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ON SOME FIXED POINT RESULTS IN d_p-COMPLETE HAUSDORFF SPACES

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Abstract

In this paper we define d_p - complete topological spaces for $p \geq 2$. A d_2 - complete topological space is a generalized d-complete topological space introduced by Troy L. Hicks. Some fixed point theorems for self-maps of d_p - complete Hausdorff topological spaces are established which generalize the results of Troy L. Hicks.

Keywords: d_p - complete topological spaces, d-complete topological spaces, orbitally lower semi continuous and orbitally continuous maps.

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1 . Introduction

Troy L. Hicks [5] has introduced *d-complete* topological spaces, attributing the basic ideas of these spaces to Kasahara ([8], [9]) and Iseki [7] as follows:

1.1 Definition: A topological space (*X*, *t*) is said to be *d- complete* if there is a mapping $d : X \times X \to [0, \infty)$ such that (i) $d(x, y) = 0 \Leftrightarrow x = y$ and (ii) $< x_n >$ is a sequence in *X* such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent implies that $< x_n >$ converges in (*X*, *t*).

In this paper we introduce d_2 - *complete* topological spaces as a generalization of *d*-complete topological spaces. In fact , we define d_p - complete topological spaces for any integer $p \geq 2$. For a non-empty set X, let X^p be its *p*-fold cartesian product.

1.2 Definition: A topological space (*X*, *t*) is said to be d_p - *complete* if there is a mapping $d_p : X^p \to [0, \infty)$ such that (i) $d_p(x_1, x_2, \ldots, x_p) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_p$ and (ii) $< x_n >$ is a sequence in X with $\lim_{n\to\infty} d_p(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+p-1}) = 0$ implies that $< x_n >$ converges to some point in (*X*, *t*). A d_p - complete topological space is denoted by (*X*, *t*, d_p)

1.3 Remark: The function *d* in the Definition 1.1 and the function d_2 (the case *p* = 2) in Definition 1.2 are both defined on $X \times X$ and satisfy condition (i) of the definitions which are identical. Since the convergence of an infinite series $\sum_{n=1}^{\infty} \alpha_n$ of real numbers implies that $\lim_{n\to\infty} \alpha_n = 0$, but not conversely; it follows that every *d*-

complete topological space is d_2 - complete, but not conversely. Therefore the class of d_2 - complete topological spaces is wider than the class of *d*-complete spaces and hence a separate study of fixed point theorems of self-maps on d_2 - complete topological spaces is meaningful.

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2 . Preliminaries

Let *X* be a non-empty set. A mapping $d_p : X^p \to [0, \infty)$ is called a p *non-negative* on X provided $d_p(x_1, x_2, \ldots, x_p) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_p$.

2.1 **Definition:** Suppose (*X*, *t*) is a topological space and d_p is a *p* non- negative on *X*. A sequence $< x_n >$ in X is said to be a d_p - *Cauchy sequence* if $d_p(x_n, x_{n+1}, \ldots, x_{n+p-1}) \to 0$ as $n \to \infty$.

In view of Definition 2.1, a topological space (*X*, *t*) is d_p - complete if there is a *p*-non- negative d_p on *X* such that every d_p - Cauchy sequence in *X* converges to some point in (*X*, *t*).

If *T* is a self map of a non-empty set *X* and $x \in X$, then the *orbit of x*, $O_T(x)$ is given by $O_T(x) = \{x, Tx, T^2x, \ldots\}$. If *T* is a self map of a topological space *X*, then a mapping $G : X \to [0, \infty)$ is said to be *T-orbitally lower semi-continuous* (resp. *T-orbitally continuous*) at $x^* \in X$ if $< x_n >$ is a sequence in $O_T(x)$ for some $x \in X$ with $x_n \to x^*$ as $n \to \infty$ then $G(x^*) \le \liminf_{n\to\infty} G(x_n)$ (resp. $G(x^*) = \lim_{n\to\infty} G(x_n)$). A self map *T* of topological space *X* is said to be *w-continuous* at $x \in X$ if $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

If d_p is a *p* non-negative on a non-empty set *X*, and $T : X \to X$ then we write, for simplicity of notation, that

(2.2) $$G_p(x) := d_p(x, Tx, T^2x, \ldots, T^{p-1}x) \text{ for } x \in X$$

Clearly we have

(2.3) $G_p(x) = 0$ if and only if *x* is a fixed point of *T* .

(2.4) $$H_p(x, y) = d_p(x, y, Ty, T^2y, \ldots, T^{p-2}y) \text{ for } x, y \in X$$

(2.5) $$E_p(x, y) = d_p(x, y, y, \ldots, y) \text{ for } x, y \in X$$

Clearly

(2.6) $$H_p(x, Tx) = G_p(x) \text{ and } E_p(x, x) = 0.$$

3 . Main results

3.1 **Theorem:** Suppose (X, t, d_p) is a d_p- complete Hausdorff topological space and T: X→ X is a w-continuous map such that $G_p(Tx) \le k(G_p(x))$ for all $x \in X$, where $k : [0,\infty) \to [0,\infty)$ is a non-decreasing function with k(0) = 0. Then T has a fixed point iff there is an $x \in X$ such that $\lim_{n\to\infty} k^n(G_p(x))=0$. In this case $\lim_{n\to\infty} T^n x = z$ and $Tz=z$.

Proof : Suppose T has a fixed point, say z, so that $z = Tz = \ldots = T^{p-1} z$ and hence $G_p(z) = 0$ which gives $k^n(G_p(z)) = 0$ for all $n \ge 1$.

$$\Rightarrow \lim_{n\to\infty} k^n(G_p(z))=0.$$

Conversely, suppose that $\lim_{n\to\infty} k^n(G_p(x))=0$ for some $x \in X$.

