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A GENERALISED DENSITY

by

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Abstract: The object of the present paper is to study asymptotic density associated with a class of three parameter matrices which are regular and almost positive.

Key words: regular, coregular and almost positive matrices, asymptotic density.

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1. Definitions and Notations:

Let $\mathcal{A}$ denote the sequence of matrices $A^i = (a_{nk}(i))$ of complex numbers $\mathbb{C}$. For a sequence $x = (x_k) \in \mathbb{C}$, we write $\mathcal{A}x$ for$(A^i_n(x))^{\infty}_{n.i=0}$ where

$$A^i_n(x) = \sum_k a_{nk}(i)x_k \tag{1.1}$$

if it exists for each n and $i \geq 0$. (The summation without limits is from 0 to ∞.)

The matrix defined by

$$a_{nk}(i) = \begin{cases} 1, & n = k, \quad \text{for all } i \\ 0, & n \neq k, \quad \text{for all } i \end{cases}$$

is the identity matrix.

A sequence x is said to be summable to the value s by the method $\mathcal{A}$ if $\lim_n A^i_n(x) = s$ uniformly in i and in that case, we write $\mathcal{A}x \to s$. In the case

$$a_{nk}(i) = \begin{cases} \frac{1}{n+1}, & i \leq k \leq i+n \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$\mathcal{A}$ reduces to the method of $\widehat{c}$ defined by Lorentz[5] yielding the space $\widehat{c}$ of almost convergent sequences. If $\mathcal{A} = A = (a_{nk})$, then we obtain the usual summability method A. It is significant to note that there does not exist any regular method A equivalent to the method $\widehat{c}$ (see Lorentz[5]).In the case

$$a_{nk}(i) = \frac{1}{n+1}\sum_{r=i}^{n+i} a_{rk} \tag{1.3}$$

$\mathcal{A}$ reduces to the almost summability method introduced by King[4]. Thus the method $\mathcal{A}$ provides a comprehensive generalised method.

The method $\mathcal{A}$ is called *conservative* if $x \to s \Rightarrow \mathcal{A}x \to s'$; *regular* if $s = s'$. The following characterisation of conservative matrices is due to Stieglitz[7].

The method $\mathcal{A}$ is *conservative* if and only if the following conditions hold

$$\|\mathcal{A}\| = \sup_{n,i}\sum_k |a_{nk}(i)| < \infty \tag{1.4}$$

$$\exists\ a_k \in \mathbb{C} : \lim_n a_{n_k}(i) = a_k \text{ uniformly in } i \tag{1.5}$$

$$\exists\ a \in \mathbb{C} : \lim_n \sum_k a_{n_k}(i) = a \text{ uniformly in } i \tag{1.6}$$

The method $\mathcal{A}$ is *regular*, if further $a_k = 0, a = 1$. We write

$$\alpha(\mathcal{A}) = a - \sum_k a_k \tag{1.7}$$

The method $\mathcal{A}$ is *conull* if $\alpha(\mathcal{A}) = 0$, otherwise *coregular*.

Throughout the paper the hypothesis (1.4) shall mean that there exists a constant $K > 0$ such that

$$\sum_k |a_{n_k}(i)| \leq K \text{ (for all } n\text{, for all } i)$$

and the series $\sum_k a_{nk}(i)$ converges uniformly in i and for each n.

Let us now take $\mathcal{A}$ as the sequence of matrices $A^i = (a_{nk}(i))$ of real numbers in the following definitions and in main results.

We write

$$a^+_{nk}(i) = \max(a_{nk}(i), 0),\ \ a^-_{nk}(i) = \max(-a_{nk}(i), 0)$$

The method $\mathcal{A} = (a_{nk}(i))$ of real numbers is called *almost positive* if and only if

$$\lim_n \sum_k a^-_{nk}(i) = 0 \text{ uniformly in } i \tag{1.8}$$

This is parallel to the definition of almost positive matrix $A = (a_{nk})$ given by Simons [6] Let $P(N)$ denote the power set of all positive integers $N = \{1, 2, 3...\}$.

