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NEW GENERALIZED ENTROPIC MODELS FOR FUZZY DISTRIBUTIONS AND THEIR PROPERTIES

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ABSTRACT

The present communication deals with the development of new measures of entropy for the fuzzy distributions. Such measures have been authenticated by the study of their essential and most desirable properties. Moreover, the graphical presentation of such mathematical models has been included for their complete study.

Keywords: Probability distribution, Entropy, Fuzzy set, Fuzzy information, Concave function, Increasing function, Decreasing function.

INTRODUCTION

In the nineteenth century the concept of entropy was introduced into thermodynamics. The tendency of the systems to become more disordered over time is described by the second law of thermodynamics, which states that the entropy of the system cannot spontaneously decrease. It was Shannon [14] who founded the subject of information theory by introducing the concept of entropy into communication theory. It was then realized that entropy is a property of any stochastic system and today it finds widespread applications in Statistics, Operations Research Techniques, Information Processing and Computing.

Brissaud [2] remarked that "Entropy is a basic physical quantity that has led to various, and sometimes apparently conflicting, interpretations. It has been successively assimilated to different concepts such as disorder and information. Associated with every probability distribution $P=\left(p_1,\, p_2,\, ...,\, p_n\right)$ where $p_1,\, p_2,\, ...,\, p_n$, are the probabilities of n outcomes, Shannon [14] introduced the following measure of entropy:

$$H(P) = -\sum_{i=1}^{n} p_i \text{ In } p_i \qquad (1.1)$$

After Shannon's [14] measure of entropy, Renyi [12] introduced entropy of order α, given by the following expression:

$$H_\alpha(P) = \frac{1}{1-\alpha} In\left(\sum_{i=1}^{n} p_i^{\alpha} \Big/ \sum_{i=1}^{n} p_i\right),\ \alpha \neq 1,\ \alpha > 0 \qquad (1.2)$$

Havrada and Charvat [6] introduced another measure called the non-additive measure of entropy, given by:

$$H^{\alpha}(P)=\frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha}-1\right), \alpha\neq 1, \alpha>0. \qquad (1.3)$$

Burg [3] developed a non-parametric measure of entropy, given by

$$H^{1}(P)=\sum_{i=1}^{n}\log p_i \qquad (1.4)$$

Many other probabilistic measures of entropy have been discussed by Kapur [8], Herremoes [7], Nanda and Paul [10], Asadi, Ebrahimi, Hamedani and Soofi [1] etc.

In practice, exact values of model parameters are rare in most engineering, data processing, and biological systems modeling and application. Normally, uncertainties arise due to incomplete information reflected in uncertain model parameters. This is often the case in price and bidding in market oriented power system operation and planning, in internet search engines, with the transfer rates in dynamic epidemiological models, and with the amount of carbohydrates, proteins and fat in ingested meals and gastroparese factor in human glucose metabolic models. A fruitful approach to handle parameter uncertainties is the use of fuzzy numbers and arithmetic. Fuzzy numbers capture our intuitive conceptions of approximate numbers and imprecise quantities such as about five and around three and five, and play a significant role in applications, such as estimation, prediction, classification, decision-making optimization and control. In these cases, fuzzy numbers represent uncertain parameters, with their support and shape derived from either experimental data or expert knowledge.

The theory of fuzzy sets which was introduced by Zadeh [15] received a good response from different quarters and after its introduction, many researchers started working around this field. Thus, keeping in view the idea of fuzzy sets, De Luca and Termini [4] introduced a measure of fuzzy entropy corresponding to Shannon's [14] measure. This fuzzy entropy is given by

$$H(A)=-\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\log\left(1-\mu_A(x_i)\right)\right] \qquad (1.5)$$

After this development, a large number of measures of fuzzy entropy were discussed, characterized and generalized by various authors. Kapur [9] introduced the following measure of fuzzy entropy:

$$H_{\alpha,\beta}(A)=\frac{1}{\beta-\alpha}\log\frac{\sum_{i=1}^{n}\left\{\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}}{\sum_{i=1}^{n}\left\{\mu_A^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}\right\}}; \alpha\geq 1, \beta\leq 1 \qquad (1.6)$$

Some other measures of fuzzy entropy and their generalizations have been studied by Zadeh [16], Ebanks [5], Rudas [13], Kapur [9], Zimmermann [17], Pal and Bezdek [11] etc. In section 2, we have proposed some new generalized measures of fuzzy entropy based upon real parameters, discussed their properties and presented these measures graphically.

