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(α, β) – Cut of intuitionistic fuzzy modules - II

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Abstract

For any intuitionistic fuzzy set A = { < x , $\mu_A(x)$, $\nu_A(x)$ > : x $\in$X} of a set X , the crisp subset $C_{\alpha,\beta}(A)$ = { x $\in$X : $\mu_A(x) \geq \alpha$, $\nu_A(x) \leq \beta$ } of X called the (α, β) – cut of A is studied by the author in [6] and [7] when A is intuitionistic fuzzy subgroup of a group G and when A is intuitionistic fuzzy submodule of an R-module M respectively. Basnet in [4] studied the (α, β)- cut set when A is intuitionistic fuzzy ideal of a ring R. This paper is a continuation of author's earlier paper [7]. In this paper, some results concerning the sum and the product of two intuitionistic fuzzy submodules are obtained. A relationship between the (α, β)- cut set of sum and product of two intuitionistic fuzzy submodules with the (α, β)- cut sets of the two intuitionistic fuzzy submodules is established.

Keywords: Intuitionistic fuzzy submodule (IFSM), (α, β) – Cut set.

Mathematics Subject Classification: 03F55 , 08A72 ,16D10

1. Introduction:

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] as a generalization of that of fuzzy sets and it is a very effective tool to study the case of vagueness. Further many researches applied this notion in various branches of mathematics especially in algebra and defined intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings, and intuitionistic fuzzy sublattices, intuitionistic fuzzy submodules and so forth.

2. Preliminaries:

In this section we recall some definitions and results which will be used later

Definition (2.1)[1] Let X be a fixed non-empty set. **An Intuitionistic fuzzy set** (IFS) A of X is an object of the following form A = { < x , $\mu_A(x)$, $\nu_A(x)$ > : x $\in$X}, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and degree of non-membership of the element x $\in$X respectively and for any x $\in$X , we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark (2.2)(i): When $\mu_A(x) + \nu_A(x) = 1$, i.e. when $\nu_A(x) = 1 - \mu_A(x) = \mu^c_A(x)$. Then A is called **fuzzy set.**

(ii) We write A = (μ_A , ν_A) to denote the IFS A = { < x, μ_A(x), ν_A(x) > : x $\in$X}.

Definition (2.3)[7 **]** Let M be a modules over a ring R. An IFS A = (μ_A , ν_A) of M is called intuitionistic fuzzy (left) submodule (IFSM) if

(i) $\mu_A(0) = 1$, $\nu_A(0) = 0$
(ii) $\mu_A(x + y) \geq \min\{ \mu_A(x) , \mu_A(y)\}$ and $\nu_A(x + y) \leq \max \{\nu_A(x) , \nu_A(y)\}$, $\forall$ x , y $\in$ M
(iii) $\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)$, $\forall$ x $\in$ M , r $\in$ R

If we replace the condition (iii) with $\mu_A(xr) \geq \mu_A(x)$ and $\nu_A(xr) \leq \nu_A(x)$, $\forall$ x $\in$ M , r $\in$ R , it is called intuitionistic fuzzy (right) module. When R is commutative ring , then these two modules coincides. From this onward , R will be a commutative ring with unity.

Definition(2.4)[5**,** 4**]** Let A = (μ_A , ν_A) and B = (μ_B , ν_B) be two intuitionistic fuzzy submodules of an R-module M. , then their sum A + B = (μ_{A+B} , ν_{A+B}) is defined as

$$\mu_{A+B}(x) = \underset{x=a+b}{Sup}\{\min\{\mu_A(a),\ \mu_B(b)\}\} \text{ and } \nu_{A+B}(x) = \underset{x=a+b}{Inf}\{\max\{\nu_A(a),\nu_B(b)\}\} \text{ , for all x} \in M$$

Theorem(2.5) Let A and B be two intuitionistic fuzzy submodules of an R-module M. Then the sum A + B of A and B is also intuitionistic fuzzy submodule of M.

