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Characterization of *-Isomorphisms in an Indefinite Inner Product Space using Indefinite Matrix Product

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Abstract

In this paper, we obtain a representation for star-isomorphisms on $\mathbb{C}^{n\times n}$ being viewed as an indefinite inner product space. This is done by using a new matrix product on $\mathbb{C}^{n\times n}$.

Keywords: Indefinite matrix product; Indefinite inner product spaces; $*-$isomorphisms.

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1 Introduction

Let $< .,. >$ denote the usual Euclidean inner product in $\mathbb{C}^n$. An indefinite inner product is defined by $[x, y] =< x, Ny >$, where $N \in \mathbb{C}^{n\times n}$ is hermitian and $N = N^{-1}$. Such a matrix N is called a weight. A space with an indefinite inner product is called an indefinite inner product space. In the rest of the paper, whenever $\mathbb{C}^n$ is referred to as an indefinite inner product space, it will be assumed that N is the corresponding weight.

Investigation of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors. This gives rise to a mismatch as there are two different values for the indefinite product of vectors. To overcome this mismatch, an indefinite matrix product was defined in [1]. For completeness, we recall this definition: Let A, B be two complex matrices of order $m \times n$ and $n \times p$, respectively. The indefinite matrix product of A and B is defined by $A \circ B = ANB$, where N is a weight as defined above. This not only rectifies the defect mentioned earlier, but also enables us to at least recover some of the results in indefinite inner product spaces in a manner similar to the ones in the Euclidean space. We refer to [1], where it has been established that the indefinite matrix product is more appropriate than the usual matrix product, in the setting of indefinite inner product spaces, in the context of certain specific questions on these spaces. We also refer to [3] for a Farkas type theorem for the new matrix product.

The objective of this article is to obtain a representation theorem for $*-$isomorphisms on $\mathbb{C}^{n\times n}$ endowed with the new matrix product. The approach adopted here parallels the techniques employed in [2] and [4]. However, the arguments here are much more involved than those mentioned above. Our approach also has the advantage that it is intrinsic, in the sense that we have developed the proof completely using the new matrix product. It is also pertinent to point out that while the results in the references mentioned earlier hold for $*-$isomorphisms on spaces of operators, we have been able to prove our results only for $*-$isomorphisms on spaces of complex matrices. This stems primarily from the difficulty in defining rank 1 operators similar to the rank 1 matrices given in definition 2.6.

The paper is organized as follows: In the next section, we fix the notation and collect preliminary results that are used in the sequel. In section 3, we prove the main results.

2 Preliminaries

The adjoint $A^{[*]}$ of a matrix A is defined by $A^{[*]} = N_1A^*N_2$, where A^* denotes the complex conjugate transpose of A and $N_1,\ N_2$ are given weights on $\mathbb{C}^{m\times m}$, $\mathbb{C}^{n\times n}$, respectively. It then follows that $[A\circ x, y] = [x, (I\circ A\circ I)^{[*]}\circ y]$ for all $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$. For a matrix $A \in \mathbb{C}^{m\times n}$, we define the range space $\mathcal{R}(A)$ of A, by $\mathcal{R}(A) = \{A \circ x : x \in \mathbb{C}^n\}$ and the null space $\mathcal{N}(A)$ of A, by $\mathcal{N}(A) = \{x \in \mathbb{C}^n : A \circ x = \mathbf{0}\}$. For a subspace M of $\mathbb{C}^n$, $M^{[\perp]}$ will denote the subspace $M^{[\perp]} = \{y \in \mathbb{C}^n : [x, y] = 0, \forall x \in M\}$. A square matrix A is called idempotent if $A \circ A = A$; an orthogonal projection if A is idempotent and $A = (I \circ A \circ I)^{[*]}$. In particular, if A is an orthogonal projection then $[A \circ x, y] = [x, A \circ y]$. Note that if A is idempotent and if $x \in R(A)$, then $A \circ x = x$.

The proof of the following result is routine.

Lemma 2.1. *If A is idempotent then $\mathcal{R}(A) = \mathcal{N}(N - A)$.*

A version of the Farkas lemma for indefinite inner product spaces was proved in [3]. We state that next.