A function $\delta : P(N) \to [0, 1]$ is called *lower asymptotic density* if the following axioms hold (Freedman and Sember [3]): For $E, F \subset P(N)$

$$D_1 : A\Delta B \text{ is finite } \Rightarrow \delta(E) = \delta(F)$$

$$D_2 : E \cap F = \phi \Rightarrow \delta(E) + \delta(F) \leq \delta(E \cup F)$$

$$D_3 : \delta(E) + \delta(F) \leq 1 + \delta(E \cap F)$$

$$D_4 : \delta(N) = 1.$$

The *upper asymptotic density*, denoted as $\overline{\delta}$, associated with δ is defined as

$$\overline{\delta}(E) = 1 - \delta(E^{'})$$

where $E^{'}$ is the complement of E. In case $\delta(E) = \overline{\delta}(E)$, then δ is called *natural density or asymptotic density or* just *density*.

2.Introduction:

Let χ_E be the characteristic function of the set E.Then for $E \subseteq N$

$$\delta_*(E) = \liminf_n \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \tag{2.1}$$

provides an example of lower asymptotic density as it can be easily verified (see [3]) that conditions $D_1 - D_4$ stated above are fulfilled by $\delta_*(E)$.

The upper asymptotic density is given by

$$\begin{aligned} \overline{\delta}_*(E) = 1 - \delta_*(E^{'}) = 1 - \liminf_n \frac{1}{n} \sum_{k=1}^{n} \chi_{E^{'}}(k) \\ = \limsup_n \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) = \delta^*(E) \quad \text{(say)} \end{aligned} \tag{2.2}$$

We have the natural density $\delta(E)$ if $\delta_*(E) = \delta(E) = \delta^*(E)$ and in that case

$$\delta(E) = \lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \tag{2.3}$$

The above example provided the motivation for Freedman and Sember [3] to consider lower asymptotic density for regular and non-negative summability method.

In the same paper, they considered yet another example of lower density, called *uniform density* introduced as follows:

$$u(E) = \lim_{n\to\infty} \min_{m\geq 0} \frac{1}{n} \sum_{i=m+1}^{m+n} \chi_E(i) \tag{2.4}$$

It may be noted that the transformation (1.2) is associated with the concept of almost convergence and it is known that no summability method is equivalent to (1.2) (see Lorentz [5]). Therefore we now propose a kind of generalised density which yields simultaneously as corollaries both density defined by (2.1) and uniform density defined by (2.4).

3. The Main Theorems:

We now extend the concept of density and uniform density for class of regular and almost positive matrices of real numbers in the following theorems.

***Theorem 1*:** Let $\mathcal{A} = (a_{nk}(i))$ be almost positive and regular.

Then for $E \subseteq N$

(a) The function $\underline{\delta}_{\mathcal{A}} : P(N) \to [0,1]$, given by

$$\underline{\delta}_{\mathcal{A}}(E) = \liminf_n \ \inf_i \sum_k a_{nk}(i)\chi_E(k) \tag{3.1}$$

is a lower asymptotic density

(b) The upper asymptotic density is given by

$$\overline{\delta}_{\mathcal{A}}(E) = \limsup_n \ \sup_i \sum_k a_{nk}(i)\chi_E(k) \tag{3.2}$$

(c) The density $\delta_{\mathcal{A}}(E)$ is ensured when $\underline{\delta}_{\mathcal{A}}(E) = \delta_{\mathcal{A}}(E) = \overline{\delta}_{\mathcal{A}}(E)$; that is

$$\delta_{\mathcal{A}}(E) = \lim_n \sum_k a_{nk}(i)\chi_E(k), \text{uniformly in } i \tag{3.3}$$

is a asymptotic density.

***Theorem 2* :** Let $\mathcal{A} = (a_{nk}(i))$ be coregular and $\mathcal{B} = (b_{nk}(i)) = \left(\dfrac{a_{nk}(i) - a_k}{\alpha(\mathcal{A})}\right)$ be almost positive. Then for $E \subseteq N$

(a) The function $\underline{\delta}_{\mathcal{B}} : P(N) \to [0,1]$ given by

$$\underline{\delta}_{\mathcal{B}}(E) = \liminf_n \ \inf_i \sum_k b_{nk}(i)\chi_E(k)$$

is a lower asymptotic density.

