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HOMOTOPY PROPERTIES OF DIGITAL SIMPLE CLOSED CURVES

Annie Kurien K*, M. S. Samuel**

***Department of Mathematics, Mar Thoma College for Women, Perumbavoor.**

**** Department of Computer Applications, MACFAST, Tiruvalla.**

Abstract:

In this paper we discuss some properties of digital simple closed curves and prove that a digital simple closed curve of more than four points is not contractible.

Keywords:

Digital image, Digital topology, Digital simple closed curve

1 Introduction

A digital image is a set X of lattice points that model a continuous object Y, where Y is a subset of a Euclidean space. Digital topology is concerned with developing a mathematical theory of such discrete objects so that digital images have topological properties that mirror those of the Euclidean objects they model; Applications of digital topology have been found shape description and in image processing operations such as thinning and skeletonization.

2 Preliminaries

Let Z be the set of integers. Z^d is the set of lattice points in d-dimensional Euclidean space. Let $X \subset Z^d$ and let k-be some adjacency relation for the members of X. Then the pair (X, k) is said to be binary digital image. For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2 \ldots. p_d)$, $q = (q_1, q_2 \ldots q_d) \in Z^d$, p and q are c_l adjacent if

1 there are atmost l indices i such that $|p_i - q_i| = 1$ and

2 for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$. C_l denotes the number of points $q \in Z^d$ that are C_l adjacent to a given point $p \in Z^d$. Thus in Z we have $c_1=2$, in Z^2 we have $c_1=4$ and $c_2=8$ and so on.

2.1 Proposition (2)

Let $X \subset Z^{d_0}$ and $Y \subset Z^{d_1}$ be digital images with k_0 adjacency and k_1 adjacency respectively. Then the function f: $X \rightarrow Y$ is (k_0, k_1) continuous if and only if for every k_0 adjacent points $\{x_0, x_1\}$ of X either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$are k_1 adjacent in Y.

2.2 Definition [2]

Let $X \subset Z^{d_0}$ and $Y \subset Z^{d_1}$ be digital images with k_0 adjacency and k_1 -adjacency respectively. Two (k_0, k_1) continuous functions f, g: $X \rightarrow Y$ are said to be digitally (k_0, k_1) homotopic in Y if there is a positive integer m and a function H: $X \times [0, m]_Z \rightarrow Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$
- for all $x \in X$ the induced function $H_x: [0, m]_Z \rightarrow Y$ defined by $H_x(t) = H(x, t)$ for all $t \in [0, m]_Z$ is $(2, k_1)$ continuous and
- for all $t \in [0,m]_Z$ the induced function $H_t: X \rightarrow Y$ defined by $H_t(x) = H(x, t)$ for all $x \in X$ is (k_0, k_1) continuous.

2.3 Definition (4)

A digital simple closed k-curve X is required to satisfy the following . (X, k) is a digital image and the following property (SCC) is satisfied for some positive integer m.

(SCC) There is a (2, k) continuous function f: $[0, m-1]_Z \rightarrow X$ such that

- f is one - to - one and onto and

- for all $t \in [0, m\text{-}1]_Z$, the set of k-neighbors of f(t) in f $[0, m\text{-}1]_Z$ is {f((t-1) mod m), f ((t+1) mod m) }.

2.4 Definition(4)

If $S = \{x_i\}_0^{m-1}$ where $x_i = f(i)$ for all $i \in [0, m\text{-}1]_Z$, then the points of S are circularly ordered.

3 Homotopy properties of digital simple closed curves

Proposition (3.1)

Let S_a be a digital simple closed K_a-curve, $a \in \{0, 1\}$ Let $f : S_0 \to S_1$ be a (k_0, k_1) continuous function. If $|S_0| = |S_1|$, then the following are equivalent.

(a) f is one to one

(b) f is onto

(c) f is a (k_0, k_1) isomorphism

Proof:

Since S_0 is a finite set (a)$\Rightarrow$ (b)

(c) follows from (a) & (b) [definition of isomorphism]

(b)$\Rightarrow$ (c)

Let $S_a = \{x_{a,\ i}\}_0^{n-1}$ where the points S_a are circularly ordered, $a \in \{0, 1\}$. Let $x_{1,\ u} \in S_1$ and let $x_{0,\ v} = f^{-1}(x_{1,u})$. Then the k_1 neighbors of $x_{i,\ u}$ in S_1 are $x_{1,\ (u-1) \bmod n}$ and $x_{1,\ (u+1) \bmod n}$ and the k_0- neighbors of $x_{0,\ v}$ in S_0 are $x_{0,\ (v-1) \bmod n}$ and $x_{0,\ (v+1) \bmod n}$. Since f is a continuous bijection, choice of $x_{0,\ v}$ implies

$$f(\{x_{0,\ (v-1) \bmod n},\ x_{0,\ (v+1) \bmod n}\}) = \{x_{1,\ (u-1) \bmod n},\ x_{1,\ (u+1) \bmod n}\}.$$

Thus, $f^{-1}(\{x_{1,\ (u-1) \bmod n}, x_{1,\ (u+1) \bmod n}\}) = \{x_{0,\ (v-1) \bmod n}, , x_{0,\ (v+1) \bmod n}\}$.

Since u is arbitrary f^{-1} is (k_1, k_0) continuous, so f is a (k_0, k_1) isomorphism.

Theorem 3.2.

Let S be a simple closed k curve and let $H : S \times [0, m]_Z \to S$ be a (k, k) homotopy between an isomorphism H_0 and Hm= f, where $f(S) \neq S$, then $|S| = 4$.

Proof

Let $S= \{x_i\}_0^{n-1}$ where the points of S are circularly ordered. There exists $\omega \in [1, m]_Z$ such that $\omega = \min\{t \in [0, m]_Z \mid H_t (S) \neq S\}$.

Without loss of generality, $x_1 \notin H_\omega (S)$. Then the induced function $H_{\omega-1}$ is a bijection, so there exists $x_u \in S$ such that $H (x_{u, \omega-1}) = x_1$. By proposition 3.1, $H_{\omega-1}(\{ x_{(u-1) \bmod n}, x_{(u+1) \bmod n}\})= \{x_0, x_2\}$ and the continuity property of Homotopy implies $H(x_{u, \omega}) \in \{x_0, x_2\}$. Without loss of generality, $H (x_{(u-1) \bmod n, \omega-1})= x_0$ …….(1) and $H(x_{u, \omega}) = x_2$ ……. (2)

Suppose n > 4. Equ. (2) implies $H (x_{(u-1) \bmod n, \omega}) \in \{x_1, x_2, x_3\}$ but this is impossible since

1 $H(x_{(u-1) \bmod n, \omega}) \notin x_1$

2 $H(x_{(u-1) \bmod n, \omega}) \notin \{x_2, x_3\}$ from equ (1) because n>4 implies neither x_2 nor x_3 is k-adjacent to x_0.

The contradiction arose from the assumption that n > 4. Therefore we must have $n \leq 4$. Since a digital simple closed curve is assumed to have at least 4 points, we must have n = 4

4 Conclusion:

We have shown that digital simple closed curves of more than 4 points are not contractible.

5 References

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