Theorem 2.2 ([3], Theorem 2.5). *For $A \in \mathbb{C}^{m\times n}$, $\mathcal{R}(I \circ A)^{[\perp]} = \mathcal{N}(A^{[*]})$.*

Lemma 2.3. *Let A be such that $A = (I \circ A \circ I)^{[*]}$. Then $\mathcal{R}(A)^{[\perp]} = \mathcal{N}(A)$.*

Proof. As $A = (I \circ A \circ I)^{[*]}$, we have $(I \circ A)^{[*]} = I \circ A$. Set $B = I \circ A$., so that $A = I \circ B$. Now, $\mathcal{R}(A)^{[\perp]} = \mathcal{R}(I \circ B)^{[\perp]} = \mathcal{N}(B^{[*]})$, by Theorem 2.2. Finally, $\mathcal{N}(B^{[*]}) = \mathcal{N}((I \circ A)^{[*]}) = \mathcal{N}(I \circ A) = \mathcal{N}(A)$, completing the proof. □

Lemma 2.4. *Let A be a rank 1 matrix satisfying $A = (I \circ A \circ I)^{[*]}$. Then $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are orthogonal complementary subspaces. In this case, $[x, x] \neq 0$, for all nonzero $x \in \mathcal{R}(A)$.*

Proof. Let $x \in \mathcal{R}(A) \cap \mathcal{N}(A)$. Then $x = A \circ x = \mathbf{0}$, showing that $\mathcal{R}(A) \cap \mathcal{N}(A) = \{\mathbf{0}\}$. For any $x \in \mathbb{C}^n$, $x = A \circ x + (N - A) \circ x \in \mathcal{R}(A) + \mathcal{N}(A)$. Thus $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are complementary subspaces. Let $x \in \mathcal{R}(A)$ and $y \in \mathcal{N}(A)$. Then, $x = A \circ x$ and so $[x, y] = [A \circ x, y] = [x, A \circ y] = 0$. Thus $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are orthogonal. Let $\mathbf{0} \neq x \in \mathcal{R}(A)$ so that $span\{x\} = \mathcal{R}(A)$. Let $z \in \mathcal{R}(A)$. Then $z = \alpha x$. If $[x, x] = 0$, then $[z, x] = 0$ and so $x \in \mathcal{R}(A)^{[\perp]} = \mathcal{N}(A)$ (by Lemma 2.3). This implies $x = \mathbf{0}$, a contradiction. Thus $[x, x] \neq 0$. □

Lemma 2.5. *Let A and B be orthogonal projections. Then*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B) \Leftrightarrow A \circ B = A = B \circ A.$

(ii) $\mathcal{R}(A)[\perp]\mathcal{R}(B) \Leftrightarrow A \circ B = \mathbf{0} = B \circ A.$

Proof. (i) Let $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. For an arbitrary x, $A \circ x \in \mathcal{R}(A) \subseteq \mathcal{R}(B)$. Since B is idempotent, $B \circ A \circ x = A \circ x$ for all x. Thus $B \circ A = A$. Taking the adjoint on both sides, we get $A^{[*]} \circ B^{[*]} = A^{[*]}$. Then, we have $I \circ A \circ I \circ I \circ B \circ I = I \circ A \circ I$. Thus $A \circ B = A$. The Converse is easy to see.
(ii) Let $\mathcal{R}(A)[\perp]\mathcal{R}(B)$. Then for all x, y $[B \circ A \circ x, y] = [A \circ x, B \circ y] = 0$. Thus $B \circ A = \mathbf{0}$. Similarly, $A \circ B = \mathbf{0}$. Conversely, let $x \in \mathcal{R}(A)$ and $y \in \mathcal{R}(B)$. Then $[x, y] = [A \circ x, B \circ y] = [x, A \circ B \circ y] = 0$. Thus $\mathcal{R}(A)[\perp]\mathcal{R}(B)$. □

To study the representation of a star-isomorphism on a general matrix A, we make use of rank 1 matrices. These are defined, next.