Proof. Let x , y $\in$ M be any two elements and let min{ μ_{A+B}(x) , μ_{A+B}(y)} = α (say)

Let ε > 0 be given , then

$$\alpha-\varepsilon < \mu_{A+B}(x) = \underset{x=a+b}{Sup}\{\min\{\mu_A(a),\ \mu_B(b)\}\} \text{ and } \alpha-\varepsilon < \mu_{A+B}(y) = \underset{y=c+d}{Sup}\{\min\{\mu_A(c),\ \mu_B(d)\}\}$$

so there exists a representation x = $a+b$, y = c + d , where $a,\ b,\ c,\ d \in$ M such that

$$\alpha-\varepsilon < \min\{\mu_A(a),\ \mu_B(b)\} \text{ and } \alpha-\varepsilon < \min\{\mu_A(c),\ \mu_B(d)\}$$

$$\Rightarrow \alpha-\varepsilon < \mu_A(a) \ , \alpha-\varepsilon < \mu_B(b) \text{ and } \alpha-\varepsilon < \mu_A(c) \ , \alpha-\varepsilon < \mu_B(d)$$

$$\Rightarrow \alpha-\varepsilon < \min\{\mu_A(a) , \mu_A(c)\} \leq \mu_A(a + c) \text{ and } \alpha-\varepsilon < \min\{\mu_B(b) , \mu_B(d)\} \leq \mu_B(b + d)$$

Thus , we get x + y = $(a + b)+(c+d) = (a + c)+(b+d)$ such that

$$\Rightarrow \alpha-\varepsilon < \min\{\mu_A(a + c),\ \mu_B(b + d)\}$$

$$\Rightarrow \alpha-\varepsilon < \underset{x=(a+c)+(b+d)}{Sup}\{\min\{\mu_A(a+c),\ \mu_B(b+d)\}\} = \mu_{A+B}(x+y)$$

Since ε is arbitrary, it follows that $\mu_{A+B}(x+y) \geq \alpha = \min\{\mu_{A+B}(x),\ \mu_{A+B}(y)\}$.

Similarly, we can show that $\nu_{A+B}(x+y) \leq \max\{\nu_{A+B}(x),\ \nu_{A+B}(y)\}$

Further, let $\beta = \max\{\mu_{A+B}(x), \mu_{A+B}(y)\} = \mu_{A+B}(x)$, and let ε >0, then

β-ε < $\mu_{A+B}(x) = \underset{x=a+b}{Sup}\{\min\{\mu_A(a),\ \mu_B(b)\}\}$, *so* there exists a representation

$x = a+b$ such that β-ε < $\min\{\mu_A(a),\ \mu_B(b)\} \Rightarrow \beta$-$\varepsilon$ <$\mu_A(a)$, β-ε <$\mu_B(b)$

$\Rightarrow$ $\beta - \varepsilon < \mu_A(a) \le \mu_A(ra)$, $\beta - \varepsilon < \mu_B(b) \le \mu_B(rb)$, for any $r \in R$

$\Rightarrow$ $\beta - \varepsilon < \min\{ \mu_A(ra) , \mu_B(rb) \}$, for any $r \in R$

Now , $rx = r(a + b) = ra + rb$ so that $\beta - \varepsilon < \min\{ \mu_A(ra) , \mu_B(rb) \}$

$$\Rightarrow \quad \beta\text{-}\varepsilon < \underset{rx = r(a+b)}{Sup} \left\{\min\{\mu_A(ra),\ \mu_B(rb)\}\right\} = \mu_{A+B}(rx)$$

Since ε is arbitrary , it follows that

$\mu_{A+B}(rx) \ge \beta = \mu_{A+B}(x)$. Similarly, we can show that $\nu_{A+B}(rx) \le \nu_{A+B}(x)$

Moreover, it is easy to check that $\mu_{A+B}(0) = 1$ and $\nu_{A+B}(0) = 0$.

