} \right] \end{array} \right) H_{p_r, q_r; W_r}^{0, n_r; X_r} \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right) \rho_i^{(j,1)}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right) \rho_i^{(j,r)}} \right] \end{array} \right)$$

$${}^P F_Q \left[(A_P); (B_Q); - \sum_{k=1}^l g_k \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\mu_i^{(k)}} (v_i - x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right) \tau_i^{(j,k)}} \right] \right] dx_1 \cdots dx_s$$

$$= \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \right]$$

$$\sum_{G=1}^M \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} a_u \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M, N}(\eta_{G, g})}{B_G g!} y_1^{K_1} \cdots y_u^{K_u} \prod_{i=1}^s \left[(v_i - u_i)^{\eta_{G, g}(a_i + b_i) + \sum_{j=1}^u K_j (\alpha_i^{(j)} + \beta_i^{(j)})} \right]$$

$$z^{\eta_{G, g}} \prod_{i=1}^s \left[\prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G, g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(l, k)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G, g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j, l)}} \right]$$

$$H_{p_r+sT+P+2s;X}^{p_r+sT+P+2s;q_r+sT+Q+s;Y} \left(\begin{array}{c|c} \begin{array}{c} z_1 w_1 \\ \vdots \\ z_r w_r \\ g_1 W_1 \\ \vdots \\ g_l W_l \\ G_1 \\ \vdots \\ G_T \end{array} & \begin{array}{c} A^*, \mathfrak{A}; \mathfrak{A}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ B^*, \mathfrak{B}; \mathfrak{B}_1 \end{array} \end{array} \right) \quad (6.1)$$

under the same notations and conditions that (5.13) with $U_r = V_r = A = B = 0$

Formula 2

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^s \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^T (u_i^{(j)} x_i + v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\mathfrak{N}_{P_i, Q_i, c_i; r'}^{M, N} \left[z \prod_{i=1}^s \left[\frac{(x_i - u_i)^{a_i} (v_i - x_i)^{b_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{c_i^{(j)}}} \right] \right] R_n^{\alpha, \beta} \left[Z \prod_{j=1}^s \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\theta_i^{(j)}}} \right] \right]$$

$$S_{N_1, \cdots, N_u}^{M_1, \cdots, M_u} \left(\begin{array}{c} y_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha'_i} (v_i - x_i)^{\beta'_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\xi_i^{(j,1)}}} \right] \\ \vdots \\ y_u \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\alpha_i^{(u)}} (v_i - x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\xi_i^{(j,u)}}} \right] \end{array} \right)$$

$$H_{p_r, q_r; W_r}^{0, n_r; X_r} \left(\begin{array}{c} z_1 \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta'_i} (v_i - x_i)^{\eta'_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\rho_i^{(j,1)}}} \right] \\ \vdots \\ z_r \prod_{i=1}^s \left[\frac{(x_i - u_i)^{\delta_i^{(r)}} (v_i - x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\rho_i^{(j,r)}}} \right] \end{array} \right) dx_1 \cdots dx_s$$

$$= \prod_{i=1}^s \left[(v_i - u_i)^{\alpha_i + \beta_i - 1} \prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{\sigma_i^{(j)}} \right]$$

$$\sum_{G=1}^M \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} a_u \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M, N}(\eta_{G, g})}{B_G g!} y_1^{K_1} \cdots y_u^{K_u}$$

$$\sum_{w,v,u,t,e,k_1,k_2} \psi'(w,v,u,t,e,k_1,k_2) Z^R \prod_{i=1}^s \left[(v_i - u_i)^{\eta_{G,g}(a_i+b_i) + \sum_{j=1}^u K_j(\alpha_i^{(j)} + \beta_i^{(j)})} \right]$$

$$\prod_{i=1}^s \left[\prod_{j=1}^W \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G,g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j,l)}} \prod_{j=W+1}^T \left(u_i U_i^{(j)} + V_i^{(j)} \right)^{-\eta_{G,g} c_i^{(j)} - \sum_{l=1}^u K_l \xi_i^{(j,l)}} \right]$$

$$H_{p_r+sT+2s;X}^{p_r+sT+2s;q_r+sT+s;Y} \left(\begin{array}{c|c} z_1 w_1 & A^*, \mathfrak{A}; \mathfrak{A}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r w_r & \cdot \\ & \cdot \\ G_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ G_T & B^*, \mathfrak{B}; \mathfrak{B}_1 \end{array} \right) \quad (6.2)$$

under the same notations and conditions that (5.13) with $U_r = V_r = A = B = 0$

7. Conclusion

In this paper we have evaluated two generalized multiple Eulerian integrals involving the multivariable I-functions defined by Prasad [4] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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