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An M^X/G/1 Queue with Two-Stage Heterogeneous Service and Multiple Working Vacation

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Abstract

In this paper, we study an M^X/G/1 queue with two stages of heterogeneous service and multiple working vacation. Using supplementary variable technique we derive the probability generating function for the number of customers in the system and the average number of customers in the system. Some special cases of interest are discussed.

Keywords: Batch arrivals, Two-Stage Heterogeneous Service, Working Vacation, Supplementary Variable Technique.

AMS Subject Classification Number: 60K25, 60K30.

1. Introduction

Queues with server vacations occur in many engineering systems such as data switching systems, computer communication networks and telecommunications systems. The $M/G/1$ queue with vacation time has been studied by number of authors. To mention a few references we would name Cooper (1981), Levy et.al. (1975), Yukata Baba (1986), Teghem (1986) and Doshi (1990).

In many queueing situations all arriving customers require the main service (First Stage) and only some of them may require the subsidiary service (Second Stage) provided by the server. Such type of queueing model was introduced by Madan. Madan (2000) investigated an $M/G/1$ queueing system with second optional service. The single server vacation queueing models with second optional service were analyzed by many authors including Kalyanaraman et.al. (2008) and Thangaraj (2010).

Recently a class of semi-vacation policies has been introduced by Servi and Finn. Such a vacation is called working vacation(WV). The server works at a lower rate rather than completely stops service during a vacation. Servi and Finn (2002) studied an $M/M/1$ queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution.Some other notable works were done by Wu and Takagi (2006), Tian et.al. (2008) and Afthab Begum (2011).

In this paper we consider the group arrival $M^X/G/1$ queue with Two-stage Heterogeneous service and multiple working vacation(MWV). The organization of the paper is as follows. In section 2 we described the model. In section 3 we obtained the steady state probability generating function. In section 4 performance measures are obtained and in section 5 some particular cases have been discussed.

2. The Model description

We consider an $M^X/G/1$ queue with Two-stage Heterogeneous service and multiple working vacation (MWV). It is assumed that customers arrive in groups according to a time homogeneous Poisson process with rate $\lambda(>0)$. The group size X is a random variable and $P_r(X=n)=g_n, n=1,2,3\ldots$ with probability generating function $X(z)=\sum_{k=1}^{\infty} g_k z^k$ and the first and second factorial moments of X are defined by $g^{(1)}=E(X)=X'(1)$ and $g^{(2)}=E(X(X-1))=X''(1)$ respectively. The service discipline is FCFS. Each arriving customer undergoes the first essential stage (FES)of service which has general distribution with distribution function $S_{b_1}(x)$, the probability density function $s_{b_1}(x)$ and the Laplace-Stieltjes transform(LST) $S_{b_1}^*(\theta)$ where S_{b_1} is the service time of the first essential stage of service.

After completion of FES of service the customer may opt for the second optional stage (SOS)of service with probability p or the customer may leave the system without taking the SOS of service with probability $q(p+q=1)$. The SOS of service follows the general distribution with the distribution function $S_{b_2}(x)$, the probability density function $s_{b_2}(x)$ and the LST $S_{b_2}^*(\theta)$ where S_{b_2} is the service time of the second optional stage of service.

Whenever the system becomes empty at a service completion instant, the server starts a working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant if there are customers in the system the server will start a new busy period. Otherwise he takes another working vacation.This type of vacation policy is called Multiple Working Vacation.

During the working vacation period the server also provides two stages of service. The first essential stage of service time S_{v_1} of a typical customer follows a general distribution with the distribution function $S_{v_1}(x)[s_{v_1}(x)$, the probability density function and $S^*_{v_1}(\theta)$, the LST] and the SOS of service time S_{v_2} also follows a general distribution with the distribution function $S_{v_2}(x)[s_{v_2}(x)$, the probability density function and $S^*_{v_2}(\theta)$,the LST].

Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with different distribution at the beginning of the following service period. Inter arrival times, service times and working vacation times are mutually independent of each other.

3. The System Size Distribution at a Random Epoch

The system size distribution at an arbitrary time will be treated by the supplementary variable technique.That is, from the joint distribution of the queue length and the remaining service time of the customer in service if the server is busy, or the remaining service time of the customer if the server is on WV. Assuming that the system is in steady state condition. Let us define the following random variables.

$N(t)$ - the system size at time t.

$S^0_{b_1}(t)$ - the remaining service time for the FES of service in not WV period.

$S^0_{b_2}(t)$ - the remaining service time for the SOS of service in not WV period.

$S^0_{v_1}(t)$ - the remaining service time for the FES of service in WV period.