Definition 2.6. *Let $x, y \in \mathbb{C}^n$. We define $A_{x,y} \in \mathbb{C}^{n\times n}$ by*

$$A_{x,y} \circ u = [u, y] I \circ x, \quad u \in \mathbb{C}^n$$

and $P_x = sgn(x) I \circ A_{x,x}$, where I denotes the identity matrix of order n and $sgn(x) = \begin{cases} 1, & \text{if } [x,x] \geq 0 \\ -1, & \text{otherwise.} \end{cases}$

Remark 2.7. *From the definition we observe that $P_x \circ x = x$ and $P_x \circ u = sgn(x)[u, x]x$.*

Now we present some properties of $A_{x,y}$. The proof of the first result to follow is routine and is hence omitted.

Lemma 2.8. *$A_{x,y}$ is a rank one matrix and $A_{\alpha x, \beta y} = \alpha\bar{\beta} A_{x,y}$, where α and β are scalars.*

Lemma 2.9. *$A^{[*]}_{x,y} = A_{y,x}$.*

Proof. Let $u, v \in \mathbb{C}^n$. Then

$$\begin{aligned}
[u, A^{[*]}_{x,y} \circ v] &= [I \circ A_{x,y} \circ I \circ u, v] \\
&= [I \circ A_{x,y} \circ w, v], \text{ where } w = I \circ u \\
&= [w, y][x, v] \\
&= [I \circ u, [v, x] y] \\
&= [u, [v, x] I \circ y] \\
&= [u, A_{y,x} \circ v].
\end{aligned}$$

□

Lemma 2.10.

(i) If $[x,x] = \pm 1$*, then* $I \circ A_{x,z} \circ I \circ A_{y,x} = sgn(x)[y,z]P_x$.

(ii) If $[z,z] = \pm 1$*, then* $I \circ A_{x,z} \circ I \circ A_{z,y} = sgn(z) I \circ A_{x,y}$. *In addition, if* $[x,x] = \pm 1$*, then* $I \circ A_{x,z} \circ I \circ A_{z,x} = sgn(x)sgn(z)P_x$.

Proof. (i)

$$\begin{aligned} I \circ A_{x,z} \circ I \circ A_{y,x} \circ u &= I \circ A_{x,z} \circ I \circ [u,x] I \circ y \\ &= [u,x] I \circ A_{x,z} \circ y \\ &= [u,x][y,z]x \\ &= sgn(x)[y,z]P_x \circ u. \end{aligned}$$

Thus $I \circ A_{x,z} \circ I \circ A_{y,x} = sgn(x)[y,z]P_x$.

(ii) can be proved in a similar manner. □

Lemma 2.11. *Let* $A \in \mathbb{C}^{n \times n}$ *with rank 1. Then there exist non-zero vectors* x, y *such that* $A = I \circ A_{x,y}$.

Proof. Let $\mathcal{R}(A) = span\{x\}$. If $[x,x] = \pm 1$, let $y = sgn(x)(I \circ A \circ I)^{[*]} \circ x$. Now, we have for any $u \in \mathbb{C}^n$, $A \circ u = \beta x$. Then $[A \circ u, x] = [\beta x, x] = \beta sgn(x)$, so that $\beta = sgn(x)[A \circ u, x]$. Thus $A \circ u = sgn(x)[A \circ u, x]x$ $= sgn(x)[u, (I \circ A \circ I)^{[*]} \circ x]x = [u,y]x = I \circ [u,y] I \circ x = I \circ A_{x,y} \circ u$. Hence $A = I \circ A_{x,y}$.

If $[x,x] = 0$ then for any $u \in \mathbb{C}^n$, let $A \circ u = \beta x$. Choose y such that $\beta = [u,y]$. Then $A \circ u = [u,y]x$. Also, $I \circ A_{x,y} \circ u = I \circ [u,y] I \circ x = [u,y]x$. Thus $A = I \circ A_{x,y}$, in this case too. □

Next, we discuss some properties of P_x.