Hence A + B is intuitionistic fuzzy submodule of M.

Definition(2.6)[4 **]** Let A = (μ_A , ν_A) and B = (μ_B , ν_B) be two intuitionistic fuzzy submodules of an R-module M. , then their product AB = (μ_{AB} , ν_{AB}) is defined as

$$\mu_{AB}(x) = \underset{x = \sum_{i<\infty} a_i b_i}{Sup} \min_i \left\{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\right\} \text{ and } \nu_{AB}(x) = \underset{x = \sum_{i<\infty} a_i b_i}{Inf} \max_i \left\{\max\{\nu_A(a_i), \nu_B(b_i)\}\right\} \text{ , for all } x \in M$$

Theorem(2.7) Let A and B be two intuitionistic fuzzy submodules of an R-module M. Then the product AB of A and B is also intuitionistic fuzzy submodule of M.

Proof. Let x , y $\in$ M be any two elements and let $\min\{ \mu_{AB}(x) , \mu_{AB}(y)\} = \alpha$ (say)

Let $\varepsilon > 0$ be given , then

$$\alpha - \varepsilon < \mu_{AB}(x) = \underset{x = \sum_{i<\infty} a_i b_i}{Sup} \min_i \left\{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\right\} \text{ and } \alpha - \varepsilon < \mu_{AB}(y) = \underset{y = \sum_{i<\infty} p_i q_i}{Sup} \min_i \left\{\min\{\mu_A(p_i),\ \mu_B(q_i)\}\right\}$$

$$\Rightarrow \alpha - \varepsilon < \min_i \left\{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\right\} \text{ and } \alpha - \varepsilon < \min_i \left\{\min\{\mu_A(p_i),\ \mu_B(q_i)\}\right\} \text{ , for all i .}$$

$$\Rightarrow \alpha - \varepsilon < \min\{\mu_A(a_i),\ \mu_B(b_i)\} \text{ and } \alpha - \varepsilon < \min\{\mu_A(p_i),\ \mu_B(q_i)\} \text{ , for all i .}$$

$$\Rightarrow \alpha - \varepsilon < \mu_A(a_i) \text{ , } \alpha - \varepsilon < \mu_B(b_i) \text{ and } \alpha - \varepsilon < \mu_A(p_i) \text{ , } \alpha - \varepsilon < \mu_B(q_i) \text{ , for all i .}$$

$$\Rightarrow \alpha - \varepsilon < \min\{\mu_A(a_i),\ \mu_A(p_i)\} \le \mu_A(a_i + p_i) \text{ and } \alpha - \varepsilon < \min\{\mu_B(b_i),\ \mu_B(q_i)\} \le \mu_B(b_i + q_i) \text{ , for all i .}$$

Thus , we get $x + y = \sum (a_i b_i + p_i q_i)$, where a_i , b_i, p_i , $q_i \in M$ such that

$$\alpha - \varepsilon < \min\{\mu_A(a_i + p_i), \mu_B(b_i + q_i)\} \text{ , for all i } \Rightarrow \alpha - \varepsilon < \min_i \left\{\min\{\mu_A(a_i + p_i),\ \mu_B(b_i + q_i)\}\right\}$$

$$\Rightarrow \alpha - \varepsilon < \underset{x+y = \sum_{i<\infty}(a_i b_i + p_i q_i)}{Sup} \min_i \left\{\min\{\mu_A(a_i + p_i), \mu_B(b_i + q_i)\}\right\} = \mu_{AB}(x + y)$$

Since $\varepsilon > 0$ is arbitrary , so we have $\mu_{AB}(x+y) \ge \alpha = \min\{\mu_{AB}(x) \ , \mu_{AB}(y)\}$