(b) The upper asymptotic density is given by

$$\overline{\delta}_{\mathcal{B}}(E) = \limsup_n \ \sup_i \sum_k b_{nk}(i)\chi_E(k)$$

4. Lemmas:

We need the following Lemmas for the proof of above theorems.

***Lemma 1* ([2], Theorem 2):** Let $\mathcal{A} = (a_{nk}(i))$ be conservative and $\|\mathcal{A}\| < \infty$. Then

$$\limsup_n \ \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leq \frac{|\alpha(\mathcal{A})| + \alpha(\mathcal{A})}{2} \limsup x - \frac{|\alpha(\mathcal{A})| - \alpha(\mathcal{A})}{2} \liminf x$$

if and only if

$$\limsup_n \ \sup_i \sum_k |a_{nk}(i) - a_k| = |\alpha(\mathcal{A})| \tag{4.1}$$

***Lemma 2*:** Let $\|\mathcal{A}\| < \infty$. Then

$$\limsup_n \ \sup_i \sum_k a_{nk}(i)x_k \leq \limsup_n \ x_n \tag{4.2}$$

if and only if $\mathcal{A}$ is regular and almost positive.

Proof: (Sufficiency) By definition

$$\sum_k |a_{nk}(i)| = \sum_k a^+_{nk}(i) + \sum_k a^-_{nk}(i) \tag{4.3}$$

$$\sum_k a_{nk}(i) = \sum_k a^+_{nk}(i) - \sum_k a^-_{nk}(i) \tag{4.4}$$

By definition of almost positiveness and identities (4.3) and (4.4) it follows that a regular method is almost positive if and only if

$$\limsup_n \sup_i \sum_k |a_{nk}(i)| = 1 \tag{4.5}$$

Now it follows from Lemma 1 (since $\alpha(\mathcal{A}) = 1$)that the inequality holds.

(Necessity) This follows from Lemma 1 by taking $\alpha(\mathcal{A}) = 1$

***Lemma 3*:**

$$\limsup_n \ \sup_i \sum_k a_{nk}(i)\chi_E(k) \leq 1 \tag{4.6}$$

if and only if $\mathcal{A}$ is regular and almost positive.

Proof: This follows from Lemma 2 by taking $x_k = \chi_E(k)$ and noting that $\limsup_k \ \chi_E(k) = 1$.

5. Proof of Theorem 1:

To prove that $\delta_{\mathcal{A}}(E)$ is a lower density we have to prove that $\delta_{\mathcal{A}}(E)$ satisfies the conditions $D_1 - D_4$.

$E \Delta F$ be finite $\Rightarrow \chi_E(k) = \chi_F(k), k > K$ for some integer K.

$$\Rightarrow \sum_k a_{nk}(i)(\chi_E(k) - \chi_F(k))$$
$$= \sum_{k=0}^{K} a_{nk}(i)(\chi_E(k) - \chi_F(k)) \tag{5.1}$$

The expression (5.1) is bounded by $\sum_{k=0}^{K} |a_{nk}(i)|$ which converges to 0 as $n \to \infty$ uniformly in i.Hence it follows that

$$\underline{\delta}_{\mathcal{A}}(E) = \underline{\delta}_{\mathcal{A}}(F)$$

This proves the requirement D_1.Next

$$E \cap F = \phi \Rightarrow \chi_{E \cup F} = \chi_E + \chi_F \tag{5.2}$$

Since the functional $\underline{\delta}_{\mathcal{A}}(E)$ is super additive, we have

$$\underline{\delta}_{\mathcal{A}}(E \cup F) \geq \underline{\delta}_{\mathcal{A}}(E) + \underline{\delta}_{\mathcal{A}}(F)$$

This proves D_2.

