Int Jr. of Mathematical Sciences & Applications
Vol. 7, No. 1, (January-June, 2017)
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ISSN No: 2230-9888
www.ijmsa.yolasite.com

On general multiple Eulerian integrals involving the multivariable I-function of Prathima, a general class of polynomials and the I-function of one variable

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ABSTRACT
The object of this paper is first,to evaluate a general multiple Eulerian integrals with general integrands involving the product of a multivariable I-function defined by Prathima et al [4], a general class of polynomials, the $\bar{I}$-function of one variable and generalized hypergeometric function with general arguments. The second multiple Eulerian integral contain a general class of polynomials, the general polynomial set, the $\bar{I}$ - function of one variable and multivariable I-function with general arguments. Our integral formulas are interesting and unified nature.

Keywords :Multivariable H-function, class of polynomial, general polynomials set, multiple Eulerian integral, $\bar{I}$-function of one variable, multivariable I-function.

2010 Mathematics Subject Classification. 33C45, 33C60, 26D20

1. Introduction

In this paper, we evaluate two multiple Eulerian integrals involving the multivariable I-function defined by Prathima et al [4], the $\bar{I}$-function of one variable defined by Rathie [6] and multivariable class of polynomials with general arguments. The multivariable I-function defined by Prathima et al [4] is a extension of the multivariable H-function defined by Srivastava et al [10]. We will use the contracted form.
The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r), \theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\bar{\delta}_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\bar{\delta}_j^{(i)}s_i\right)} \tag{1.4}$$

where $i=1,\cdots,r$. Also $z_i \neq 0$ for $i=1,\cdots,r$

The parameters $m_j,n_j,p_j,q_j(j=1,\cdots,r),n,p,q$ are non negative integers (for more details, see Nambisan [6])

$\alpha_j^{(i)}(j=1,\cdots,p;i=1,\cdots,r),\beta_j^{(i)}(j=1,\cdots,q;i=1,\cdots,r),\gamma_j^{(i)}(j=1,\cdots,p_i;i=1,\cdots,r)$ and $\delta_j^{(i)}$

$(j=1,\cdots,q_i;i=1,\cdots,r)$ are assumed to be positive quantities for standardisation purpose.

$a_j(j=1,\cdots,p),b_j(j=1,\cdots,q),c_j^{(i)}(j=1,\cdots,p_i,i=1,\cdots,r),d_j^{(i)}(j=1,\cdots,q_i,i=1,\cdots,r)$ are complex numbers.

The exposants $A_j(j=1,\cdots,p),B_j(j=1,\cdots,q),C_j^{(i)}(j=1,\cdots,p_i;i=1,\cdots,r),D_j^{(i)}(j=1,\cdots,q_i;i=1,\cdots,r)$

of various gamma function involved in (2.2) and (2.3) may take non integer values.

The contour L_i in the complex s_i-plane is of Mellin Barnes type which runs from $c-i\infty$ to $c+i\infty$ (c real) with indentation, if necessary, in such a manner that all singularities of $\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right),j=1,\cdots,m_i$lie to the right and $\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}-\gamma_j^{(i)}s_i\right),j=1,\cdots,n_i$ are to the left of L_i.

Following the result of Braaksma [2] the I-function of r variables is analytic if :

$$U_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\bar{\delta}_j^{(i)},i=1,\cdots,r \tag{1.5}$$

The integral (2.1) converges absolutely if

$|arg(z_k)|<\frac{1}{2}\Delta_k\pi,k=1,\cdots,r$ where

$$\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\bar{\delta}_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}>0 \tag{1.6}$$

The $\bar{I}$- function , introduced by Rathie [6], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\bar{I}(z)=I_{p,q}^{m,n}\left(\mathrm{z}\ \middle|\ \begin{matrix}(\mathrm{a}_j,\alpha_i;A_j)_{n,n+1},(a_j,\alpha_i;A_j)_p\\(\mathrm{b}_j,\beta_j;1)_{m,m+1},(b_j,\beta_j;B_j)_q\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\Omega_{p,q}^{m,n}(s)z^{-s}\mathrm{d}s \tag{1.7}$$

for all z different to 0 and

$$\Omega\,_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma^(b_j+\beta_js)\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j-\alpha_js)}{\prod_{j=n+1}^{p}\Gamma^{A_i}(a_j+\alpha_js)\prod_{j=m+1}^{q}\Gamma^{B_j}(1-b_j-\beta_js)} \tag{1.8}$$