Similarly, we can show that $\nu_{AB}(x+y) \le \max\{\nu_{AB}(x), \nu_{AB}(y)\}$

Further, let $\beta = \max\{\mu_{AB}(x), \mu_{AB}(y)\} = \mu_{AB}(x)$, and let $\varepsilon > 0$, then

$\beta\text{-}\varepsilon < \mu_{AB}(x) = \underset{x=\sum_{i<\infty} a_i b_i}{Sup} \min_i \{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\}$, *so* there exists a representation

$x = \sum_{i<\infty} a_i b_i$ such that $\beta\text{-}\varepsilon < \min_i \{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\}$

$\Rightarrow \quad \beta\text{-}\varepsilon < \min\{\mu_A(a_i),\ \mu_B(b_i)\}$, for all i.

$\le \min\{\mu_A(ra_i),\ \mu_B(b_i)\}$, for all i and for all $r \in R$

Hence $\beta\text{-}\varepsilon < \min_i \{\min\{\mu_A(ra_i),\ \mu_B(b_i)\}\} < \underset{x=\sum_{i<\infty}(ra_i)b_i}{Sup} \min_i \{\min\{\mu_A(ra_i),\ \mu_B(b_i)\}\} = \mu_{AB}(rx)$

As $\varepsilon > 0$ is arbitrary , so we have $\mu_{AB}(rx) \ge \beta = \mu_{AB}(x)$

Similarly, we can show that $\nu_{AB}(rx) \le \nu_{AB}(x)$

Also , it can be easily checked that $\mu_{AB}(0) = 1$ and $\nu_{AB}(0) = 0$

Hence AB is intuitionistic fuzzy submodule of M.

3. (α , β) – Cut of Intuitionistic fuzzy set

Definition (3.1)[6 **]** Let A be Intuitionistic fuzzy set of a universe set X . Then (α , β)-cut of A is a crisp subset $C_{\alpha,\beta}(A)$ of the IFS A is given by

$$C_{\alpha,\beta}(A) = \{ x : x \in X \text{ such that } \mu_A(x) \ge \alpha,\ \nu_A(x) \le \beta \},$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \le 1$.

Proposition (3.2)[6 **]** If A and B be two IFS's of a universe set X , then following holds

(i) $C_{\alpha,\beta}(A) \subseteq C_{\delta,\theta}(A)$ if $\alpha \ge \delta$ and $\beta \le \theta$

(ii) $C_{1-\beta,\beta}(A) \subseteq C_{\alpha,\beta}(A) \subseteq C_{\alpha,1-\alpha}(A)$

(iii) $A \subseteq B$ implies $C_{\alpha,\beta}(A) \subseteq C_{\alpha,\beta}(B)$

(iv) $C_{\alpha,\beta}(A \cap B) = C_{\alpha,\beta}(A) \cap C_{\alpha,\beta}(B)$

(v) $C_{\alpha,\beta}(A \cup B) \supseteq C_{\alpha,\beta}(A) \cup C_{\alpha,\beta}(B)$ equality hold if $\alpha + \beta = 1$

(vi) $C_{\alpha,\beta}(\cap A_i) = \cap C_{\alpha,\beta}(A_i)$

(vii) $C_{0,1}(A) = X$.

Theorem(3.3)[7 **]** If $A = (\mu_A, \nu_A)$ be IFS of an R-module M , then A is a IFSM of M if and only if $C_{\alpha,\beta}(A)$ is submodule of M , for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$, where $\mu_A(0) \ge \alpha$, $\nu_A(0) \le \beta$.

