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On Generalized Function $G_{\rho,\eta,\gamma}[a,z]$ And It's Fractional Calculus

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Abstract

The present paper is devoted to the study of the generalized function $G_{\rho,\eta,\gamma}[a,z]$ defined in [3] and its fractional calculus. The importance of the study of this function lies in the fact that it provides the generalization of the generalized Mittag-Leffler function $E^{\gamma}_{\rho,\mu}(z)$ [6], classical Mittag-Leffler $E_{\rho,\mu}(z)$ and $E_{\rho}(z)$, and the Kummer's confluent hypergeometric function $\phi(\gamma;\mu;z)$. The differentiation and integration formulae for the function $G_{\rho,\eta,\gamma}[a,z]$ are also established. We have also studied the fractional integral operator $(G_{\rho,\eta,\gamma,\omega,a+}\psi)(x)$ with such a function in the kernel. The compositions with Riemann-Liouville fractional integral and differential operator are also derived. As applications of our main-results some known results for generalized Mittag-Leffler function due to Kilbas et al.[2]are cited. The results involving the R-function [4] are also obtained as special cases of our main findings.

Keywords: R-function, Generalized Mittag-Leffler function, Riemann-Liouville fractional calculus, Generalized fractional integral operators.

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1. Introduction

The Riemann-Liouville fractional integral I^{α}_{a+} and fractional derivative D^{α}_{a+} of order α are defined by [1],[7]:

$$(I^{\alpha}_{a+}\psi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\alpha}}dt, \quad (\alpha\in\mathbb{C}, \quad Re(\alpha)>0) \tag{1}$$

$$\text{and} \quad (D^{\alpha}_{a+}\psi)(x) = \left(\frac{d}{dx}\right)^n \left(I^{n-\alpha}_{a+} \ \psi\right)(x), \quad (\alpha\in\mathbb{C}, \quad Re(\alpha)>0)\,; n = Re(\alpha)+1. \tag{2}$$

respectively.The special G and R - Function are defined by [3,5]:

$$G_{\rho,\eta,r}[a,z] = z^{r\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(r)_n(az^{\rho})^n}{\Gamma(n\rho+\rho r-\eta)n!}, \qquad Re(\rho r-\eta)>0, \tag{3}$$

At $r=1$, and z replaced by $(z-c)$ it reduces to

$$R_{\rho,\eta}[a,c,z] = (z-c)^{\rho-\eta-1}\sum_{n=0}^{\infty}\frac{[a(z-c)^{\rho}]^n}{\Gamma(n\rho+\rho-\eta)}, \qquad \rho\geq 0, \quad \rho\geq\eta. \tag{4}$$

The generalized Mittag-Leffler function defines by [1]:

$$E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k + \mu) k!} \ , \qquad (\rho, \mu, \gamma \in \mathbb{C}, \quad Re(\rho) > 0)\,. \tag{5}$$

Where at $\gamma = 1$, $E^{1}_{\rho,\mu}(z)$ coincides with the classical Mittag-Leffler function $E_{\rho,\mu}(z)$ and in particular $E_{1,1}(z) = e^z$ and when $\rho = 1$ it coincides with Kummer's confluent hyper geometric function $\phi(\gamma, \mu; z)$ with the exactness to the constant multiplier $[\Gamma(\mu)]^{-1}$

The present paper is organized as follows. The section-2 contains four properties of the function defined in (3). In first two results the differentiation and integration of the $G_{\rho,\eta,\gamma}[a, z]$ function are obtained while in the next two results of the section, the Riemann-Liouville fractional integral and differential operators are operated on the $G_{\rho,\eta,\gamma}[a, z]$ function. The section-3, deals with the study of fractional integral operator $(G_{\rho,\eta,\gamma,\omega;a+}\psi)(x)$ involving the function $G_{\rho,\eta,\gamma}[a, z]$ in the kernel.In this section two theorems are established for boundedness of this integral operator. The compositions of the operator $(G_{\rho,\eta,\gamma,\omega;a+}\psi)(x)$ with the Riemann-Liouville fractional integral and differential operators are established in the section-4.

