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A NOTE ON THE ELLIPTIC PROBABILITY

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***Abstract**: Is it still possible to say something original about an extensively studied subject as the Probability? The Elliptic Probability seems a positive response. This concept appeared in a seminal paper regarding a new physical approach to chances aimed to understand the differences between the Gambling Mathematics and the Standard Probability Theory. After having analyzed a random binary test on a semi-empirical basis, it was figured out that the standard probability axioms need the linear time whereas the gambling strategies require a manifold non-linear temporal frame. A third-way model supporting a hypothetical recursive time and supplying new formulas was then introduced. Motivated by geometrical reasons, a three-fold classification of chances in Flat (Standard Probability), Hyperbolic (Gambling Mathematics) and Elliptic (New Physical Approach to Probability) was suggested to the reader for meditation and improvement but an unexpectedly rapid feedback on such argument has imposed this elucidating note.*

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1. INTRODUCTION

In a seminal work [10] we analyzed a binary event game from different perspectives in order to investigate the well-consolidated predictive patterns of the Standard Probability Theory (*SPT* for brevity) and of the Gambling Mathematics (*GM* for short); we there introduced a semi-empirical model called New Physical Approach to Probability (acronym *NPAP*).

We found that, however motivated (by the Principle of Indifference or by Symmetry), the *SPT* has the physical implication of a *linear time* and its probability of success, after n consecutive failures, is $p_{n+1}(E)=\frac{1}{2}$

We noticed that the *GM* is grounded on what we defined "Principle of Sufficient Reason" which implies a *non-linear* temporal frame not precisely identifiable for the wide range of strategies. The mainstream *GM's* probability of success, after n consecutive failures, was individuated in the interval $\frac{1}{2}<p_{n+1}(E)\le\frac{2^n}{1+2^n}$..

Based on the "Principle of Certainty" (*i.e.*, the axiom that the success must occur after a finite number of trials) the *NPAP* assumes a recursive time and its probability of success, after n consecutive failures, is $p_{n+1}(E)=\frac{m+n}{2m}$ with $m\ge 23$ integer; it was left as choice the generalization $p_{n+1}(E)=\frac{2m-\sqrt[k]{m^k-n^k}}{2m}$ (with $k\in\mathbb{Z}^+$).

Among the other open questions [10] we advanced a geometric reclassification of the *SPT*, *GM*, *NPAP* as, respectively, Flat Probability (*FP* for brevity), Hyperbolic Probability (*HP* for short) and Elliptic Probability (acronym *EP*). Since the last topic has quickly aroused a wide interest [11, 12] this paper is a due clarification concerning it and the occasion to show some explanatory graphs too.

2. A RANDOM BINARY GAME

We imagined [10-12] to flip an unbiased coin *n* times supposing to place bets on each toss (also called "trial") and to continue betting on the same outcome until the first success.

In particular, we examined how much we should bet on the next trial if the unfavorable outcomes have always occurred since we started the game, a problem diversely tackled by the *SPT*, *GM* and *NPAP*.

Let $\bar{E}$ be the unfavorable event, arbitrarily "Heads" or "Tails", and let $n(\bar{E})$ be the number of unfavorable events already occurred since the first trial.

Let *n* be the number of times the coin has already been flipped; according to our premise:

$$n = n(\bar{E}) \tag{1}$$

Throughout the paper, *n* will be the independent variable (on the *x*-axis).

Let $p_{n+1}(\bar{E})$ be the probability of failure in the next trial, after *n* consecutive failures; it is the only kind of probability *p* we need:

$$p = p_{n+1}(\bar{E}) \tag{2}$$

Throughout the paper, *p* will be the dependent variable (on the *y*-axis).

Now we try to elucidate the classification into *FP*, *HP*, *EP* by determining the *p*-distribution for each of the corresponding approaches *SPT*, *GM* and *NPAP*.

3. THE SPT AS "FLAT" PROBABILITY

3.1 Axioms and Formulas of the SPT

A repeated coin flipping is a Bernoulli process [2] where successive trials are independent of each other; past outcomes do not influence future events, *i.e.*, the probability of success or failure in the next trial is exactly the same of the first toss.

In the *SPT* each outcome is assigned a constant and equal probability by the "Principle of Indifference" [17] also known as the "Principle of Insufficient Reason":

$$p = \frac{1}{2} \tag{3}$$

We inferred [10] that all the memoryless random processes [15] require the *linear time*, a traditional physical concept not free from paradoxes [18] when applied to chances.

3.2 SPT and Euclidean Geometry

The graph describing the solutions of the *Eq.*(3) is a set of points on the horizontal line

$$y = \frac{1}{2} \tag{4}$$

which never intersects the null probability axis $y = 0$.

Although not specifying how, the possible geometric interpretation of the *SPT* as *flat* probability was proposed [10] because there is a unique line parallel to the *x*-axis passing for the point $A(0, \frac{1}{2})$ and it recalls Euclid's fifth postulate [14] in the Playfair's formulation [22].