Theorem(3.4) Let A and B be two IFSM of an R-module M , then

$$C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B) \subseteq C_{\alpha,\beta}(A + B) \text{ and the equality holds if } \alpha + \beta = 1$$

Proof. Let $x = y + z \in C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B)$, where $y \in C_{\alpha,\beta}(A)$ and $z \in C_{\alpha,\beta}(B)$

$\Rightarrow \quad \mu_A(y) \ge \alpha$, $\nu_A(y) \le \beta$ and $\mu_B(z) \ge \alpha$, $\nu_B(z) \le \beta$

$\Rightarrow$ min $\{\mu_A(y), \mu_B(z)\} \geq \alpha$ and $\max\{\nu_A(y), \nu_B(z)\} \leq \beta$

$$\Rightarrow \quad \underset{x=y+z}{Sup}\{\min\{\mu_A(y), \mu_B(z)\} \geq \alpha \quad \text{and} \quad \underset{x=y+z}{Inf}\{\max\{\nu_A(y), \nu_B(z)\} \leq \beta$$

$$\Rightarrow \quad \mu_{A+B}(x) \geq \alpha \quad \text{and} \quad \nu_{A+B}(x) \leq \beta \quad \text{i.e.} \quad x \in C_{\alpha,\beta}(A+B)$$

Thus $C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B) \subseteq C_{\alpha,\beta}(A + B)$.

For the other part , let $\alpha + \beta = 1$ and $x \in C_{\alpha,\beta}(A + B)$. Then

$$\mu_{A+B}(x) \geq \alpha \quad \text{and} \quad \nu_{A+B}(x) \leq \beta \quad \text{i.e.} \quad x \in C_{\alpha,\beta}(A+B)$$

$$\Rightarrow \quad \mu_{A+B}(x) \geq \alpha \quad \text{and} \quad \nu_{A+B}(x) \leq \beta \quad \text{i.e.} \quad \underset{x=y+z}{Sup}\{\min\{\mu_A(y), \mu_B(z)\}\} \geq \alpha$$

$$\Rightarrow \quad \min\{\mu_A(a), \mu_B(b)\} \geq \alpha \quad \text{, for some } x = a+b$$

$$\Rightarrow \quad \mu_A(a) \geq \alpha \quad \text{and} \quad \mu_B(b) \geq \alpha$$

$$\Rightarrow \quad \nu_A(a) \leq 1 - \mu_A(a) \leq 1-\alpha = \beta \quad \text{and} \quad \nu_B(b) \leq 1 - \mu_B(b) \leq 1-\alpha = \beta$$

$$\Rightarrow \quad a \in C_{\alpha,\beta}(A) \quad \text{and} \quad b \in C_{\alpha,\beta}(B) \quad \text{and so} \quad x = a+b \in C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B)$$

Hence $\quad C_{\alpha,\beta}(A+B) = C_{\alpha,\beta}(A) + C_{\alpha,\beta}(B)$.

Remark (3.5) In the above theorem the equality does not hold as can be seen from the following example:

Example (3.6) Consider the $\mathbb{Z}$- module $\mathbb{Z}_6$, where $\mathbb{Z}_6 = \{ 0 , 1 , 2 , 3, 4, 5 \}$ and define

$$\mu_A(0) = 1 , \quad \mu_A(4) = 0.6 = \mu_A(2) , \quad \mu_A(1) = \mu_A(3) = \mu_A(5) = 0$$
$$\nu_A(0) = 0 , \quad \nu_A(4) = 0.3 = \nu_A(2) , \quad \nu_A(1) = \nu_A(3) = \nu_A(5) = 1$$

$$\textit{and} \quad \mu_B(0) = 1 , \quad \mu_B(4) = \mu_B(2) = \mu_B(1) = \mu_B(5) = 0 , \quad \mu_B(3) = 0.5$$
$$\nu_B(0) = 0 , \quad \nu_B(4) = \nu_B(2) = \nu_B(1) = \nu_B(5) = 1 , \quad \nu_B(3) = 0.3$$

Clearly, A and B are IFSM's of $\mathbb{Z}$- module $\mathbb{Z}_6$. Now $A + B = (\mu_{A+B} , \nu_{A+B})$, where

$$\mu_{A+B}(x) = \underset{x=a+b}{Sup}\{\min\{\mu_A(a), \mu_B(b)\}\} \quad \text{and} \quad \nu_{A+B}(x) = \underset{x=a+b}{Inf}\{\max\{\nu_A(a), \nu_B(b)\}\} \quad \text{, for all } x \in Z_6$$