$S^0_{v_2}(t)$ - the remaining service time for the SOS of service in WV period.

$$Y(t) = \begin{cases} 0 & \text{if the server is idle on vacation} \\ 1 & \text{if the server is busy for giving FES of service in not WV period} \\ 2 & \text{if the server is busy for giving SOS of service in not WV period} \\ 3 & \text{if the server is busy for giving FES of service in WV period} \\ 4 & \text{if the server is busy for giving SOS of service in WV period} \end{cases}$$

so that the supplementary variables $S_{b_1}^0(t), S_{b_2}^0(t), S_{v_1}^0(t)$ and $S_{v_2}^0(t)$ are introduced in order to obtain bivariate Markov Process $\{N(t), \partial(t); t \geq 0\}$, where

$$\partial(t) = \begin{cases} S_{b_1}^0(t) & \text{if } Y(t) = 1 \\ S_{b_2}^0(t) & \text{if } Y(t) = 2 \\ S_{v_1}^0(t) & \text{if } Y(t) = 3 \\ S_{v_2}^0(t) & \text{if } Y(t) = 4 \end{cases}$$

We define the following limiting probabilities:

$$Q_0 = \lim_{t\to\infty} \Pr\{N(t) = 0, Y(t) = 0\}$$

$$Q_{n,1}(x) = \lim_{t\to\infty} \Pr\{N(t) = n, Y(t) = 3, x < S_{v_1}^0(t) \leq x + dx\}, n \geq 1.$$

$$Q_{n,2}(x) = \lim_{t\to\infty} \Pr\{N(t) = n, Y(t) = 4, x < S_{v_2}^0(t) \leq x + dx\}, n \geq 1.$$

$$P_{n,1}(x) = \lim_{t\to\infty} \Pr\{N(t) = n, Y(t) = 1, x < S_{b_1}^0(t) \leq x + dx\}, n \geq 1.$$

$$P_{n,2}(x) = \lim_{t\to\infty} \Pr\{N(t) = n, Y(t) = 2, x < S_{b_2}^0(t) \leq x + dx\}, n \geq 1.$$

By considering the steady state, we have the following system of the differential difference equations.

$$\lambda Q_0 = qP_{1,1}(0) + P_{1,2}(0) + qQ_{1,1}(0) + Q_{1,2}(0) \tag{1}$$

$$-\frac{d}{dx}Q_{1,1}(x) = -(\lambda+\eta)Q_{1,1}(x) + qQ_{2,1}(0)s_{v_1}(x) + Q_{2,2}(0)s_{v_1}(x) + \lambda g_1 Q_0 s_{v_1}(x) \tag{2}$$

$$-\frac{d}{dx}Q_{n,1}(x) = -(\lambda+\eta)Q_{n,1}(x) + qQ_{n+1,1}(0)s_{v_1}(x) + Q_{n+1,2}(0)s_{v_1}(x) + Q_0\lambda g_n s_{v_1}(x) + \lambda\sum_{k=1}^{n-1} Q_{n-k,1}(x)g_k; n>1 \tag{3}$$

$$-\frac{d}{dx}Q_{1,2}(x) = -(\lambda+\eta)Q_{1,2}(x) + pQ_{1,1}(0)s_{v_2}(x) \tag{4}$$

$$-\frac{d}{dx}Q_{n,2}(x) = -(\lambda+\eta)Q_{n,2}(x) + pQ_{n,1}(0)s_{v_2}(x) + \lambda\sum_{k=1}^{n-1} Q_{n-k,2}(x)g_k; n>1 \tag{5}$$

$$-\frac{d}{dx}P_{1,1}(x) = -\lambda P_{1,1}(x) + qP_{2,1}(0)s_{b_1}(x) + P_{2,2}(0)s_{b_1}(x) + \eta s_{b_1}(x)\int_0^\infty Q_{1,1}(y)dy \tag{6}$$

$$-\frac{d}{dx}P_{n,1}(x) = -\lambda P_{n,1}(x) + qP_{n+1,1}(0)s_{b_1}(x) + P_{n+1,2}(0)s_{b_1}(x) + \eta s_{b_1}(x)\int_0^\infty Q_{n,1}(y)dy + \lambda\sum_{k=1}^{n-1} P_{n-k,1}(x)g_k; n>1 \tag{7}$$

$$-\frac{d}{dx}P_{1,2}(x) = -\lambda P_{1,2}(x) + pP_{1,1}(0)s_{b_2}(x) + \eta s_{b_2}(x)\int_0^\infty Q_{1,2}(y)dy \tag{8}$$

$$-\frac{d}{dx}P_{n,2}(x) = -\lambda P_{n,2}(x) + pP_{n,1}(0)s_{b_2}(x) + \eta s_{b_2}(x)\int_0^\infty Q_{n,2}(y)dy + \lambda\sum_{k=1}^{n-1} P_{n-k,2}(x)g_k; n>1 \tag{9}$$