2 NEW GENERALIZED MEASURES OF FUZZY ENTROPY AND THEIR VALIDITY

I. Firstly, we propose a new parametric measure of fuzzy entropy as given by the following mathematical expression:

$$\begin{aligned}H_{\alpha}(A)=&-\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\log\mu_A(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\log\left(1-\mu_A(x_i)\right)\right]\\&+\frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^{\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\right], \alpha>1\end{aligned} \qquad (2.1)$$

To prove that the measure introduced in equation (2.1) is a correct measure of fuzzy entropy, we study its essential properties as follows:

1. $H_\alpha(A)$ is a concave function of $\mu_A(x_i)$.

Proof: We have

$$\frac{d^2H_\alpha(A)}{d\mu_A^2(x_i)}=-\begin{bmatrix}(\alpha-1)\mu_A^{\alpha-2}(x_i)+\alpha\mu_A^{\alpha-2}(x_i)\\ +\alpha\mu_A^{\alpha-2}(x_i)\log\mu_A(x_i)+(\alpha-1)(1-\mu_A(x_i))^{\alpha-2}\\ +\alpha(1-\mu_A(x_i))^{\alpha-2}+\alpha(\alpha-1)(1-\mu_A(x_i))^{\alpha-2}\log(1-\mu_A(x_i))\end{bmatrix}$$

$$+\frac{\alpha}{1-\alpha}\left[\frac{\left(\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right)(\alpha-1)\left(\mu_A^{\alpha-2}(x_i)+(1-\mu_A(x_i))^{\alpha-2}\right)-\alpha\left(\mu_A^{\alpha-1}(x_i)-(1-\mu_A(x_i))^{\alpha-1}\right)^2}{\left(\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right)^2}\right]$$

$$=-\begin{bmatrix}(\alpha-1)\left(\mu_A^{\alpha-2}(x_i)+(1-\mu_A(x_i))^{\alpha-2}\right)+\alpha\mu_A^{\alpha-2}(x_i)\left(1+\log\mu_A^{(\alpha-1)}(x_i)\right)\\ +\alpha(1-\mu_A(x_i))^{\alpha-2}\left(1+\log(1-\mu_A(x_i))^{(\alpha-1)}\right)\end{bmatrix}$$

$$-\frac{\alpha}{\alpha-1}\left[\frac{\begin{matrix}(\alpha-1)\mu_A^{\alpha}(x_i)(1-\mu_A(x_i))^{\alpha-2}+(\alpha-1)\mu_A^{\alpha-2}(x_i)(1-\mu_A(x_i))^{\alpha}\\ -\mu_A^{2\alpha-2}(x_i)-(1-\mu_A(x_i))^{2\alpha-2}-2\alpha\mu_A^{\alpha-1}(x_i)(1-\mu_A(x_i))^{\alpha-1}\end{matrix}}{\left(\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right)^2}\right]<0\ for\ \alpha>1$$

Thus, $H_\alpha(A)$ is a concave function.

2. $H_\alpha(A)$ does not change when $\mu_A(x_i)$ is replaced by $1-\mu_A(x_i)$

3. $H_\alpha(A)$ is an increasing function of $\mu_A(x_i)$ for $0\le\mu_A(x_i)\le\frac{1}{2}$

4. $H_\alpha(A)$ is a decreasing function of $\mu_A(x_i)$ for $\frac{1}{2}\le\mu_A(x_i)\le1$