When the poles of $\Gamma(b_j-\beta_js),j=1,\cdots,m$, are simples the integral (1.8) can be evaluate with the help of the Residue Theorem. We obtain

$$\bar{I}(z)=\sum_{G=1}^{m}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{p,q}^{m,n}(s)}{B_Gg!}z^s \tag{1.9}$$

with $s=\eta_{G,g}=\dfrac{b_G+g}{B_G}, p<q, |z|<1$ and $\Omega_{p,q}^{m,n}(s)$ is given in (1.8)

For more detail, see Rathie [6].

The generalized polynomials defined by Srivastava [8], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]} \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}$$

$$A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \tag{1.10}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_u,K_u]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_u = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u] \tag{1.11}$$

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

2. Sequence of functions

Agarwal and Chaubey [1], Salim [5] and several others have studied a general sequence of functions. In the present document we shall study the following useful series formula for a general sequence of functions.

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-sx^r}] = \sum_{w,v,u,t,e,k_1,k_2,}\psi(w,v,u,t,e,k_1,k_2)x^R \tag{2.1}$$

where $$\sum_{w,v,u,t,e,k_1,k_2} = \sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{c=0}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \tag{2.2}$$

and the infinite series on the right side (2.1) is absolutely convergent, $R=ln+qv+pt+rw+k_1r+k_2q$ (2.3)

and $$\psi(w,v,u,t,e,k_1,k_2) = \frac{(-)^{t+w+k_2}(-v)_u(-t)_e(\alpha)_t l^n}{w!v!u!t!e!K_nk_1!k_2!}\frac{s^{w+k_1}F^{\gamma n-t}}{(1-\alpha-t)_e}(\alpha-\gamma n)_e$$

$$(-\beta-\delta n)_v g^{v+k_2}h^{\delta n-v-k_2}(v-\delta n)_{k_2}E^t\left(\frac{pe+rw+\lambda+qn}{l}\right)_n \tag{2.4}$$

where K_n is a sequence of constants. This function will note $R_n^{\alpha,\beta}[x]$
By suitably specializing the parameters involving in (2.1), a general sequence of function reduced to generalized polynomials set studied by Raizada [7], a class of polynomials introduced by Fujiwara [3] and several others authors.

3. Integral representation of generalized hypergeometric function

The following generalized hypergeometric function in terms of multiple contour integrals is also required [10 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

The equivalent form of Eulerian beta integral is :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t=(b-a)^{\alpha+\beta}B(\alpha,\beta)(Re(\alpha)>0,Re(\beta)>0,a\neq b) \tag{3.2}$$

4. First integral

We note :

$$A=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p};B=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} \tag{4.1}$$

$$C=(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r};(1,0;1);\cdots;(1,0;1);(1,0;1);\cdots;(1,0;1) \tag{4.2}$$

$$D=(d_j^{(1)},\bar{\delta}_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\bar{\delta}_j^{(r)};D_j^{(r)})_{1,q_r};(0,1;1);\cdots;(0,1;1);(0,1;1);\cdots;(0,1;1) \tag{4.3}$$

$$X=m_1,n_1;\cdots;m_r,n_r;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{4.4}$$

$$Y=p_1,q_1;\cdots;p_r,q_r;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{4.5}$$

$$A^*=[1+\sigma_i'-\eta_{G,g}c_i'-\sum_{j=1}^{u}K_j\xi_j^{(1,j)};\rho_i^{(1,1)},\cdots,\rho_i^{(1,r)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},1,0,\cdots,0;1]_{1,s},\cdots,$$

$$[1+\sigma_i^{(T)}-\eta_{G,g}c_i^{(T)}-\sum_{j=1}^{u}K_j\xi_j^{(T,j)};\rho_i^{(T,1)},\cdots,\rho_i^{(T,r)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},0,\cdots,0,1;1]_{1,s},$$

$$[1-A_j;0,\cdots,0,1,\cdots,1,0,\cdots,0;1]_{1,P},$$

$$[1-\alpha_i-\eta_{G,g}a_i-\sum_{j=1}^{u}K_j\alpha_i^{(j)};\delta_i',\cdots,\delta_i^{(r)},\mu_i',\cdots,\mu_i^{(l)},1,\cdots,1,0,\cdots,0;1]_{1,s},$$