Let $x_n = T^n x$ for $n \ge 0$ with $x_0 = x$. Then

$$d_p(x_n, x_{n+1}, \ldots, x_{n+p-1}) = d_p(T^n x, T^{n+1}x, \ldots, T^{n+p-1}x)$$
$$= G_p(T^n x)$$
$$\le k(G_p(T^{n-1}x)) \le k^2\left(G_p\left(T^{n-2}x\right)\right) \le \ldots\ldots \le k^n\left(G_p(x)\right) \to 0 \text{ as } n \to \infty$$ which gives

(x_n) is a d_p- Cauchy sequence in X and hence converges, say to z. That is, $\lim_{n\to\infty} T^n x = z$ exists. Since

$x_n \to z$ as $n \to \infty$ and since T is w- continuous, we get that $Tx_n \to Tz$ as $n \to \infty$. That is,

$x_{n+1} \to Tz$ as $n \to \infty$ which gives $x_n \to Tz$ as $n \to \infty$.

Since X is Hausdorff, we get Tz =z.

3.2 Remark: It may be noted that in view of Remark 1.3, the result proved by Troy L. Hicks ([5], Theorem 2,pp.437) is a particular case of Theorem 3.1 .

3.3 Theorem : Suppose (X,t,d_p) is a d_p-complete Hausdorff topological space and $T : X \to X$ such that there is an $x \in X$ with $G_p(Ty) \le k(G_p(y))$ for all $y \in O_T(x)$, where $k : [0,\infty) \to [0,\infty)$ is a non-decreasing function with $k(0) = 0$. Suppose $k^n(G_p(x)) \to 0$ as $n \to \infty$.Then

a) $\lim_{n\to\infty} T^n x = z$ exists and

b) Tz = z iff $G_p(x)$ is T - orbitally lower semi-continuous at z.

Proof : (a) Consider $d_p(T^n x, T^{n+1}x, \ldots\ldots, T^{n+p-1}x)$

$= G_p(T^n x) \le k(G_p(T^{n-1}x)) \le k^2(G_p(T^{n-2}x))\ldots\ldots \le k^n(G_p(x)) \to 0$ as $n \to \infty$ by hypothesis, which gives that $(T^n x)$ is a d_p-Cauchy sequence in X and hence converges, say, to z.

(b) Suppose Tz = z and $(T^n x)$ is a sequence in $O_T(x)$ with $\lim_{n\to\infty} T^n x = z$.

Then $G_p(z) = 0 \le \liminf_{n\to\infty} G_p(T^n x)$ which gives the T-orbitally lower semi-continuity of $G_p(x)$ at z.

Conversely, suppose that $G_p(x)$ is T-orbitally lower semi-continuous at z.

By proof of part (a), $(T^n x)$ is a d_p-Cauchy sequence and hence $0 = \lim_{n\to\infty} G_p(T^n x)$.

This together with the T-orbitally lower semi-continuity of $G_p(x)$ at z imply Tz=z by Theorem 3.1.

3.4 Remark: The result of Troy L. Hicks ([5], Theorem 3, pp.437, 438) is a particular case of Theorem 3.3.

We now present some consequences of Theorems 3.1 and 3.3.

3.5 Corollary: Suppose (X, t, d_p) is a d_p-complete Hausdorff topological space, $T : X \to X$ is w-continuous and satisfying $H_p(Tx, Ty) \le \lambda H_p(x, y)$ for all $x, y \in X$, where $0 < \lambda < 1$. Then T has a unique fixed point z and $z = \lim_{n\to\infty} T^n x$ for any $x \in X$.

Proof : Taking $k(t) = \lambda t$, where $0 < \lambda < 1$.

Then $k^n(G_p(x)) = \lambda^n G_p(x)$ and therefore, by Theorem 3.1, we have $x_n \to z$ as $n \to \infty$, where Tz = z.

Suppose z and $z^*$are two fixed points and taking $x = z$, $y = z^*$ in the inequality, we get $H_p(Tz, Tz^*) \le \lambda H_p(z, z^*) \Rightarrow H_p(z, z^*) \le \lambda H_p(z, z^*)$, which gives $H_p(z, z^*) = 0$. That is, $z = z^*$, proving the uniqueness.

3.6 Remark: The result proved by Troy L. Hicks ([5], Corollary 2,pp.438) is a particular case of Corolloary 3.5.

3.7 Corollary: Suppose (X, t, d_p) is a d_p-complete Hausdorff topological space, $T : X \to X$ and $H_p(Tx, Ty) \le \phi(H_p(x,y)) H_p(x,y)$ for all $x, y \in X$, where $\phi : [0, \infty) \to [0,1)$ is non-decreasing. If $x_n = T^n y$, then

a) $\lim_{n\to\infty} x_n = z$ exists and

b) Tz = z iff $G_p(x)$ is T-orbitally lower semi-continuous at z.

Proof :Taking $k(t) = t\,\phi(t)$.

We find by induction that $k^n(t) \leq t\,(\phi(t))^n$ and the inequality of the theorem gives

$H_p(Tx, Ty) \leq k(H_p(x,y))$ for all $x, y \in X$.

Now, taking $y = Tx$ in the above inequality, we get

$H_p(Tx, T^2x) \leq k(H_p(x, Tx)$ for all $x \in X$

$\Rightarrow G_p(Tx) \leq k(G_p(x))$ for all $x \in X$.

Since $\phi(t) < 1$, $k^n(G_p(x)) \to 0$ as $n \to \infty$ and therefore the theorem follows from Theorem 3.3.

3.8 Remark : Note that, the result of Troy L. Hicks ([5], pp.438) is a particular case of Corollary 3.7.

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