Lemma 2.12. *If* $[x,x]=\pm 1$, *then* P_x *is an orthogonal projection.*

Proof. Let $u\in\mathbb{R}^n$. Then $P_x\circ P_x\circ u = P_x\circ sgn(x)[u,x]x = sgn(x)[u,x]P_x\circ x = sgn(x)[u,x]x = P_x\circ u$. Thus P_x is idempotent. Also, $[(I\circ P_x\circ I)^{[*]}\circ u, v] = [u, P_x\circ v] = sgn(x)[u,x][x,v] = [sgn(x)[u,x]x, v] = [P_x\circ u, v]$. Thus $(I\circ P_x\circ I)^{[*]} = P_x$. □

The next result is the converse of Lemma 2.12.

Lemma 2.13. *If* P *is an orthogonal projection of rank 1, then there exists* $x\in\mathbb{C}^n$ *such that* $P=P_x$ *with* $[x,x]=\pm 1$.

Proof. Let $\mathcal{R}(P) = span\{x\}$. Since P is an orthogonal projection, by Lemma 2.4, $[x,x]\neq 0$. Without loss of generality we may assume that $[x,x]=\pm 1$.

Let $u\in\mathbb{C}^n$. Then $P\circ u=\alpha x$, where $\alpha = sgn(x)[P\circ u, x]$. So, $P\circ u = sgn(x)[P\circ u,x]x = sgn(x)[u,P\circ x]x = sgn(x)[u,x]x = P_x\circ u$, where we have used the fact that $P\circ x = x$, since $x\in\mathcal{R}(P)$. Thus $P=P_x$. □

3 Main Results

In this section we establish the main results, viz., Theorem 3.4 and Corollary 3.5. First we prove a couple of preliminary results. We begin with the definition of a $*-$isomorphism.

Definition 3.1. *A linear map* $\Phi:\mathbb{C}^{n\times n}\to\mathbb{C}^{n\times n}$ *is called an isomorphism if it is one-one, onto and* $\Phi(A\circ B)=\Phi(A)\circ\Phi(B)$ *for all* $A,B\in\mathbb{C}^{n\times n}$. *An isomorphism* Φ *is called a* $*-$*isomorphism if* $[\Phi(A)]^{[*]} = I\circ\Phi(I\circ A^{[*]}\circ I)\circ I$ *for all* $A\in\mathbb{C}^{n\times n}$.

Lemma 3.2. *Let* $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ *be a* $*-$*isomorphism. If* $[x, x] = \pm 1$, *then* $\Phi(P_x)$ *is an orthogonal projection of rank 1.*

Proof. Let $[x, x] = \pm 1$. Then by Lemma 2.12, P_x is an orthogonal projection. We have $\Phi(P_x) = \Phi(P_x \circ P_x) = \Phi(P_x) \circ \Phi(P_x)$. Also, $(I \circ \Phi(P_x) \circ I)^{[*]} = I \circ (\Phi(P_x))^{[*]} \circ I = I \circ I \circ \Phi(I \circ P_x^{[*]} \circ I) \circ I \circ I = \Phi(P_x)$. Thus $\Phi(P_x)$ is an orthogonal projection.

Let the rank of $\Phi(P_x)$ be more than 1. Let y, z be two linearly independent vectors such that $[y, y] = \pm 1$, $[z, z] = \pm 1$ and $span\{y, z\} \subseteq \mathcal{R}(\Phi(P_x))$. Consider P_y, P_z. Since y and z are linearly independent, $P_y \neq \alpha P_z$ for any scalar α. Then $\mathcal{R}(P_y) = span\{y\} \subseteq \mathcal{R}(\Phi(P_x))$. Thus by Lemma 2.5, $P_y \circ \Phi(P_x) = P_y$. As Φ is a $*-$isomorphism, $\Phi^{[-1]}$ is also a $*-$isomorphism. So, $\Phi^{[-1]}(P_y)$ is idempotent and $\Phi^{[-1]}(P_y) = (I \circ \Phi^{[-1]}(P_y) \circ I)^{[*]}$ so that $\Phi^{[-1]}(P_y)$ is an orthogonal projection. Also, $\Phi^{[-1]}(P_y) \circ P_x = \Phi^{[-1]}(P_y)$. Thus $\mathcal{R}(\Phi^{[-1]}(P_y)) \subseteq \mathcal{R}(P_x)$, again by Lemma 2.5. Similarly, $\mathcal{R}(\Phi^{[-1]}(P_z)) \subseteq \mathcal{R}(P_x)$ (Note that $\Phi^{[-1]}(P_y) \neq \alpha \Phi^{[-1]}(P_z)$ for any scalar α.). If $\Phi^{[-1]}(P_y) = \mathbf{0}$, then $y = \mathbf{0}$, contradicting $[y, y] = \pm 1$. Thus $\mathcal{R}(\Phi^{[-1]}(P_y)) \neq \{\mathbf{0}\}$ and $\mathcal{R}(\Phi^{[-1]}(P_z)) \neq \{\mathbf{0}\}$, similarly. Thus there exist vectors $u \in \mathcal{R}(\Phi^{[-1]}(P_y))$ and $v \in \mathcal{R}(\Phi^{[-1]}(P_z))$. Then $u, v \in \mathcal{R}(P_x) = span\{x\}$. If v is a multiple of u, then it follows that z is a multiple of y, a contradiction. So, u, v are linearly independent. However, this contradicts the fact that $u, v \in span\{x\}$. So rank of $\Phi(P_x)$ is 1. □