W-items (T-W)-items

$$[1-\beta_i-\eta_{G,g}b_i-\sum_{j=1}^{u}K_j\beta_i^{(j)};\eta_i',\cdots,\eta_i^{(r)},\theta_i',\cdots,\theta_i^{(l)},0,\cdots,0,1,\cdots,1;1]_{1,s} \tag{4.6}$$

W-items (T-W)-items

$$B^* = [1+\sigma'_i - \eta_{G,g}c'_i - \sum_{j=1}^{u} K_j \xi_i^{(1,j)}; \rho_i^{(1,1)}, \cdots, \rho_i^{(1,r)}, \tau_i^{(1,1)}, \cdots, \tau_i^{(1,l)}, 0, \cdots, 0; 1]_{1,s}, \cdots,$$

$$[1+\sigma_i^{(T)} - \eta_{G,g}c_i^{(T)} - \sum_{j=1}^{u} K_j \xi_i^{(T,j)}; \rho_i^{(T,1)}, \cdots, \rho_i^{(T,r)}, \tau_i^{(T,1)}, \cdots, \tau_i^{(T,l)}, 0, \cdots, 0; 1]_{1,s},$$

$$[1-B_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0; 1]_{1,Q},$$

$$[1-\alpha_i-\beta_i - \eta_{G,g}(a_i+b_i) - \sum_{j=1}^{u} K_j(\alpha_i^{(j)}+\beta_i^{(j)}); (\delta'_i+\eta'_i), \cdots, (\delta_i^{(r)}+\eta_i^{(r)})$$

$$(\mu'_i+\theta'_i), \cdots, (\mu_i^{(l)}+\theta_i^{(l)}), 1, \cdots, 1; 1]_{1,s} \tag{4.7}$$

We have the following multiple Eulerian integral and we obtain the I-function of $(r+l+T)-$variables

$$\int_{u_1}^{v_1} \cdots \int_{u_s}^{v_s} \prod_{i=1}^{s} \left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1} \prod_{j=1}^{T} (u_i^{(j)}x_i+v_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\bar{I}_{p,q}^{m,n} \left[z \prod_{i=1}^{s} \left[\frac{(x_i-u_i)^{a_i}(v_i-x_i)^{b_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{c_i^{(j)}}} \right] \right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u} \left(\begin{array}{c} \mathrm{y}_1 \prod_{i=1}^{s} \left[\frac{(x_i-u_i)^{\alpha'_i}(v_i-x_i)^{\beta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_u \prod_{i=1}^{s} \left[\frac{(x_i-u_i)^{\alpha_i^{(u)}}(v_i-x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,u)}}} \right] \end{array} \right)$$

$$I \left(\begin{array}{c} \mathrm{z}_1 \prod_{i=1}^{s} \left[\frac{(x_i-u_i)^{\delta'_i}(v_i-x_i)^{\eta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \prod_{i=1}^{s} \left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}} \right] \end{array} \right)$$

$$
{}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\mu_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s
$$

$$
=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{i=1}^{s}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]
$$

$$
\sum_{G=1}^{m}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}a_u\frac{(-)^g\Omega_{p,q}^{m,n}(\eta_{G,g})}{B_Gg!}y_1^{K_1}\cdots y_u^{K_u}\prod_{i=1}^{s}\left[(v_i-u_i)^{\eta_{G,g}(a_i+b_i)+\sum_{j=1}^{u}K_j(\alpha_i^{(j)}+\beta_i^{(j)})}\right]
$$

$$
z^{\eta_{G,g}}\prod_{i=1}^{s}\left[\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\right]
$$

$$
I_{p+sT+P+2s;q+sT+Q+s;Y}^{0,n+sT+P+2s;X}\left(\begin{array}{c|c}
\mathrm{z}_1w_1 & \mathrm{A}\ ;\mathrm{A}^*:C\\
\cdots & \cdot\\
\cdots & \cdot\\
\mathrm{z}_rw_r & \cdot\\
\mathrm{g}_1W_1 & \cdot\\
\cdots & \cdot\\
\cdots & \cdot\\
\mathrm{g}_lW_l & \cdot\\
\mathrm{G}_1 & \cdot\\
\cdots & \cdot\\
\cdots & \cdot\\
\mathrm{G}_T & \mathrm{B}\ ;\mathrm{B}^*:D
\end{array}\right) \tag{4.8}
$$