We define the Laplace Stieltjes Transforms and the probability generating functions as follows,

$$S_{b_i}^*(\theta) = \int_0^\infty e^{-\theta x}s_{b_i}(x)dx; \qquad i=1,2$$

$$S_{v_i}^*(\theta) = \int_0^\infty e^{-\theta x}s_{v_i}(x)dx; \qquad i=1,2$$

$$Q_{n,i}^*(\theta) = \int_0^\infty e^{-\theta x}Q_{n,i}(x)dx; \qquad i=1,2$$

$$P_{n,i}^*(\theta) = \int_0^\infty e^{-\theta x}P_{n,i}(x)dx; \qquad i=1,2$$

$$Q_i^*(z,\theta) = \sum_{n=1}^\infty z^n Q_{n,i}^*(\theta); \qquad i=1,2$$

$$Q_i(z,0) = \sum_{n=1}^\infty z^n Q_{n,i}(0); \qquad i=1,2$$

$$P_i^*(z,\theta) = \sum_{n=1}^\infty z^n P_{n,i}^*(\theta); \qquad i=1,2$$

$P_i(z,0) = \sum_{n=1}^{\infty} z^n P_{n,i}(0); \quad i = 1,2$

Taking the LST of (2) to (9) we have

$$\theta Q_{1,1}^*(\theta) - Q_{1,1}(0) = (\lambda+\eta)Q_{1,1}^*(\theta) - qQ_{2,1}(0)S_{v_1}^*(\theta) - Q_{2,2}(0)S_{v_1}^*(\theta) - \lambda g_1 Q_0 S_{v_1}^*(\theta) \quad (10)$$

$$\theta Q_{n,1}^*(\theta) - Q_{n,1}(0) = (\lambda+\eta)Q_{n,1}^*(\theta) - qQ_{n+1,1}(0)S_{v_1}^*(\theta) - Q_{n+1,2}(0)S_{v_1}^*(\theta) - \lambda g_n Q_0 S_{v_1}^*(\theta) - \lambda \sum_{k=1}^{n-1} Q_{n-k,1}^*(\theta) g_k; n > 1 \quad (11)$$

$$\theta Q_{1,2}^*(\theta) - Q_{1,2}(0) = (\lambda+\eta)Q_{1,2}^*(\theta) - pQ_{1,1}(0)S_{v_2}^*(\theta) \quad (12)$$

$$\theta Q_{n,2}^*(\theta) - Q_{n,2}(0) = (\lambda+\eta)Q_{n,2}^*(\theta) - pQ_{n,1}(0)S_{v_2}^*(\theta) - \lambda \sum_{k=1}^{n-1} Q_{n-k,2}^*(\theta) g_k; n > 1 \quad (13)$$

$$\theta P_{1,1}^*(\theta) - P_{1,1}(0) = \lambda P_{1,1}^*(\theta) - qP_{2,1}(0)S_{b_1}^*(\theta) - P_{2,2}(0)S_{b_1}^*(\theta) - \eta S_{b_1}^*(\theta)\int_0^{\infty} Q_{1,1}(y)dy \quad (14)$$

$$\theta P_{n,1}^*(\theta) - P_{n,1}(0) = \lambda P_{n,1}^*(\theta) - qP_{n+1,1}(0)S_{b_1}^*(\theta) - P_{n+1,2}(0)S_{b_1}^*(\theta) - \eta S_{b_1}^*(\theta)\int_0^{\infty} Q_{n,1}(y)dy - \lambda \sum_{k=1}^{n-1} P_{n-k,1}^*(\theta) g_k; n > 1 \quad (15)$$

$$\theta P_{1,2}^*(\theta) - P_{1,2}(0) = \lambda P_{1,2}^*(\theta) - pP_{1,1}(0)S_{b_2}^*(\theta) - \eta S_{b_2}^*(\theta)\int_0^{\infty} Q_{1,2}(y)dy \quad (16)$$

$$\theta P_{n,2}^*(\theta) - P_{n,2}(0) = \lambda P_{n,2}^*(\theta) - pP_{n,1}(0)S_{b_2}^*(\theta) - \eta S_{b_2}^*(\theta)\int_0^{\infty} Q_{n,2}(y)dy - \lambda \sum_{k=1}^{n-1} P_{n-k,2}^*(\theta) g_k; n > 1 \quad (17)$$

z^n times (11) summing over n from 2 to ∞ is added up with z times (10) we get

$$[\theta - (\lambda - \lambda X(z) + \eta)]Q_1^*(z,\theta) = Q_1(z,0)[1 - \frac{q}{z}S_{v_1}^*(\theta)] + S_{v_1}^*(\theta)\left[qQ_{1,1}(0) - \frac{Q_2(z,0)}{z} + Q_{1,2}(0) - \lambda Q_0 X(z)\right] \quad (18)$$

z^n times (13) summing over n from 2 to ∞ is added up with z times (12) we get

$$[\theta - (\lambda - \lambda X(z) + \eta)]Q_2^*(z,\theta) = Q_2(z,0) - pS_{v_2}^*(\theta)Q_1(z,0) \quad (19)$$