Lemma 3.3. *Let* $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ *be a* $*-$*isomorphism. Then for each* $x \in \mathbb{C}^n$ *with* $[x, x] = \pm 1$, *there exists* $\bar{x} \in \mathbb{C}^n$ *with* $[\bar{x}, \bar{x}] = \pm 1$ *such that* $\Phi(P_x) = P_{\bar{x}}$.

Proof. $\Phi(P_x)$ is an orthogonal projection of rank 1, by Lemma 3.2. By Lemma 2.13, there exists $\bar{x} \in \mathbb{C}^n$ such that $\Phi(P_x) = P_{\bar{x}}$ with $[\bar{x}, \bar{x}] = \pm 1$. □

Theorem 3.4. *Let* $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ *be a* $*-$*isomorphism. Then there exists* $U \in \mathbb{C}^{n\times n}$ *such that* $\Phi(A) = c(I \circ U) \circ A \circ (U \circ I)^{[*]}$ *for all rank 1 matrices* $A \in \mathbb{C}^{n\times n}$ *with* $(I \circ U)^{[*]} \circ (U \circ I) = cN = (I \circ U) \circ (U \circ I)^{[*]}$, *where* $c = \pm 1$.

Proof. Let $x_0 \in \mathbb{C}^n$ such that $[x_0, x_0] = \pm 1$. By Lemma 3.3, there exists $\bar{x_0} \in \mathbb{C}^n$ such that $\Phi(P_{x_0}) = P_{\bar{x}_0}$ and $[\bar{x}_0, \bar{x}_0] = \pm 1$.

Define $U : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ by

$$U \circ y = I \circ \Phi(I \circ A_{y,x_0}) \circ \bar{x_0}, \quad y \in \mathbb{C}^n.$$

Then

$$\begin{aligned} U \circ x_0 &= I \circ \Phi(I \circ A_{x_0,x_0}) \circ \bar{x_0} \\ &= sgn(x_0) I \circ \Phi(P_{x_0}) \circ \bar{x_0} \\ &= sgn(x_0) I \circ \bar{x}_0. \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} [U \circ y, U \circ z] &= [I \circ \Phi(I \circ A_{y,x_0}) \circ \bar{x}_0, I \circ \Phi(I \circ A_{z,x_0}) \circ \bar{x}_0] \\ &= [I \circ (I \circ \Phi(I \circ A_{z,x_0}))^{[*]} \circ I \circ I \circ \Phi(I \circ A_{y,x_0}) \circ \bar{x}_0, \bar{x}_0] \\ &= [\Phi(I \circ A_{z,x_0}^{[*]}) \circ \Phi(I \circ A_{y,x_0}) \circ \bar{x}_0, \bar{x}_0] \\ &= [\Phi(I \circ A_{x_0,z}) \circ \Phi(I \circ A_{y,x_0}) \circ \bar{x}_0, \bar{x}_0], \text{ (by Lemma 2.9)} \\ &= [\Phi(I \circ A_{x_0,z} \circ I \circ A_{y,x_0}) \circ \bar{x}_0, \bar{x}_0] \\ &= sgn(x_0)[y, z][\Phi(P_{x_0}) \circ \bar{x}_0, \bar{x}_0], \text{ (by (i) of Lemma 2.10)} \\ &= sgn(x_0)[y, z][P_{\bar{x}_0} \circ \bar{x}_0, \bar{x}_0] \\ &= sgn(x_0)[y, z][\bar{x}_0, \bar{x}_0] \\ &= sgn(x_0) sgn(\bar{x}_0)[y, z]. \end{aligned}$$