Where

$$
w_m=\prod_{i=1}^{s}\left[(v_i-u_i)^{\delta_i^{(m)}+\eta_i^{(m)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{(j,m)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{(j,m)}}\right],m=1,\cdots,r \tag{4.9}
$$

$$
W_k=\prod_{i=1}^{s}\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right],k=1,\cdots,l \tag{4.10}
$$

$$
G_j=\prod_{i=1}^{s}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=1,\cdots,W \tag{4.11}
$$

$$
G_j=-\prod_{i=1}^{s}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=W+1,\cdots,T \tag{4.12}
$$

Provided that :

(A) $0 \leqslant W \leqslant T; u_i, v_i \in \mathbb{R}; min\{a_i, b_i, c_i^{(j)}, \alpha_i^{(k)}, \beta_i^{(k)}, \xi_i^{(j,k)}\} > 0$, $i = 1, \cdots, s; k = 1, \cdots, u; j = 1, \cdots, T$

(B) $min\{\delta_i^{(t)}, \eta_i^{(t)}, \rho_i^{(j,t)}, \mu_i^{(k)}, \theta_i^{(k)}, \tau_i^{(j,k)}\} \geqslant 0; j = 1, \cdots, T; i = 1, \cdots, s; k = 1, \cdots, l; t = 1, \cdots, r$

(C) $\sigma_i^{(j)} \in \mathbb{R}, U_i^{(j)}, V_i^{(j)} \in \mathbb{C}, z_t, g_k \in \mathbb{C}; j = 1, \cdots, s; t = 1, \cdots, r; k = 1, \cdots, l; t = 1, \cdots, r$

(D) $max\left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}}\right] < 1, i = 1, \cdots, s; j = 1, \cdots, W$ and

$$max\left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}}\right] < 1, i = 1, \cdots, s; j = W+1, \cdots, T$$

(F) $Re\big[\alpha_i + a_i \min_{1 \leqslant j \leqslant m} \frac{b_j}{\beta_j} + \sum_{t=1}^{r} \delta_i^{(t)} \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(t)}}{\bar{\delta}_j^{(t)}}\big] > 0;$

$$Re\big[\beta_i + b_i \min_{1 \leqslant j \leqslant m} \frac{b_j}{\beta_j} + \sum_{t=1}^{r} \eta_i^{(t)} \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(t)}}{\bar{\delta}_j^{(t)}}\big] > 0; i = 1, \cdots, s$$

(E) $U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)} \leqslant 0, i = 1, \cdots, r$

(F) $\Delta_k = -\sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \bar{\delta}_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)}$

$$-\delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^{T} \rho_i^{(j,k)} > 0$$

(G) $\left|arg\left(z_i \prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{-\rho_i^{(j,k)}}\right)\right| < \frac{1}{2} \Delta_i \pi$

(H) The series occuring on the right-hand side of (4.8) are absolutely and uniformly convergent

(I) $m_j, n_j, p_j, q_j (j = 1, \cdots, r), n, p, q \in \mathbb{N}^*$ **;** $\delta_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, q_i; i = 1, \cdots, r)$

$\alpha_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, p; i = 1, \cdots, r), \beta_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, q; i = 1, \cdots, r), \gamma_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, p_i; i = 1, \cdots, r)$

$a_j (j = 1, \cdots, p), b_j (j = 1, \cdots, q), c_j^{(i)} (j = 1, \cdots, p_i, i = 1, \cdots, r), d_j^{(i)} (j = 1, \cdots, q_i, i = 1, \cdots, r) \in \mathbb{C}$

The exposants $A_j (j = 1, \cdots, p), B_j (j = 1, \cdots, q), C_j^{(i)} (j = 1, \cdots, p_i; i = 1, \cdots, r), D_j^{(i)} (j = 1, \cdots, q_i; i = 1, \cdots, r)$

of various gamma function involved in (1.3) and (1.4) may take non integer values.