In this paper we have studied the following integral operator

$$(G_{\rho,\eta,\gamma,\omega;a+}\psi)(x) = \int_a^x G_{\rho,\eta,\gamma}[\omega, (x-t)]\psi(t)\ dt, \qquad (x > a). \tag{6}$$

with $\rho, \eta, \gamma, \omega \in \mathbb{C}$, $(Re(\rho), Re(\eta) > 0)$ in particular $\gamma = 0$ and in view of (3) and (1):

$$(G_{\rho,\eta,0,\omega;a+}\psi)(x) = \left(I^{-\eta}_{a+}\psi\right) \tag{7}$$

and at $\gamma = 1$ it reduces to the following integral operator,

$$(R_{\rho,\eta,\omega;a+}\psi)(x) = \int_a^x R_{\rho,\eta}[\omega, 0, (x-t)]\psi(t)\ dt, \qquad (x > a). \tag{8}$$

Where the function $R_{\rho,\eta}[\omega, 0, (x-t)]$ in the kernel and is defined in (4).

The following series identity defined in [8] is required to established our main results,

$$\sum_{m,n=0}^{\infty} C_{m+n}(\nu)_m(\sigma)_n \frac{x^{m+n}}{m!\ n!} = \sum_{n=0}^{\infty} C_n(\upsilon + \sigma)_n \frac{x^n}{n!} \tag{9}$$

2. Some properties of function $G_{\rho,\eta,\gamma}[a, z]$

For $\rho, \eta, \gamma, \omega, \sigma, q, \alpha \in \mathbb{C}$, $(Re(\rho), Re(\eta), Re(q), Re(\alpha) > 0)$ and $n \in N$ there hold the following properties for the special function $G_{\rho,\eta,\gamma}[a, z]$ defined in (3).

Property-1

$$\left(\frac{d}{dz}\right)^n [G_{\rho,\eta,\gamma}[\omega, z]] = G_{\rho,\eta+n,\gamma}[\omega, z] \tag{10}$$

Property-2

$$\int_0^x G_{\rho,\eta,\gamma}[\omega, (x-t)] G_{\rho,\eta,\sigma}[\omega, t] dt = G_{\rho,\eta+q,\gamma+\sigma}[\omega, x] \tag{11}$$

Property-3

$$I_{a+}^{\alpha} \left(G_{\rho,\eta,\gamma}[\omega, (t-a)]\right)(x) = G_{\rho,\eta-\alpha,\gamma}[\omega, (x-a)], \quad (x > a) \tag{12}$$

Property-4

$$D_{a+}^{\alpha} \left(G_{\rho,\eta,\gamma}[\omega, (t-a)]\right)(x) = G_{\rho,\eta+\alpha,\gamma}[\omega, (x-a)], \quad (x > a) \tag{13}$$

Outline of Proofs:

To prove the result in (10) we use the definition of $G_{\rho,\eta,\gamma}[a, z]$ function in (3) and then taking term-by-term differentiation under the summation sign(which is possible in accordance with the uniform convergence of the series in (3) and then interpreting the resulting series with the help of (3) we at once arrive at the desired result in (10).

To establish the integral in (11) we denote its LHS by Δ_1 i.e.

$\Delta_1 = \int_0^x G_{\rho,\eta,\gamma}[\omega, (x-t)] G_{\rho,q,\sigma}[\omega, t] dt$

Now with the help of the definition (1.3) and then changing the order of integration and summation we have

$$\Delta_1 = \sum_{k_1,k_2=0}^{\infty} \frac{(\gamma)_{k_1} (\sigma)_{k_2}\ \omega^{k_1+k_2}}{\Gamma(\rho\gamma + \rho k_1 - \eta)\ \Gamma(\rho\sigma + \rho k_2 - q) k_1!, k_2!} \int_0^x t^{(\rho\sigma+\rho k_2-q-1)} (x-t)^{(\rho\gamma+\rho k_1-\eta-1)} dt$$

We evaluate the inner integral with the help of beta-integral and then making use of the series identity (9) and the definition (3) therein we at once arrive at the desired result in (11).

To prove the relation in (12) we denote its LHS by Δ_2 i.e.

$\Delta_2 = I_{a+}^{\alpha} G_{\rho,\eta,\gamma}[\omega, (t-a)](x)$

We now operate the Riemann-Liouville integral operator $\left(I_{a+}^{\alpha} f\right)(x)$ defined in (1) on the function $G_{\rho,\eta,\gamma}[\omega, (t-a)]$ defined in (3) and then on changing the order of integration and summation to have

$$\Delta_2 = \sum_{k=0}^{\infty} \frac{(\gamma)_k\ \omega^k}{\Gamma(\rho\gamma + \rho k - \eta)\ \Gamma(\alpha) k!} \int_0^x (x-t)^{(\alpha-1)} (t-a)^{(\rho\gamma+\rho k-\eta-1)} dt$$

On evaluating the inner integral with the help of Beta-integral and then interpreting the resulting series with the help of (3) we at once arrive at the desired result in (12).

To prove the relation in (13) we denote its LHS by Δ_3 i.e.