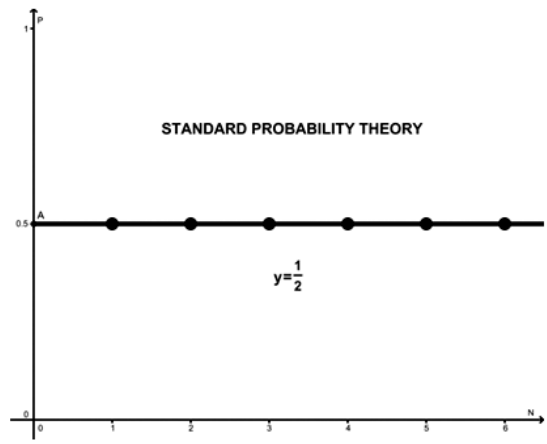

Figure 1. The Eqs. (3) and (4) of the SPT

4. THE GM AS "HYPERBOLIC" PROBABILITY

4.1 Axioms and Formulas of the GM

Despite a variety of dissimilar gambling strategies denoting its heuristic inconsistency, the GM's approach to probability obtained an epistemological absolution [10] from the accusation to be cognitively biased [26] for its expectation of regularities in the event sequences.

In fact, we noticed [10] that the *GM's* different predictions about an upcoming trial on account of the previous sequence of outcomes, known as "Gambler's fallacy" [24], are motivated by a sort of "Principle of Sufficient Reason" (opposite to the *SPT's* Principle of Insufficient Reason) by which there is a reason (not the certainty) to exclude that one out of two incompatible events can occur infinite times in a row while the other event never.

We understood [10] that this axiom involves a *non-linear time*, an alternative hypothesis not to be *a priori* excluded [3-9, 13, 21, 25] but indefinable when applied to the *GM's* contradictory strategies.

As best representative of the *GM's* manifold probabilities we chose the "Martingale", an old-fashioned strategy where we double the bet after each losing toss:

$$p = \frac{1}{1+2^n}, \text{ with } n \in \mathbb{N} \tag{5}$$

The *Eq.*(5) is the lower bound of the following interval containing all the gambling behaviors based on the belief in the "fairness" of the coin [27]:

$$\frac{1}{1+2^n} \leq p < \frac{1}{2}, \text{ with } n \in \mathbb{N} \tag{6}$$

4.2 GM and Hyperbolic Geometry

The graph describing the solutions of the *Eq.* (7) is a set of points on the curve:

$$y = \frac{1}{1+2^x} \tag{7}$$

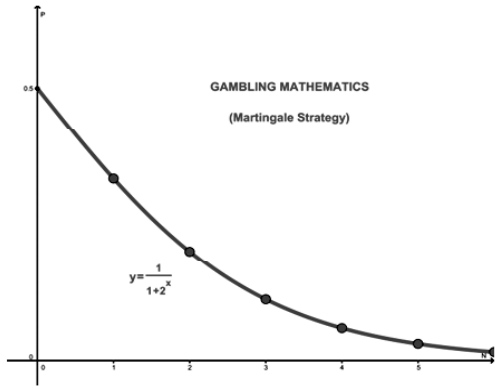

Figure 2: The Eqs. (5) and (7) of the GM

Never intersecting the null probability axis $y = 0$ are also the solutions of the *Eq.*(6) in the following range of curves:

$$\frac{1}{1+2^x} \leq y < \frac{1}{2} \tag{8}$$

Although not specifying how, the possible geometric interpretation of the *GM* as *hyperbolic* probability was proposed [10] because the *x*-axis is an unreachable asymptote for infinitely many curves and it recalls the hyperbolic geometry [1] by Lobachevsky [19, 20].

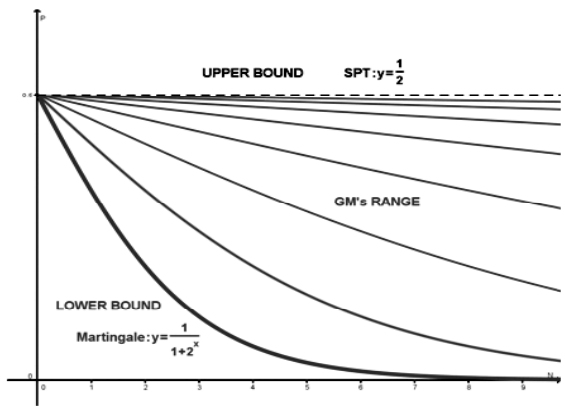

Figure 3: GM's Range According to the Eq.(8).

5. THE NPAP AS "ELLIPTIC" PROBABILITY

5.1 Axioms and Formulas of the NPAP

The *NPAP's* different predictions about an upcoming trial on the basis of the previous sequence of outcomes is motivated by what we called "Principle of Certainty", *i.e.*, the *certainty* to exclude that one out of two incompatible events can occur infinite times in a row while the other event never [10].

Different from both the *SPT's* Principle of Insufficient Reason and the *GM's* Principle of Sufficient Reason, the *NPAP's* axiom assumes a non-linear time recursively structured (albeit not necessarily cyclical).