(J) $P \leqslant Q + 1$. The equality holds, when , in addition,

either $P > Q$ and $\sum_{k=1}^{l} \left|g_k \left(\prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{-\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}} < 1 \qquad (u_i \leqslant x_i \leqslant v_i; i = 1, \cdots, s)$

or $P \leqslant Q$ and $\max_{1\leqslant k\leqslant l}\left[\left|\left(g_k\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{-\tau_i^{(j,k)}}\right)\right|\right]<1 \qquad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,s)$

Proof

To establish the formula (4.8), we first use series representation (1.9) and (1.10) for $\bar{I}(z)$ and $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[.]$ respectively, we use contour integral representation with the help of (1.1) for the multivariable I-function occuring in its left-hand side and use the contour integral representation with the help of (3.1) for the generalized hypergeometric function ${}_PF_Q(.)$. Changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) . Now we write :

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}$$

where $K_i^{(j)}=\eta_i^{(j)}-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}-\sum_{t=1}^{r}\rho_i^{(j,t)}\zeta_t-\sum_{k=1}^{l}\tau_i^{(j,k)}\zeta_k \qquad i=1,\cdots,s; j=1,\cdots,T$ (4.13)

and express the factor occuring in R.H.S. Of (4.8) in terms of following Mellin-Barnes contour integral with the help of the result [9, page 18, eq.(2.6.4)]

$$\frac{1}{(2\pi\omega)^W}\int_{L_1'}\cdots\int_{L_W'}\prod_{j=1}^{W}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i'^{(j)}}}{\Gamma(-K_i^{(j)})}\right]$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\Bigg]\,\mathrm{d}\zeta_1'\cdots\mathrm{d}\zeta_W' \tag{4.14}$$

and

$$\frac{1}{(2\pi\omega)^{T-W}}\int_{L_{W+1}'}\cdots\int_{L_T'}\prod_{j=W+1}^{T}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]$$

$$\prod_{j=W+1}^{T}\left[-\frac{(U_i^{(j)}(v_i-x_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\Bigg]\,\mathrm{d}\zeta_{W+1}'\cdots\mathrm{d}\zeta_T' \tag{4.15}$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost **x**-integral with the help of (3.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r+l+T)-$ variables, we obtain the formula (4.8).

5. Second integral

We note :

$$A=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}; B=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} \tag{5.1}$$

$$C=(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r};(1,0;1);\cdots;(1,0;1) \tag{5.2}$$

$$D=(d_j^{(1)},\bar{\delta}_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\bar{\delta}_j^{(r)};D_j^{(r)})_{1,q_r};(0,1;1);\cdots;(0,1;1) \tag{5.3}$$

$$A^{*}=[1+\sigma_{i}'-\theta_{i}'R-\eta_{G,g}c_{i}'-\sum_{j=1}^{u}K_{j}\xi_{i}^{(1,j)};\rho_{i}^{(1,1)},\cdots,\rho_{i}^{(1,r)},1,0,\cdots,0;1]_{1,s},\cdots,$$

$$[1+\sigma_{i}^{(T)}-\theta_{i}^{(T)}R-\eta_{G,g}c_{i}^{(T)}-\sum_{j=1}^{u}K_{j}\xi_{i}^{(T,j)};\rho_{i}^{(T,1)},\cdots,\rho_{i}^{(T,r)},0,\cdots,0,1;1]_{1,s},$$

$$[1-\alpha_{i}-\zeta_{i}R-\eta_{G,g}a_{i}-\sum_{j=1}^{u}K_{j}\alpha_{i}^{(j)};\delta_{i}',\cdots,\delta_{i}^{(r)},\underbrace{1,\cdots,1}_{\text{W-items}},\underbrace{0,\cdots,0}_{\text{(T-W)-items}};1]_{1,s},$$

$$[1-\beta_{i}-\lambda_{i}R-\eta_{G,g}b_{i}-\sum_{j=1}^{u}K_{j}\beta_{i}^{(j)};\eta_{i}',\cdots,\eta_{i}^{(r)},\underbrace{0,\cdots,0}_{\text{W-items}},\underbrace{1,\cdots,1}_{\text{(T-W)-items}};1]_{1,s} \tag{5.4}$$

$$B^{*}=[1+\sigma_{i}'-\theta_{i}'R-\eta_{G,g}c_{i}'-\sum_{j=1}^{u}K_{j}\xi_{i}^{(1,j)};\rho_{i}^{(1,1)},\cdots,\rho_{i}^{(1,r)},0,\cdots,0;1]_{1,s},\cdots,$$