Since $\chi_{E \cap F} = \chi_E + \chi_F - \chi_{E \cup F}$; we have

$$\begin{aligned}1 + \underline{\delta}_{\mathcal{A}}(E \cap F) &\geq 1 + \liminf_n \inf_i \sum_k a_{nk}(i)\chi_E(k) \\ &+ \liminf_n \inf_i \sum_k a_{nk}(i)\chi_F(k) \\ &+ \liminf_n \inf_i \left(-\sum_k a_{nk}(i)\chi_{E \cup F}(k) \right) \\ &= 1 + \underline{\delta}_{\mathcal{A}}(E) + \underline{\delta}_{\mathcal{A}}(F) - \overline{\delta}_{\mathcal{A}}(E \cup F) \\ &\geq \underline{\delta}_{\mathcal{A}}(E) + \underline{\delta}_{\mathcal{A}}(F) \qquad \text{(by Lemma 3)}\end{aligned}$$

This proves D_3.

Lastly

$$\underline{\delta}_{\mathcal{A}}(N) = \liminf_n \inf_i \sum_k a_{nk}(i)\chi_N(k) = \liminf_n \inf_i \sum_k a_{nk}(i) = 1$$

This proves Theorem 1 (a).

Now since $\chi_{E'} = 1 - \chi_E$, it follows that

$$\begin{aligned}\overline{\delta}_{\mathcal{A}}(E) &= 1 - \delta_{\mathcal{A}}(E^{'}) \\ &= 1 - \liminf_n \inf_i \sum_k a_{nk}(i)(1 - \chi_E(k)) \\ &= \limsup_n \sup_i \sum_k a_{nk}(i)\chi_E(k)\end{aligned}$$

This proves Theorem 1(b).

In the case $\underline{\delta}_{\mathcal{A}}(E) = \overline{\delta}_{\mathcal{A}}(E)$ then it can be easily shown that Theorem 1(c) holds.

6. Proof of Theorem 2:

By definition

$$\lim_n \sum_k b_{nk}(i) = \lim_n \sum_k \frac{(a_{nk}(i) - a_k)}{\alpha(\mathcal{A})} = 1 \tag{6.1}$$

uniformly in i, and

$$\lim_n b_{nk}(i) = \lim_n \frac{(a_{nk}(i) - a_k)}{\alpha(\mathcal{A})} = 0 \tag{6.2}$$

uniformly in i, since $\alpha(A) \neq 0$. Lastly

$$\sup_{n,i} \sum_k |b_{nk}(i)| \leq \sup_{n,i} \sum_k |a_{nk}(i)| + \sum_k |a_k| < \infty \tag{6.3}$$

So from (6.1), (6.2) and (6.3), we can conclude that $\mathcal{B} = (b_{nk}(i)) = \left(\frac{a_{nk}(i) - a_k}{\alpha(\mathcal{A})}\right)$ is a regular method. Also it is given that $\mathcal{B}$ is almost positive. Hence by Theorem 1 of the present work we can easily establish Theorem 2.

7. A Corollary:

Let $\mathcal{A} = A = (a_{nk})$ be a regular and almost positive matrix. Then for $E \subseteq N$

$$\begin{aligned}\underline{\delta}_{\mathcal{A}}(E) &= \liminf_n \inf_i \sum_k a_{nk}\chi_E(k+i) \\ &= \liminf_n \inf_i \sum_{k=i}^{\infty} a_{n,k-i}\chi_E(k)\end{aligned} \tag{7.1}$$

is a lower asymptotic density.

(b) The upper asymptotic density is given by

$$\overline{\delta}_{\mathcal{A}}(E) = \limsup_{n} \sup_{i} \sum_{k} a_{nk}\chi_E(k+i) \tag{7.2}$$

Now we can easily verify the corollary .

Remark:

In the special case

$$a_{n,k} = \begin{cases} \frac{1}{n+1}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

$\sum_k a_{nk}\chi_E(k+i)$ reduces to $\dfrac{1}{n+1}\sum_{k=i}^{n+i}\chi_E(k)$ and consequently $\underline{\delta}_{\mathcal{A}}(E)$ and $\overline{\delta}_{\mathcal{A}}(E)$ (in (7.1) and(7.2)) respectively represent lower and upper uniform density(see [1],[3]).

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