Inserting $\theta = (\lambda - \lambda X(z) + \eta)$ in (18) and (19) we get

$$Q_1(z,0) = \left[\frac{zS_{v_1}^*(\lambda - \lambda X(z) + \eta)[\lambda Q_0 X(z) - qQ_{1,1}(0) - Q_{1,2}(0)]}{z - (1-p)S_{v_1}^*(\lambda - \lambda X(z) + \eta) - pS_{v_1}^*(\lambda - \lambda X(z) + \eta)S_{v_2}^*(\lambda - \lambda X(z) + \eta)}\right] \quad (20)$$

The denominator of the above equation has a unique root z_1 in $(0,1)$. Therefore

$qQ_{1,1}(0)+Q_{1,2}(0)=\lambda Q_0 X(z_1)$

Substituting this in (20) we have

$$Q_1(z,0)=\left[\frac{\lambda z Q_0 S^*_{v_1}(\lambda-\lambda X(z)+\eta)(X(z)-X(z_1))}{z-(1-p)S^*_{v_1}(\lambda-\lambda X(z)+\eta)-pS^*_{v_1}(\lambda-\lambda X(z)+\eta)S^*_{v_2}(\lambda-\lambda X(z)+\eta)}\right] \tag{21}$$

and

$$Q_2(z,0)=\left[\frac{pS^*_{v_2}(\lambda-\lambda X(z)+\eta)\lambda z Q_0(X(z)-X(z_1))S^*_{v_1}(\lambda-\lambda X(z)+\eta)}{z-(1-p)S^*_{v_1}(\lambda-\lambda X(z)+\eta)-pS^*_{v_1}(\lambda-\lambda X(z)+\eta)S^*_{v_2}(\lambda-\lambda X(z)+\eta)}\right] \tag{22}$$

Substituting (21) and (22) in (18) and (19) and putting $\theta=0$, we have

$$Q^*_1(z,0)=\left[\frac{\lambda z Q_0(X(z)-X(z_1))[1-S^*_{v_1}(\lambda-\lambda X(z)+\eta)]}{(\lambda-\lambda X(z)+\eta)[z-(1-p)S^*_{v_1}(\lambda-\lambda X(z)+\eta)-pS^*_{v_1}(\lambda-\lambda X(z)+\eta)S^*_{v_2}(\lambda-\lambda X(z)+\eta)]}\right] \tag{23}$$

$$Q^*_2(z,0)=\left[\frac{p\lambda Q_0 z(X(z)-X(z_1))S^*_{v_1}(\lambda-\lambda X(z)+\eta)[1-S^*_{v_2}(\lambda-\lambda X(z)+\eta)]}{(\lambda-\lambda X(z)+\eta)[z-(1-p)S^*_{v_1}(\lambda-\lambda X(z)+\eta)-pS^*_{v_1}(\lambda-\lambda X(z)+\eta)S^*_{v_2}(\lambda-\lambda X(z)+\eta)]}\right] \tag{24}$$

z^n times (15) summing over n from 2 to ∞ is added up with z times (14) we get

$$\begin{aligned}[\theta-(\lambda-\lambda X(z))]P^*_1(z,\theta)=P_1(z,0)&\left[\frac{z-qS^*_{b_1}(\theta)}{z}\right]\\ &-S^*_{b_1}(\theta)\left[\eta Q^*_1(z,0)-qP_{1,1}(0)+\frac{P_2(z,0)}{z}-P_{1,2}(0)\right]\end{aligned} \tag{25}$$

z^n times (17) summing over n from 2 to ∞ is added up with z times (16) we get

$$[\theta-(\lambda-\lambda X(z))]P^*_2(z,\theta)=P_2(z,0)-pS^*_{b_2}(\theta)P_1(z,0)-\eta S^*_{b_2}(\theta)Q^*_2(z,0) \tag{26}$$

Inserting $\theta=(\lambda-\lambda X(z))$ in (25) and (26) and also substituting $qP_{1,1}(0)+P_{1,2}(0)$

$=\lambda Q_0(1-X(z_1))$ in (25) we get

$$P_1(z,0)=\left[\frac{S^*_{b_1}(\lambda-\lambda X(z))[\eta z Q^*_1(z,0)+\eta S^*_{b_2}(\lambda-\lambda X(z))Q^*_2(z,0)-\lambda z Q_0(1-X(z_1))]}{z-(1-p)S^*_{b_1}(\lambda-\lambda X(z))-pS^*_{b_1}(\lambda-\lambda X(z))S^*_{b_2}(\lambda-\lambda X(z))}\right] \tag{27}$$

and

$$P_2(z,0)=\frac{\left[\begin{array}{l} S^*_{b_2}(\lambda-\lambda X(z))\Big\{pzS^*_{b_1}(\lambda-\lambda X(z))[\eta Q^*_1(z,0)-\lambda Q_0(1-X(z_1))] \\ \qquad\qquad +\eta Q^*_2(z,0)[z-(1-p)S^*_{b_1}(\lambda-\lambda X(z))]\Big\}\end{array}\right]}{z-(1-p)S^*_{b_1}(\lambda-\lambda X(z))-pS^*_{b_1}(\lambda-\lambda X(z))S^*_{b_2}(\lambda-\lambda X(z))} \tag{28}$$