$$[1+\sigma_{i}^{(T)}-\theta_{i}^{(T)}R-\eta_{G,g}c_{i}^{(T)}-\sum_{j=1}^{u}K_{j}\xi_{i}^{(T,j)};\rho_{i}^{(T,1)},\cdots,\rho_{i}^{(T,r)},0,\cdots,0;1]_{1,s},$$

$$[1-\alpha_{i}-\beta_{i}\,(\zeta_{i}+\lambda_{i})R-\eta_{G,g}(a_{i}+b_{i})\;-\sum_{j=1}^{u}K_{j}(\alpha_{i}^{(j)}+\beta_{i}^{(j)});(\delta_{i}'+\eta_{i}'),\cdots,(\delta_{i}^{(r)}+\eta_{i}^{(r)})$$

$$(\mu_{i}'+\theta_{i}'),\cdots,(\mu_{i}^{(l)}+\theta_{i}^{(l)}),1,\cdots,1;1]_{1,s} \tag{5.5}$$

We have the following multiple Eulerian integral

$$\int_{u_1}^{v_1}\cdots\int_{u_s}^{v_s}\prod_{i=1}^{s}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(u_i^{(j)}x_i+v_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$\bar{I}_{p,q}^{m,n}\left[z\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{a_i}(v_i-x_i)^{b_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{c_i^{(j)}}}\right]\right]R_n^{\alpha,\beta}\left[Z\prod_{j=1}^{s}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c}\mathrm{y}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\alpha'_i}(v_i-x_i)^{\beta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_u\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\alpha_i^{(u)}}(v_i-x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,u)}}}\right]\end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1 \prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta'_i}(v_i-x_i)^{\eta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right] \end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\prod_{i=1}^{s}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{G=1}^{m}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}a_u\frac{(-)^g\Omega_{p,q}^{m,n}(\eta_{G,g})}{B_Gg!}y_1^{K_1}\cdots y_u^{K_u}$$

$$\sum_{w,v,u,t,e,k_1,k_2}\psi'(w,v,u,t,e,k_1,k_2)Z^R\prod_{i=1}^{s}\left[(v_i-u_i)^{\eta_{G,g}(a_i+b_i)+\sum_{j=1}^{u}K_j(\alpha_i^{(j)}+\beta_i^{(j)})}\right]$$

$$\prod_{i=1}^{s}\left[\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\right]$$

$$I_{p+sT+2s;q+sT+s;Y}^{0,n+sT+2s;X}\left(\begin{array}{c|c} \mathrm{z}_1w_1 & \mathrm{A}\ ;\mathrm{A}^*:C \\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \mathrm{z}_rw_r & \cdot \\ & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot & \cdot \\ \mathrm{G}_T & \mathrm{B}\ ;\mathrm{B}^*:D \end{array}\right) \tag{5.6}$$

where

$$\psi'(w,v,u,t,e,k_1,k_2)=\frac{\psi(w,v,u,t,e,k_1,k_2,)\prod_{i=1}^{s}(v_i-u_i)^{(\zeta_i+\lambda_i)R}}{\prod_{i=1}^{s}\left[\prod_{j=1}^{W}(u_iU_i^{(j)}+V_i^{(j)})^{\theta_i^{(j)}R}\prod_{j=W+1}^{T}(u_iU_i^{(j)}+V_i^{(j)})^{\theta_i^{(j)}R}\right]} \tag{5.7}$$

$\psi(w,v,u,t,e,k_1,k_2)$ and R are given by (2.4) and (2.3) respectively.

$$w_l=\prod_{i=1}^{s}\left[(v_i-u_i)^{\delta_i^{(l)}+\eta_i^{(l)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{(j,l)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{(j,l)}}\right],l=1,\cdots,r \tag{5.8}$$

$$G_j = \prod_{i=1}^{s} \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = 1, \cdots, W \tag{5.9}$$

$$G_j = -\prod_{i=1}^{s} \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right], j = W+1, \cdots, T \tag{5.10}$$

Provided that :

(A) $0 \leqslant W \leqslant T; u_i, v_i \in \mathbb{R}; min\{a_i, b_i, c_i^{(j)}, \alpha_i^{(k)}, \beta_i^{(k)}, \xi_i^{(j,k)}\} > 0$, $i = 1, \cdots, s; k = 1, \cdots, u; j = 1, \cdots, T$