Therefore , we get

$$\mu_{A+B}(0) = Sup\begin{Bmatrix}\{\min\{\mu_A(0),\ \mu_B(0)\}.\{\min\{\mu_A(1),\ \mu_B(5)\},\{\min\{\mu_A(2),\ \mu_B(4)\},\\ \{\min\{\mu_A(3),\ \mu_B(3)\},\{\min\{\mu_A(4),\ \mu_B(2)\},\{\min\{\mu_A(5),\ \mu_B(1)\}\end{Bmatrix}$$

= Sup{ min{1,1}, min{0,0},min{0.6,0},min{0,0.5},min{0.6,0},min{0,0}}

= Sup{1, 0, 0, 0, 0, 0 }=1

$$\mu_{A+B}(1) = Sup\begin{Bmatrix}\{\min\{\mu_A(0),\ \mu_B(1)\}.\{\min\{\mu_A(1),\ \mu_B(0)\},\{\min\{\mu_A(2),\ \mu_B(5)\},\\ \{\min\{\mu_A(3),\ \mu_B(4)\},\{\min\{\mu_A(4),\ \mu_B(3)\},\{\min\{\mu_A(5),\ \mu_B(2)\}\end{Bmatrix}$$

= Sup{ min{1,0}, min{0,1},min{0.6,0},min{0,0},min{0.6,0.5},min{0,0}}

= Sup{0, 0, 0, 0, 0.5, 0 }=0.5

Similarly, we can find $\mu_{A+B}(2) = 0.6$, $\mu_{A+B}(3) = 0.5$, $\mu_{A+B}(4) = 0.6$, $\mu_{A+B}(5) = 0.5$

Also , $\nu_{A+B}(0) = Inf\begin{Bmatrix}\max\{\nu_A(0),\ \nu_B(0)\},\max\{\nu_A(1),\ \nu_B(5)\},\max\{\nu_A(2),\ \nu_B(4)\},\\ \max\{\nu_A(3),\ \nu_B(3)\},\max\{\nu_A(4),\ \nu_B(2)\},\max\{\nu_A(5),\ \nu_B(1)\}\end{Bmatrix}$

= *Inf*{ max{0,0}, max{1,1},max{0.3,1},max{1,0.3},max{0.3,1},max{1,1}}

= *Inf*{0, 1, 1, 1, 1, 1 }= 0

Similarly, we can find $\nu_{A+B}(1) = 0.3$, $\nu_{A+B}(2) = 0.6$, $\nu_{A+B}(3) = 0.3$, $\nu_{A+B}(4) = 0.3$, $\nu_{A+B}(5) = 0.3$

Now, $C_{0.5,0.3}(A+B) = \{0,1,3,4,5\}$. Also , $C_{0.5,0.3}(A) + C_{0.5,0.3}(B) = \{0,4,2\} + \{1,3\} = \{1,3,5\}$

Clearly, $C_{0.5,0.3}(A) + C_{0.5,0.3}(B) \subset C_{0.5,0.3}(A+B)$.

Theorem (3.7) Let A and B be two IFSM of an R-module M , then

$C_{\alpha,\beta}(A)C_{\alpha,\beta}(B) \subseteq C_{\alpha,\beta}(AB)$ and the equality holds if $\alpha + \beta = 1$

Proof. Now AB = (μ_{AB} , ν_{AB}) , where

$\mu_{AB}(x) = \underset{x=\sum_{i<\infty} a_i b_i}{Sup} \min_i \{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\}$ and $\nu_{AB}(x) = \underset{x=\sum_{i<\infty} a_i b_i}{Inf} \max_i \{\max\{\nu_A(a_i), \nu_B(b_i)\}\}$, for all $x \in M$