Substituting (23),(24),(27) and (28) in (25) and (26) and inserting $\theta=0$, we get

$$\begin{aligned}P^*_1(z,0)=\frac{\lambda Q_0}{D_1(z)D_2(z)}\Big\{&z[1-S^*_{b_1}(\lambda-\lambda X(z))]\{\eta z(X(z)-X(z_1))[1-S^*_{v_1}(\lambda-\lambda X(z)+\eta)]\\ &+p\eta S^*_{b_2}(\lambda-\lambda X(z))(X(z)-X(z_1))S^*_{v_1}(\lambda-\lambda X(z)+\eta)\\ &\times(1-S^*_{v_2}(\lambda-\lambda X(z)+\eta))-(1-X(z_1))(\lambda-\lambda X(z)+\eta)\\ &\times[z-(1-p)S^*_{v_1}(\lambda-\lambda X(z)+\eta)-pS^*_{v_1}(\lambda-\lambda X(z)+\eta)\\ &\times S^*_{v_2}(\lambda-\lambda X(z)+\eta)]\}\Big\}\end{aligned} \tag{29}$$

and

$$\begin{aligned}P^*_2(z,0)=\frac{\lambda Q_0}{D_1(z)D_2(z)}\Big\{&pz[1-S^*_{b_2}(\lambda-\lambda X(z))]\{\eta(X(z)-X(z_1))S^*_{v_1}(\lambda-\lambda X(z)+\eta)\\ &\times(1-S^*_{v_2}(\lambda-\lambda X(z)+\eta))(z-(1-p)S^*_{b_1}(\lambda-\lambda X(z)))\\ &+\eta z(X(z)-X(z_1))S^*_{b_1}(\lambda-\lambda X(z))(1-S^*_{v_1}(\lambda-\lambda X(z)+\eta))\\ &-(1-X(z_1))(\lambda-\lambda X(z)+\eta)S^*_{b_1}(\lambda-\lambda X(z))\\ &\times[z-(1-p)S^*_{v_1}(\lambda-\lambda X(z)+\eta)\\ &-pS^*_{v_1}(\lambda-\lambda X(z)+\eta)S^*_{v_2}(\lambda-\lambda X(z)+\eta)]\}\Big\}\end{aligned} \tag{30}$$

where

$$D_1(z) = (\lambda - \lambda X(z))[z - (1-p)S^*_{v_1}(\lambda - \lambda X(z) + \eta) - pS^*_{v_1}(\lambda - \lambda X(z) + \eta)S^*_{v_2}(\lambda - \lambda X(z) + \eta)] \tag{31}$$

$$D_2(z) = (\lambda - \lambda X(z) + \eta)[z - (1-p)S^*_{b_1}(\lambda - \lambda X(z)) - pS^*_{b_1}(\lambda - \lambda X(z))S^*_{b_2}(\lambda - \lambda X(z))] \tag{32}$$

We define

$$P_B(z) = P^*_1(z,0) + P^*_2(z,0) \tag{33}$$

as the probability generating function for the number of customers in the system when the server is on not WV period and

$$P_V(z) = Q^*_1(z,0) + Q^*_2(z,0) + Q_0\,. \tag{34}$$

as the probability generating function for the number of customers in the system when the server is on WV period.

Then

$$P(z) = P_B(z) + P_V(z) \tag{35}$$

as the probability generating function for the number of customers in the system. We shall now use the normalizing condition $P(1) = 1$ to determine the only unknown Q_0, which appears in (35). Substituting $z = 1$ in (35) and using L'hospital's rule we obtain

$$Q_0 = \frac{(1-\rho_b)}{\left[\dfrac{(\lambda - \lambda X(z_1) + \eta)}{\eta} - \dfrac{\lambda(1 - X(z_1))S^*_{v_1}(\eta)[E(S_{b_1}) + pS^*_{v_2}(\eta)E(S_{b_2})]}{1 - (1-p)S^*_{v_1}(\eta) - pS^*_{v_1}(\eta)S^*_{v_2}(\eta)}\right]} \tag{36}$$

where $\rho_b = \lambda E(X)[E(S_{b_1}) + pE(S_{b_2})]$, $E(S_{b_1})$ and $E(S_{b_2})$ are the mean service times of stage 1 and stage 2 respectively. From (36) we obtain the system stability condition,