(B) $min\{\zeta_i, \lambda_i, \theta_i^{(j)}, \delta_i^{(l)}, \eta_i^{(l)}, \rho_i^{(j,l)}\} \geqslant 0; j = 1, \cdots, T; i = 1, \cdots, s; l = 1, \cdots, r$

(C) $Re(\alpha_i, \beta_i, v_i^{(j)}) > 0 (i = 1, \cdots, s; j = 1, \cdots, T); |\tau| < 1$

(D) $max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \cdots, s; j = 1, \cdots, W$ and

$max \left[\frac{(v_i - u_i)U_i^{(j)}}{u_i U_i^{(j)} + V_i^{(j)}} \right] < 1, i = 1, \cdots, s; j = W+1, \cdots, T$

(E) $U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)} \leqslant 0, i = 1, \cdots, r$

(F) $\Delta_k = -\sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \bar{\delta}_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)}$

$-\delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^{T} \rho_i^{(j,k)} > 0$

(G) $\left| arg \left(z_i \prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{-\rho_i^{(j,k)}} \right) \right| < \frac{1}{2} \Delta_i \pi$

(H)$m_j, n_j, p_j, q_j (j = 1, \cdots, r), n, p, q \in \mathbb{N}^*$ **;**$\bar{\delta}_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, q_i; i = 1, \cdots, r)$

$\alpha_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, p; i = 1, \cdots, r), \beta_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, q; i = 1, \cdots, r), \gamma_j^{(i)} \in \mathbb{R}_+ (j = 1, \cdots, p_i; i = 1, \cdots, r)$

$a_j (j = 1, \cdots, p), b_j (j = 1, \cdots, q), c_j^{(i)} (j = 1, \cdots, p_i, i = 1, \cdots, r), d_j^{(i)} (j = 1, \cdots, q_i, i = 1, \cdots, r) \in \mathbb{C}$

The exposants $A_j (j = 1, \cdots, p), B_j (j = 1, \cdots, q), C_j^{(i)} (j = 1, \cdots, p_i; i = 1, \cdots, r), D_j^{(i)} (j = 1, \cdots, q_i; i = 1, \cdots, r)$ of various gamma function involved in (1.3) and (1.4) may take non integer values.

(I) The series occuring on the right-hand side of (5.5) is absolutely and uniformly convergent

Proof

To establish the formula (5.6), we first use series representation (1.9), (1.10) and (3.1) for , $\bar{I}(z)$,$S_{N_1, \cdots, N_u}^{M_1, \cdots, M_u}[.]$ and $R_n^{\alpha,\beta}[.]$ respectively and the contour integral representation with the help of (1.2) for the multivariable I-function occuring in its left-hand side. Changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) . Now, we write:

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} \tag{5.10}$$

where $K_i^{(j)}=\eta_i^{(j)}-\eta_{G,g}c_i^{(j)}-R\theta_i^{(j)}-\sum_{l=1}^{u}L_l\xi_i^{(j,l)}-\sum_{t=1}^{r}\rho_i^{(j,t)}\zeta_t\,;i=1,\cdots,s;j=1,\cdots,T$ (5.11)

and express the factors occuring in R.H.S. Of (5.6) in terms of following Mellin-Barnes contour integral , we obtain :

$$\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^W}\int_{L'_1}\cdots\int_{L'_W}\prod_{j=1}^{W}\Big[\Gamma(-\zeta'_j)\Gamma(-K_i^{(j)}+\zeta'_j)$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta'_j}\Bigg]\mathrm{d}\zeta'_1\cdots\mathrm{d}\zeta'_W \tag{5.12}$$

and

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^{T-W}}\int_{L'_{W+1}}\cdots\int_{L'_T}\prod_{j=W+1}^{T}\Big[\Gamma(-\zeta'_j)\Gamma(-K_i^{(j)}+\zeta'_j)$$

$$\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}(x_i-v_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta'_j}\Bigg]\mathrm{d}\zeta'_{W+1}\cdots\mathrm{d}\zeta'_T \tag{5.13}$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost **x-**integral with the help of (3.2) and reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function of $(r+T)-$ variables, we obtain the formula (5.6).

6. Multivariable H-function

If $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$, the multivariable I-function defined by Nambisan et al [6] reduces to the multivariable H-funcction defined by Srivastava et al [11] and we have the two following formulas.