$\Delta_3 = D^{\alpha}_{a+} G_{\rho,\eta,\gamma}[\omega, (t-a)](x)$

And then use the definition of Riemann-Liouville fractional differential operator (2), we have.

$\Delta_3 = \left(\frac{d}{dz}\right)^n \left\{ I^{\eta-\alpha}_{a+} G_{\rho,\eta,\gamma}[\omega, (t-a)](x) \right\}$

This on making use of the result in (12) gives

$\Delta_3 = \left(\frac{d}{dz}\right)^n G_{\rho,\eta,\gamma}[\omega, (t-a)]$

Now with the help of the result in (13), we at once arrive at the desired result in (13).

3. Boundedness of the operator $(G_{\rho,\eta,\gamma,\omega,a+}\psi)(x)$

In the following theorems we prove the boundedness on the space $L(a,b)$ and $C(a,b)$ of the operator $(G_{\rho,\eta,\gamma,\omega,a+}\psi)(x)$ defined by (6).

Theorem-1: Let $\rho, \eta, \omega \in \mathbb{C},\ (Re(\rho), Re(\eta)) > 0$, and $b > a$ then the operator $(G_{\rho,\eta,\gamma,\omega,a+})$ is bounded on $L(a,b)$ and

$$\|G_{\rho,\eta,\gamma,\omega;a+}\psi\|_L \leqq B\|\psi\|_L \tag{14}$$

where

$$B = (b-a)^{Re(\rho)\gamma - Re(\eta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_k||\omega(b-a)^{Re(\rho)}|^k}{|\Gamma(\gamma\rho + \rho k - \eta)|[Re(\rho\gamma + \rho k) - Re(\eta)]k!} \tag{15}$$

Theorem-2: Let $\rho, \eta, \omega \in \mathbb{C},\ (Re(\rho), Re(\eta)) > 0$, and $b > a$ then the operator $(G_{\rho,\eta,\gamma,\omega,a+})$ is bounded on $C(a,b)$ and

$$\|G_{\rho,\eta,\gamma,\omega;a+}\psi\|_C \leqq B\|\psi\|_C \tag{16}$$

where B is given in equation (15).

Proof of Theorems

To prove Theorem-1, we denote its LHS by Δ_4 i.e.

$\Delta_4 = \|G_{\rho,\eta,\gamma,\omega;a+}\psi\|_L$

Now using (6) and (3) and interchanging the order of integration by the Dirichlet formula we have

$$\Delta_4 = \int_a^b \left| \int_a^x (x-t)^{\rho\gamma-\eta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k[\omega(x-t)^{\rho}]^k}{\Gamma(\gamma\rho + \rho k - \eta)k!}\ \psi(t)\ dt \right| dx$$

$$\leqq \int_a^b \left| \int_t^b (x-t)^{Re(\rho)\gamma - Re(\eta)-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k[\omega]^k(x-t)^{Re(\rho)k}}{\Gamma(\gamma\rho + \rho k - \eta)k!}\ \psi(t)\ dt \right| dx$$

On putting $x-t=U$

$$\Delta_4 \leqq \int_a^b \left| \int_0^{b-t} (U)^{Re(\rho)(\gamma+k)-Re(\eta)-1} \sum_{k=0}^{\infty} \frac{|(\gamma)_k||\omega|^k}{|\Gamma(\gamma\rho+\rho k-\eta)|k!} \, du \right| |\psi(t)| dx$$

$$\leqq \sum_{k=0}^{\infty} \frac{|(\gamma)_k||\omega|^k}{|\Gamma(\gamma\rho+\rho k-\eta)|k!} \int_a^b \left| \int_0^{b-t} (U)^{Re(\rho)(\gamma+k)-Re(\eta)-1} \, du \right| |\psi(t)| dx$$

Now interpreting the inner integral term-by-term integration and taking into account (15) we at once arrive at the desired result in (14).

To prove Theorem-2, we know that the integral operator $G_{\rho,\eta,\gamma,\omega;a+}$ is bounded on $L(a,b)$, by the theorem-1 so it is also bounded in space $C(a,b)$ of continuous function g on $[a,b]$ with a finite norm $\|g\|_c = \max_{a \leq x \leq b} |g(x)|$

Using this concept and the definition of $G_{\rho,\eta,\gamma,\omega;a+}$ (6) we have for any $x \in c[a,b]$ and $c \in [a,b]$

$$\|G_{\rho,\eta,\gamma,\omega;a+}\psi\| \ (x) \leqq \int_a^x \left| (x-t)^{\rho\gamma-\eta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k[\omega(x-t)^{\rho}]^k}{\Gamma(\gamma\rho+\rho k-\eta)k!} \, \psi(t) \right| dt$$

$$\leqq \|\psi(t)\|_c \int_a^x \left| (U)^{Re(\rho)(\gamma+k)-Re(\eta)-1} \, du \sum_{k=0}^{\infty} \frac{(\gamma)_k[\omega]^k}{|\Gamma(\gamma\rho+\rho k-\eta)|k!} \right|$$

The integral on the right hand side is less or equal to B which is defined in (15). This completes the proof of theorem-2.