$$\rho_b < 1. \tag{37}$$

4. Performance Measures

Mean System Length

Let L_v and L_b denote the mean system size during the working vacation and not working vacation period respectively. Then

$$L_v = \frac{d}{dz} P_V(z) \text{ at } z = 1$$
$$= \frac{d}{dz}\left\{\frac{[N_1(z) + N_2(z)]}{D(z)}\lambda Q_0\right\} \text{ at } z = 1 \qquad \text{where}$$

$$N_1(z) = z(X(z) - X(z_1))[1 - S_{v_1}^*(\lambda - \lambda X(z) + \eta)]$$

$$N_2(z) = pz(X(z) - X(z_1))S_{v_1}^*(\lambda - \lambda X(z) + \eta)[1 - S_{v_2}^*(\lambda - \lambda X(z) + \eta)]$$

$$D(z) = (\lambda - \lambda X(z) + \eta)[z - (1-p)S_{v_1}^*(\lambda - \lambda X(z) + \eta)$$
$$-pS_{v_1}^*(\lambda - \lambda X(z) + \eta)S_{v_2}^*(\lambda - \lambda X(z) + \eta)]$$

Therefore $$L_v = \frac{\lambda Q_0\left\{D(1)[N_1'(1) + N_2'(1)] - D'(1)[N_1(1) + N_2(1)]\right\}}{(D(1))^2} \qquad \text{where}$$

$$N_1(1) = (1 - X(z_1))(1 - S_{v_1}^*(\eta))$$

$$N_2(1) = p(1 - X(z_1))S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta))$$

$$N_1'(1) = E(X)(1 - S_{v_1}^*(\eta)) + (1 - X(z_1))[(1 - S_{v_1}^*(\eta) + \lambda E(X)S_{v_1}^{*'}(\eta))]$$

$$N_2'(1) = p\left\{(1 - S_{v_2}^*(\eta))(1 - X(z_1))[S_{v_1}^*(\eta) - \lambda E(X)S_{v_1}^{*'}(\eta)] + \right.$$
$$\left.\lambda E(X)(1 - X(z_1))S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta) + E(X)S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta))\right\}$$

$$D(1) = \eta\left\{1 - (1-p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)\right\}$$

$$D'(1) = -\lambda E(X)\left[1 - (1-p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)\right]$$

$$+\eta\Big[1+\lambda E(X)(1-p)S_{v_1}^{*'}(\eta)$$
$$+p\lambda E(X)(S_{v_1}^{*'}(\eta)S_{v_2}^{*}(\eta)+S_{v_1}^{*}(\eta)S_{v_2}^{*'}(\eta))\Big]$$

$$L_b=\frac{d}{dz}P_B(z)\ \text{ at }\ z=1$$
$$=\frac{d}{dz}\left\{\frac{(N_3(z)N_4(z)+N_5(z)N_6(z))}{D_1(z)D_2(z)}\lambda Q_0\right\}\ \text{ at }\ z=1$$

where

$$N_3(z)=z(1-S_{b_1}^{*}(\lambda-\lambda X(z)))$$

$$N_4(z)=\Big\{\eta z(X(z)-X(z_1))[1-S_{v_1}^{*}(\lambda-\lambda X(z)+\eta)]+p\eta S_{b_2}^{*}(\lambda-\lambda X(z))$$
$$\times(X(z)-X(z_1))S_{v_1}^{*}(\lambda-\lambda X(z)+\eta)(1-S_{v_2}^{*}(\lambda-\lambda X(z)+\eta))$$
$$-(1-X(z_1))(\lambda-\lambda X(z)+\eta)[z-(1-p)S_{v_1}^{*}(\lambda-\lambda X(z)+\eta)$$
$$-pS_{v_1}^{*}(\lambda-\lambda X(z)+\eta)S_{v_2}^{*}(\lambda-\lambda X(z)+\eta)]\Big\}$$

$$N_5(z)=pz(1-S_{b_2}^{*}(\lambda-\lambda X(z)))$$

$$N_6(z)=\Big\{\eta(X(z)-X(z_1))S_{v_1}^{*}(\lambda-\lambda X(z)+\eta)(1-S_{v_2}^{*}(\lambda-\lambda X(z)+\eta))$$
$$\times\Big[z-(1-p)S_{b_1}^{*}(\lambda-\lambda X(z))\Big]+\eta z(X(z)-X(z_1))S_{b_1}^{*}(\lambda-\lambda X(z))$$
$$\times(1-S_{v_1}^{*}(\lambda-\lambda X(z)+\eta))-(1-X(z_1))(\lambda-\lambda X(z)+\eta)S_{b_1}^{*}(\lambda-\lambda X(z))$$
$$\times\Big[z-(1-p)S_{v_1}^{*}(\lambda-\lambda X(z)+\eta)-pS_{v_1}^{*}(\lambda-\lambda X(z)+\eta)S_{v_2}^{*}(\lambda-\lambda X(z)+\eta)\Big]\Big\}$$