Formula 1

$$\int_{u_1}^{v_1}\cdots\int_{u_s}^{v_s}\prod_{i=1}^{s}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(u_i^{(j)}x_i+v_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$\bar{I}_{p,q}^{m,n}\left[z\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{a_i}(v_i-x_i)^{b_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{c_i^{(j)}}}\right]\right]S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c}\mathrm{y}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\alpha'_i}(v_i-x_i)^{\beta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_u\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\alpha_i^{(u)}}(v_i-x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,u)}}}\right]\end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta'_i}(v_i-x_i)^{\eta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right] \\ . \\ . \\ . \\ \mathrm{z}_r\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right] \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\mu_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{i=1}^{s}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{G=1}^{m}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}a_u\frac{(-)^g\Omega_{p,q}^{m,n}(s)}{B_Gg!}y_1^{K_1}\cdots y_u^{K_u}\prod_{i=1}^{s}\left[(v_i-u_i)^{\eta_{G,g}(a_i+b_i)+\sum_{j=1}^{u}K_j(\alpha_i^{(j)}+\beta_i^{(j)})}\right]$$

$$z^{\eta_{G,g}}\prod_{i=1}^{s}\left[\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(l,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\right]$$

$$H_{p+sT+P+2s;q+sT+Q+s;Y}^{0,n+sT+P+2s;X}\left(\begin{array}{c|c} \mathrm{z}_1w_1 & \mathrm{A}\ ;\mathrm{A}^*:C \\ \cdots & . \\ \cdots & . \\ \mathrm{z}_rw_r & . \\ \mathrm{g}_1W_1 & . \\ \cdots & . \\ \cdots & . \\ \mathrm{g}_lW_l & . \\ \mathrm{G}_1 & . \\ \cdots & . \\ \cdots & . \\ \mathrm{G}_T & \mathrm{B}\ ;\mathrm{B}^*:D \end{array}\right) \tag{6.1}$$

under the same notations and conditions that (4.8) with $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$

Formula 2

$$\int_{u_1}^{v_1}\cdots\int_{u_s}^{v_s}\prod_{i=1}^{s}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(u_i^{(j)}x_i+v_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$I_{p,q}^{m,n}\left[z\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{a_i}(v_i-x_i)^{b_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{c_i^{(j)}}}\right]\right]R_n^{\alpha,\beta}\left[Z\prod_{j=1}^{s}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\begin{pmatrix}\mathrm{y}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\alpha'_i}(v_i-x_i)^{\beta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_u\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\alpha_i^{(u)}}(v_i-x_i)^{\beta_i^{(u)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\xi_i^{(j,u)}}}\right]\end{pmatrix}$$

$$I\begin{pmatrix}\mathrm{z}_1\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta'_i}(v_i-x_i)^{\eta'_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{s}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\prod_{i=1}^{s}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{G=1}^{m}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}a_u\frac{(-)^g\Omega_{p,q}^{m,n}(\eta_{G,g})}{B_Gg!}y_1^{K_1}\cdots y_u^{K_u}$$

$$\sum_{w,v,u,t,e,k_1,k_2}\psi'(w,v,u,t,e,k_1,k_2)Z^R\prod_{i=1}^{s}\left[(v_i-u_i)^{\eta_{G,g}(a_i+b_i)+\sum_{j=1}^{u}K_j(\alpha_i^{(j)}+\beta_i^{(j)})}\right]$$

$$\prod_{i=1}^{s}\left[\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\eta_{G,g}c_i^{(j)}-\sum_{l=1}^{u}K_l\xi_i^{(j,l)}}\right]$$

$$I_{p+sT+2s;q+sT+s;Y}^{0,n+sT+2s;X}\left(\begin{array}{c|c} \mathrm{z}_1 w_1 & \mathrm{A}\ ;\mathrm{A}^* : C \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \mathrm{z}_r w_r & \cdot \\ & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ & \cdot \\ \mathrm{G}_T & \mathrm{B}\ ;\mathrm{B}^* : D \end{array}\right) \tag{6.2}$$

under the same notations and conditions that (5.6) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$

7. Conclusion

In this paper we have evaluated two generalized multiple Eulerian integrals involving the multivariable I-functions defined by Prathima et al [4] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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