5. $H_\alpha(A)=0$ when $\mu_A(x_i)=0$ or 1

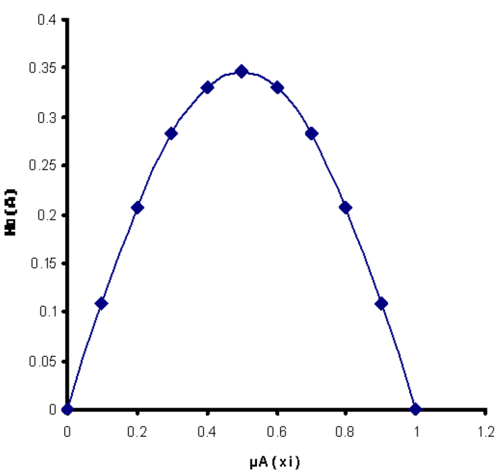
Since $H_\alpha(A)$ satisfies all the essential properties of being a measure of fuzzy entropy, it is a correct measure of fuzzy entropy.

Next, with the help of the data, we have presented the measure (2.1) graphically. For this purpose, we have computed different values of $H_\alpha(A)$ corresponding to different fuzzy values $\mu_A(x_i)$ as shown in the following Table- 2.1:

Table- 2.1

$\mu_A(x_i)$	$H_\alpha(A)$
0.0	0.00000
0.1	0.10836
0.2	0.20718
0.3	0.28312
0.4	0.33050
0.5	0.34657
0.6	0.33050
0.7	0.28312
0.8	0.20718
0.9	0.10836
1.0	0.00000

Next, we have presented the values of $H_\alpha(A)$ graphically and obtained the following Fig.-2.1 which shows that the measure introduced in equation (2.1) is a concave function.



II. Next, we propose a new parametric measure of fuzzy entropy of order a, given by the following mathematical expression:

$$H^{\alpha}(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

(2.2)

$$+\frac{2^{\alpha-1}}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - 1\right];\alpha \neq 1,\ \alpha > 1$$

We shall prove that (2.2) as a measure of fuzzy entropy.
We have

$$\frac{dH^{\alpha}(A)}{d\mu_A(x_i)} = -\left[1+\log\mu_A(x_i) - 1 - \log(1-\mu_A(x_i))\right] + \frac{2^{\alpha-1}\alpha}{1-\alpha}\left[\mu_A^{\alpha-1}(x_i) - (1-\mu_A(x_i))^{\alpha-1}\right]$$

$$= -\left[\log\mu_A(x_i) - \log(1-\mu_A(x_i))\right] + \frac{2^{\alpha-1}\alpha}{1-\alpha}\left[\mu_A^{\alpha-1}(x_i) - (1-\mu_A(x_i))^{\alpha-1}\right]$$

Also

$$\frac{d^2H^{\alpha}(A)}{d\mu_A^{2}(x_i)} = -\left[\frac{1}{\mu_A(x_i)} + \frac{1}{(1-\mu_A(x_i))}\right] + \frac{2^{\alpha-1}\alpha}{1-\alpha}\left[(\alpha-1)\mu_A^{\alpha-2}(x_i) + (\alpha-1)(1-\mu_A(x_i))^{\alpha-1}\right]$$

$$= -\left[\frac{1}{\mu_A(x_i)(1-\mu_A(x_i))} + 2^{\alpha-1}\alpha\left[\mu_A^{\alpha-2}(x_i) + (1-\mu_A(x_i))^{\alpha-1}\right]\right] < 0$$

Thus $H^{\alpha}(A)$ is concave.

Hence, $H^{\alpha}(A)$ satisfies the following properties:

(*i*) $H^{\alpha}(A)$ is a concave function of $\mu_A(x_i)$.

(*ii*) $H^{\alpha}(A)$ doesn't change when $\mu_A(x_i)$ is replaced by $1-\mu_A(x_i)$.

(iii) $H^{\alpha}(A)$ is an increasing function of $\mu_A(x_i)$ for $0 \leq \mu_A(x_i) \leq \frac{1}{2}$.

(iv) $H^{\alpha}(A)$ is a decreasing function of $\mu_A(x_i)$ for $\frac{1}{2} \leq \mu_A(x_i) \leq 1$.

(*v*) $H^{\alpha}(A) = 0$ when $\mu_A(x_i) = 0$ or 1.

Hence, $H^{\alpha}(A)$ is a valid measure of fuzzy entropy.

Proceeding as above, the measure $H^{\alpha}(A)$ can similarly be presented graphically.

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