$D_1(z)$ and $D_2(z)$ are respectively given in equations (31) and (32).

$$\text{Therefore }\ L_b=\frac{\lambda Q_0}{2[D_1'(1)]^2[D_2'(1)]^2}\Big\{D_1'(1)D_2'(1)[N_3''(1)N_4'(1)+N_3'(1)N_4''(1)$$
$$+N_5''(1)N_6'(1)+N_5'(1)N_6''(1)]-(N_3'(1)N_4'(1)$$
$$+N_5'(1)N_6'(1))[D_1''(1)D_2'(1)+D_1'(1)D_2''(1)]\Big\}$$

where

$$N_3'(1) = -\lambda E(X)E(S_{b_1})$$

$$N_3''(1) = -\Bigg[2\lambda E(X)E(S_{b_1}) + (\lambda E(X))^2 E(S_{b_1}^2) + E(S_{b_1})\lambda E(X(X-1))\Bigg]$$

$$N_4'(1) = \Bigg[\eta E(X) + \lambda E(X)(1 - X(z_1))\Bigg]\Bigg[1 - (1-p)S_{v_1}^*(\eta) - pS_{v_1}^*\eta S_{v_2}^*(\eta)\Bigg]$$
$$-\eta(1 - X(z_1))S_{v_1}^*(\eta)\Bigg[1 - p\lambda E(X)E(S_{b_2})(1 - S_{v_2}^*(\eta))\Bigg]$$

$$N_4''(1) = p\eta\lambda(1 - X(z_1))S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta))\Bigg[\lambda(E(X))^2E(S_{b_2}^2) + E(S_{b_2})E(X(X-1))\Bigg]$$
$$-2p\eta E(S_{b_2})(\lambda E(X))^2\Bigg[S_{v_1}^{*'}(\eta)(1 - S_{v_2}^*(\eta)) - S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)\Bigg]$$
$$-2p\eta\lambda(E(X))^2(1 - S_{v_2}^*(\eta))[S_{v_1}^{*'}(\eta) - E(S_{b_2})S_{v_1}^*(\eta)]$$
$$+2p\eta\lambda(E(X))^2S_{v_1}^*(\eta)S_{v_2}^{*'}\eta + 2\eta E(X)(1 - S_{v_1}^*(\eta))$$
$$+2\eta\lambda E(X)(1 - X(z_1))S_{v_1}^{*'}(\eta) + 2\eta\lambda(E(X))^2S_{v_1}^{*'}(\eta) + 2\lambda E(X)(1 - X(z_1))$$
$$\times\Bigg\{1 + (1-p)\lambda E(X)S_{v_1}^{*'}(\eta) + p\lambda E(X)(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta))\Bigg\}$$
$$+\Bigg[E(X(X-1))(\eta + \lambda(1 - X(z_1)))(1 - (1-p)S_{v_1}^*(\eta) - PS_{v_1}^*(\eta)S_{v_2}^*(\eta))\Bigg]$$

$$N_5'(1) = -p\lambda E(X)E(S_{b_2})$$

$$N_5''(1) = -p\Bigg[2\lambda E(X)E(S_{b_2}) + [\lambda E(X)]^2E(S_{b_2}^2) + E(S_{b_2})\lambda E[X(X-1)]\Bigg]$$

$$N_6'(1) = -\eta[1 - X(z_1)]\Bigg[\lambda E(X)E(S_{b_1})S_{v_1}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^*(\eta)\Bigg(1 - \lambda E(X)E(S_{b_1})\Bigg)\Bigg]$$
$$+E(X)\Bigg[\eta + \lambda(1 - X(z_1))\Bigg]\Bigg[1 - (1-p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)\Bigg]$$

$$N_6''(1) = \eta S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta))[1 - X(z_1)]\Bigg[-\lambda E(S_{b_1})E[X(X-1)] - [\lambda E(X)]^2E(S_{b_1}^2)\Bigg]$$
$$-2\eta E(S_{b_1})[\lambda E(X)]^2[1 - X(z_1)]\Bigg[S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)\Bigg]$$
$$+2\eta\lambda E(X)[1 - X(z_1)]\Bigg[S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta) + S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta)\Bigg]$$
$$+2\eta E(X)\Bigg[1 - S_{v_1}^*(\eta)S_{v_2}^*(\eta)\Bigg] + 2\eta\lambda[E(X)]^2[1 - S_{v_2}^*(\eta)]$$
$$\times\Bigg[-(1-p)E(S_{b_1})S_{v_1}^*(\eta) - pS_{v_1}^{*'}(\eta)\Bigg] + 2\eta p\lambda[E(X)]^2S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)$$
$$-2\eta\lambda E(X)E(S_{b_1})[1 - X(z_1)]S_{v_1}^*(\eta) + \Bigg[1 - (1-p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)\Bigg]$$