Also, $[U \circ y, U \circ z] = [I \circ U^{[*]} \circ I \circ U \circ y, z]$. It then follows that $I \circ U^{[*]} \circ I \circ U \circ I = sgn(x_0)sgn(\bar{x}_0)I$. Thus

$$(I \circ U)^{[*]} \circ (U \circ I) = cN, \tag{3.2}$$

where $c = sgn(x_0)sgn(\bar{x}_0)$. Similarly, we can prove $(I \circ U) \circ (U \circ I)^{[*]} = cN$.

We next prove

$$\Phi(I \circ A_{x_0,y}) = c(I \circ U) \circ (I \circ A_{x_0,y}) \circ (U \circ I)^{[*]}.$$

Let $u, y \in \mathbb{C}^n$. Then

$$\begin{aligned}
\Phi(I \circ A_{x_0,y}) \circ (I \circ U) \circ u &= \Phi(I \circ A_{x_0,y}) \circ \Phi(I \circ A_{u,x_0}) \circ \bar{x}_0 \\
&= \Phi(I \circ A_{x_0,y} \circ I \circ A_{u,x_0}) \circ \bar{x}_0 \\
&= sgn(x_0)[u, y]\Phi(P_{x_0}) \circ \bar{x}_0, \text{ (by (i) of Lemma 2.10)} \\
&= sgn(x_0)[u, y]P_{\bar{x}_0} \circ \bar{x}_0 \\
&= sgn(x_0)[u, y]\bar{x}_0.
\end{aligned}$$

Also,

$$\begin{aligned}
(I \circ U) \circ (I \circ A_{x_0,y}) \circ u &= [u, y]I \circ U \circ x_0 \\
&= [u, y]\Phi(I \circ A_{x_0,x_0}) \circ \bar{x}_0 \\
&= sgn(x_0)[u, y]\Phi(P_{x_0}) \circ \bar{x}_0 \\
&= sgn(x_0)[u, y]P_{\bar{x}_0} \circ \bar{x}_0 \\
&= sgn(x_0)[u, y]\bar{x}_0.
\end{aligned}$$

Thus, $\Phi(I \circ A_{x_0,y}) \circ (I \circ U) = (I \circ U) \circ (I \circ A_{x_0,y})$. Post multiplying by $\circ(U \circ I)^{[*]}$, we get $\Phi(I \circ A_{x_0,y}) = c(I \circ U) \circ (I \circ A_{x_0,y}) \circ (U \circ I)^{[*]}$, where we have made use of the fact that $\circ cN = cI$.

Let A be a rank 1 matrix. By Lemma 2.11, there exist non-zero vectors x, y such that $A = I \circ A_{x,y}$.

Then

$$\begin{aligned}
\Phi(A) &= \Phi(I \circ A_{x,y}) \\
&= \Phi(sgn(x_0)I \circ A_{x,x_0} \circ I \circ A_{x_0,y}) \text{ (by (ii) of Lemma 2.10)} \\
&= sgn(x_0)\Phi(I \circ A_{x_0,x}^{[*]}) \circ \Phi(I \circ A_{x_0,y}) \\
&= sgn(x_0)I \circ \{\Phi(I \circ A_{x_0,x})\}^{[*]} \circ I \circ \Phi(I \circ A_{x_0,y}) \\
&= c\ sgn(x_0)I \circ \left\{c(I \circ U) \circ (I \circ A_{x_0,x}) \circ (U \circ I)^{[*]}\right\}^{[*]} \circ I \circ \Phi(I \circ A_{x_0,y}) \\
&= c^2 sgn(x_0)I \circ U \circ I \circ A_{x_0,x}^{[*]} \circ I \circ (I \circ U)^{[*]} \circ (U \circ I) \circ A_{x_0,y} \circ (U \circ I)^{[*]} \\
&= c\ sgn(x_0)I \circ U \circ I \circ A_{x,x_0} \circ I \circ A_{x_0,y} \circ (U \circ I)^{[*]} \text{ (by Equation (3.2))} \\
&= c\ I \circ U \circ I \circ A_{x,y} \circ (U \circ I)^{[*]} \text{ (by (ii) of Lemma 2.10)} \\
&= c\ (I \circ U) \circ A \circ (U \circ I)^{[*]}.
\end{aligned}$$