4. Compositions of the operator $(G_{\rho,\eta,\gamma,\omega;a+}\psi)(x)$

Let $\rho,\eta,\gamma,\omega,\sigma,q,\alpha \in \mathbb{C}$, $(Re(\rho), Re(\eta), Re(q), Re(\alpha) > 0)$ and $n \in N$ then the following results hold for $\psi \in L(a,b)$.

Result-1 $\quad G_{\rho,\eta,\gamma,\omega;a+}[(t-a)^{\beta-1}](x) = \Gamma(\beta) G_{\rho,\eta-\beta,\gamma}[\omega,(x-a)]$

Result-2 $\quad I_{a+}^{\alpha} \ G_{\rho,\eta,\gamma,\omega;a+}\psi = G_{\rho,\eta-\alpha,\gamma,\omega;a+}\psi = G_{\rho,\eta,\gamma,\omega;a+} I_{a+}^{\alpha}\psi$

Result-3 $\quad D_{a+}^{\alpha} \ G_{\rho,\eta,\gamma,\omega;a+}\psi = G_{\rho,\eta+\alpha,\gamma,\omega;a+}\psi$

Holds for any continuous function $\psi \in c(a,b)$

Result-4 $\quad G_{\rho,\eta,\gamma,\omega;a+} \ G_{\rho,q,\sigma,\omega;a+}\psi = G_{\rho,\eta+q,\gamma+\sigma,\omega;a+}\psi$

Result-5 $\quad G_{\rho,\eta,\gamma,\omega;a+} \ G_{\rho,q,-\gamma,\omega;a+}\psi = I_{a+}^{-(\eta+q)} \ \psi$

Outline of proof:

To prove the result in (17), we denote its LHS by Δ_4 i.e.

$\Delta_4 = G_{\rho,\eta,\gamma,\omega;a+} \ [(t-a)^{\beta-1}](x)$

Now making use of definition (6) and (3) and then term-by-term integration and with the help of Beta integral we at once arrive at the result (17) on using (3) therein.

To prove the result in (18), we denote its LHS by Δ_5 i.e.

$\Delta_5 = (I_{a+}^{\alpha} G_{\rho,\eta,\gamma,\omega;a+\psi})(x)$

Use the definition (1) and (6) and applying Dirichlet formula for $x > a$ we have:

$\Delta_5 = \int_a^x \left(\left\{I_{a+}^{\alpha}\left[G_{\rho,\eta,\gamma}(\omega,\tau)\right]\right\}(x-t)\right)\psi(t)dt$

Now on using the relation (12) we at once arrive at the desired result in (18).The second relation of (18) is proved similarly.

To prove the result in (19), we denote its LHS by Δ_6 i.e.

$\Delta_6 = \left(D_{a+}^{\alpha} G_{\rho,\ \eta,\ \gamma,\ \omega;\ a+}\psi\right)(x)$

Now using the definition (2) and the result in (18) we have

$\Delta_6 = \left(\frac{d}{dx}\right)^n \left(G_{\rho,\ \eta\ -\ n\ +\ \alpha,\ \gamma,\ \omega;\ a+}\psi\right)(x)$

On applying (6) and (10) we at once arrive at the desired result in (19) in accordance with the definition (6).

To prove the result in (20), we denote its LHS by Δ_7 i.e.

$\Delta_7 = \left(G_{\rho,\ \eta,\ \gamma,\ \omega;\ a+}\ G_{\rho,\ q,\ ,\ \sigma,\ \omega;\ a+}\ \psi\right)(x)$

Now using the definition (6) we have

$\Delta_7 = \int_a^x \left[\int_0^{x-t} G_{\rho,\ \eta,\ \gamma}[\omega,\ x-t-\tau]\ G_{\rho,\ q,\ \sigma}[\omega,\ \tau]\ d\tau\right]\ \psi(t)\ dt$

On evaluating the inner integral with the help of (11) and then with the help of (6) we at once arrive at the desired result in (20).

The result in (21) is obtained by taking $\sigma = -\gamma$ in (20) and in view of the relation (7).

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