$$\times\Big\{E[X(X-1)][\eta+\lambda(1-X(z_1))]+2[\lambda E(X)]^2[1-X(z_1)]E(S_{b_1})\Big\}$$
$$+2\lambda E(X)[1-X(z_1)]\Big\{1+(1-p)S^{*\prime}_{v_1}(\eta)\lambda E(X)$$
$$+p\lambda E(X)\Big[S^{*\prime}_{v_1}(\eta)S^{*}_{v_2}(\eta)+S^{*}_{v_1}(\eta)S^{*\prime}_{v_2}(\eta)\Big]\Big\}$$

$$D_1'(1)=-\lambda E(X)\Big[1-(1-p)S^*_{v_1}(\eta)-pS^*_{v_1}(\eta)S^*_{v_2}(\eta)\Big]$$

$$D_1''(1)=-2\lambda E(X)\Big[1+\lambda E(X)(1-p)S^{*\prime}_{v_1}(\eta)+p\lambda E(X)(S^{*\prime}_{v_1}(\eta)S^*_{v_2}(\eta)+S^*_{v_1}(\eta)S^{*\prime}_{v_2}(\eta))\Big]$$
$$-\lambda E[X(X-1)]\Big[1-(1-p)S^*_{v_1}(\eta)-pS^*_{v_1}(\eta)S^*_{v_2}(\eta)\Big]$$

$$D_2'(1)=\eta\Big[1-\lambda E(X)E(S_{b_1})-p\lambda E(X)E(S_{b_2})\Big]$$

$$D_2''(1)=-2\lambda E(X)\Big[1-\lambda E(X)E(S_{b_1})-p\lambda E(X)E(S_{b_2})\Big]+\eta\Big\{E[X(X-1)][E(S_{b_1})$$
$$+pE(S_{b_2})]-[\lambda E(X)]^2[E(S^2_{b_1})+pE(S^2_{b_2})+2pE(S_{b_1})E(S_{b_2})]\Big\}$$

where $E(S_{b_1})$ and $E(S_{b_2})$ are the mean service times of stage 1 and stage 2 respectively and $E(S^2_{b_1})$ and $E(S^2_{b_2})$ are the second moments of the service times of stage 1 and stage 2 respectively.

5. Particular Cases

Case i: If no customer receives the additional second stage service then on setting $p=0$ we get $P(z)$ as

$$P(z)=P_B(z)+P_V(z) \tag{38}$$

where

$$P_B(z)=\frac{\left[\begin{array}{r}\lambda z(1-S^*_{b_1}(\lambda-\lambda X(z)))\{\eta z(X(z)-X(z_1))(1-S^*_{v_1}(\lambda-\lambda X(z)+\eta))\\ -(1-X(z_1))(\lambda-\lambda X(z)+\eta)(z-S^*_{v_1}(\lambda-\lambda X(z)+\eta))\}Q_0\end{array}\right]}{(\lambda-\lambda X(z))(\lambda-\lambda X(z)+\eta)[z-S^*_{v_1}(\lambda-\lambda X(z)+\eta)][z-S^*_{b_1}(\lambda-\lambda X(z))]}$$

$$P_V(z)=\frac{\lambda z(X(z)-X(z_1))(1-s^*_{v_1}(\lambda-\lambda X(z)+\eta))Q_0}{(\lambda-\lambda X(z)+\eta)[z-S^*_{v_1}(\lambda-\lambda X(z)+\eta)]}+Q_0$$

$$Q_0 = \frac{1-\rho_b}{\left[\dfrac{(\lambda-\lambda X(z_1)+\eta)}{\eta} - \dfrac{\lambda(1-X(z_1))S^*_{v_1}(\eta)E(S_{b_1})}{1-S^*_{v_1}(\eta)}\right]}$$

where $\rho_b = \lambda E(X)E(S_{b_1})$

equation (38) is well known generating function of the steady state queue length distribution of an $M^X/G/1$ queue with multiple working vacation studied by Aftab Begum (2011) irrespective of the notations.

Case ii: If the server never takes a vacation then taking limit as $\eta \to \infty$ we get $P(z)$ as $P(z) = P_B(z) + Q_0$ (39)

$$P_B(z) = \frac{\left\{z[S^*_{b_1}(\lambda-\lambda z)-1] + pzS^*_{b_1}(\lambda-\lambda z)[S^*_{b_2}(\lambda-\lambda z)-1]\right\}}{\left[z-(1-p)S^*_{b_1}(\lambda-\lambda z) - pS^*_{b_1}(\lambda-\lambda z)S^*_{b_2}(\lambda-\lambda z)\right]} \times Q_0; Q_0 = 1-\rho_b$$

Equation (39) is well known generating function of the steady state queue length distribution of an $M/G/1$ queue with second optional service studied by Madan (2000) (for $X(z) = z$) irrespective of the notations.

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