Thus $\Phi(A) = c(I \circ U) \circ A \circ (U \circ I)^{[*]}$ for all rank 1 matrices $A \in \mathbb{C}^{n\times n}$. □

Corollary 3.5. *Let* $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ *be a* $*-$*isomorphism. Then there exists* $U \in \mathbb{C}^{n\times n}$ *such that* $\Phi(A) = c(I \circ U) \circ A \circ (U \circ I)^{[*]}$ *for all* $A \in \mathbb{C}^{n\times n}$ *with* $(I \circ U)^{[*]} \circ (U \circ I) = cN = (I \circ U) \circ (U \circ I)^{[*]}$, *where* $c = \pm 1$. *Moreover,* U *is unique up to a scalar multiple of absolute value 1.*

Proof. By Equation (3.1), $U \circ x_0 = sgn(x_0)I \circ \bar{x}_0$. So,

$$\begin{aligned}
(U \circ I)^{[*]} \circ I \circ U \circ (sgn(x_0)x_0) &= (U \circ I)^{[*]} \circ I \circ (I \circ \bar{x}_0) \qquad (3.3) \\
&= (U \circ I)^{[*]} \circ \bar{x}_0.
\end{aligned}$$

Recall that $I \circ U^{[*]} \circ I \circ U \circ I = cI$, where $c = sgn(x_0)sgn(\bar{x}_0)$. Then $(U \circ I)^{[*]} \circ I \circ U = c\ N$, so that we have

$$\begin{aligned}
(U \circ I)^{[*]} \circ \bar{x}_0 &= (U \circ I)^{[*]} \circ I \circ U \circ (sgn(x_0)I \circ x_0) \\
&= c\ N \circ (sgn(x_0)x_0) \\
&= c\ sgn(x_0)x_0,
\end{aligned}$$

where we have used the fact that $N \circ x_0 = x_0$.

Let $x_0 \in \mathbb{C}^n$ with $[x_0, x_0] \pm 1$. Then $A \circ I \circ A_{v,x_0}$ is a rank 1 matrix for all $v \in \mathbb{C}^n$. By Theorem 3.4, there exists $U \in \mathbb{C}^{n\times n}$ such that

$$\Phi(A \circ I \circ A_{v,x_0}) = c(I \circ U) \circ A \circ I \circ A_{v,x_0} \circ (U \circ I)^{[*]},$$

with $(I \circ U)^{[*]} \circ (U \circ I) = cN = (I \circ U) \circ (U \circ I)^{[*]}$. Then

$$\begin{aligned}
\Phi(A) \circ I \circ U \circ v &= \Phi(A) \circ \Phi(I \circ A_{v,x_0}) \circ \bar{x}_0 \\
&= \Phi(A \circ I \circ A_{v,x_0}) \circ \bar{x}_0 \\
&= c(I \circ U) \circ A \circ I \circ A_{v,x_0} \circ (U \circ I)^{[*]} \circ \bar{x}_0 \\
&= sgn(x_0)(I \circ U) \circ A \circ I \circ A_{v,x_0} \circ x_0 \text{ (by Equation (3.3))} \\
&= (I \circ U) \circ A \circ v.
\end{aligned}$$

Thus $\Phi(A) = c(I \circ U) \circ A \circ (U \circ I)^{[*]}$.

Uniqueness of U follows in a manner similar to the Euclidean case (Section 5, [2]). □

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