Let $x = \sum_{i<\infty} a_i b_i \in C_{\alpha,\beta}(A)C_{\alpha,\beta}(B)$, where $a_i \in C_{\alpha,\beta}(A)$ and $b_i \in C_{\alpha,\beta}(B)$, *for all i.*

$\Rightarrow \quad \mu_A(a_i) \geq \alpha \quad , \nu_A(a_i) \leq \beta$ and $\mu_B(b_i) \geq \alpha \quad , \nu_B(b_i) \leq \beta$, *for all i*

$\Rightarrow \quad \min\{\mu_A(a_i),\ \mu_B(b_i)\} \geq \alpha$ and $\max\{\nu_A(a_i),\ \nu_B(b_i)\} \leq \beta$, *for all i*

$\Rightarrow \quad \min_i \{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\} \geq \alpha$ and $\max_i \{\max\{\nu_A(a_i), \nu_B(b_i)\}\} \leq \beta$

$\Rightarrow \quad \underset{x=\sum_{i<\infty} a_i b_i}{Sup} \min_i \{\min\{\mu_A(a_i),\ \mu_B(b_i)\}\} \geq \alpha$ and $\underset{x=\sum_{i<\infty} a_i b_i}{Inf} \max_i \{\max\{\nu_A(a_i), \nu_B(b_i)\}\} \leq \beta$

$\Rightarrow \quad \mu_{AB}(x) \geq \alpha$ and $\nu_{AB}(x) \leq \beta$ i.e. $x \in C_{\alpha,\beta}(AB)$

Thus , we get $C_{\alpha,\beta}(A)C_{\alpha,\beta}(B) \subseteq C_{\alpha,\beta}(AB)$.

For the second part, let $\alpha + \beta = 1$ and $x \in C_{\alpha,\beta}(AB)$. Then

$$\mu_{\mathrm{AB}}(x) \geq \alpha \qquad \text{and} \qquad \nu_{\mathrm{AB}}(x) \leq \beta .$$

$$\textit{Now,} \quad \mu_{\mathrm{AB}}(x) \geq \alpha \;\Rightarrow\; \underset{x=\sum_{i<\infty} a_i b_i}{Sup} \min_i \left\{ \min\{\mu_{\mathrm{A}}(a_i),\ \mu_B(b_i)\} \right\} \geq \alpha$$

$$\Rightarrow \quad \min_i \left\{ \min\{\mu_{\mathrm{A}}(a_i),\ \mu_B(b_i)\} \right\} \geq \alpha \quad \text{, for some } x = \sum_{i<\infty} a_i b_i$$

$$\Rightarrow \quad \min\{\mu_{\mathrm{A}}(a_i),\ \mu_B(b_i)\} \geq \alpha \text{ , for all i } \Rightarrow \mu_{\mathrm{A}}(a_i) \geq \alpha \text{ and } \mu_B(b_i) \geq \alpha \text{ , } \textit{for all } i$$

$$\Rightarrow \quad \nu_{\mathrm{A}}(a_i) \leq 1 - \mu_{\mathrm{A}}(a_i) \leq 1 - \alpha = \beta \text{ and } \nu_B(b_i) \leq 1 - \mu_B(b_i) \leq 1 - \alpha = \beta \text{ , } \textit{for all } i$$

$$\Rightarrow \quad a_i \in \mathrm{C}_{\alpha,\beta}(A) \text{ and } b_i \in \mathrm{C}_{\alpha,\beta}(B), \textit{ for all } i$$

$$\Rightarrow \quad x = \sum_{i<\infty} a_i b_i \in \mathrm{C}_{\alpha,\beta}(A)\mathrm{C}_{\alpha,\beta}(B)$$

Hence $\mathrm{C}_{\alpha,\beta}(AB) \subseteq \mathrm{C}_{\alpha,\beta}(A)\mathrm{C}_{\alpha,\beta}(B)$ and so the equality follows.

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