The *NPAP* is innovative because the number of failures before a success is finite and not potentially infinite as for the *SPT* and *GM*.

We established the following *NPAP's* probability:

$$p = \frac{23-n}{46}, \text{ with } n \in \mathbb{N} \tag{9}$$

We improved [10] the *Eq*.(9) as follows:

$$p = \frac{m-n}{2m}, \text{ with } m = 23, 24, 25\ldots \tag{10}$$

We eventually considered higher degree distributions with the integer parameters $k > 1$ and $m \geq 23$:

$$p = \frac{\sqrt[k]{m^k - n^k}}{2m} \tag{11}$$

We emphasized the elliptic case of the *Eq*.(11) for $k = 2$:

$$p = \frac{\sqrt{m^2 + n^2}}{2m} \tag{12}$$

5.2 NPAP and Elliptic Geometry

The graph describing the solutions of the *Eq*.(9) is a set of points on the straight line:

$$y = \frac{x}{46} + \frac{1}{2} \tag{13}$$

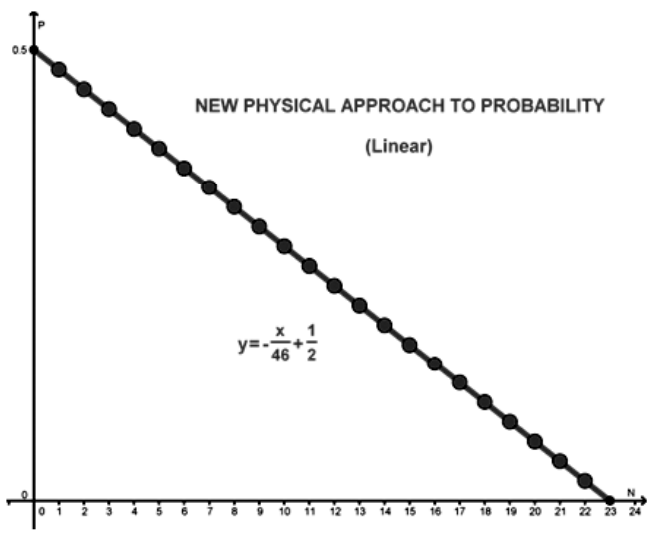

Figure 4: The Eqs.(9) and (13) of the NPAP.

Intersecting $y = 0$ in the integer roots $x = m \geq 23$, the solutions of the *Eq*.(10) are on the lines:

$$y = \frac{m-x}{2m}, \text{ with } m = 23, 24, 25 \ldots. \tag{14}$$

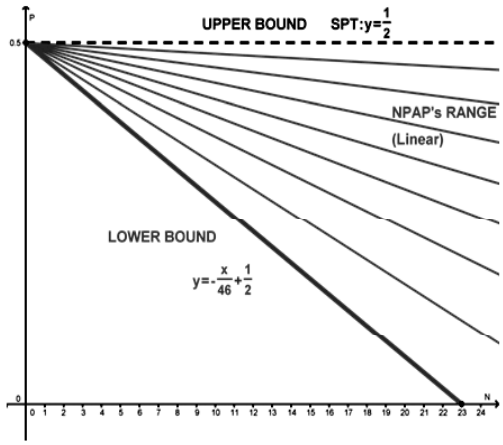

Figure 5: The Eqs. (10) and (14) of the NPAP.

The same roots are shared by the curves containing the solutions of the *Eq.*(11):

$$y = \frac{\sqrt[k]{m^k - x^k}}{2m}, \text{ with } k \in \mathbb{Z}^+ \tag{15}$$

If $k = 2$ and $m = 23$ in the Eq. (15) then the solutions of the Eqs.(11) and (12) are on the first quadrant arch of the following ellipse:

$$\left(\frac{x}{23}\right)^2 + (2y)^2 = 1 \tag{16}$$

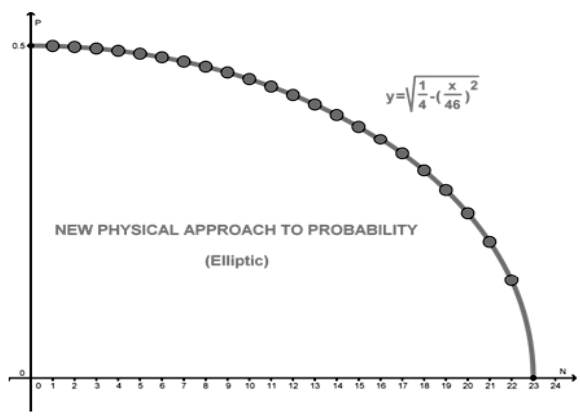

Figure 6: The Eqs.(12) and (16) of the NPAP.

Although not specifying how, the possible geometric interpretation of the *NPAP* as *elliptic* probability was proposed [10] because there is no line parallel to $y = 0$ (Figures 4 and 5) and all the curves intersect the null probability axis (Figure 6); the whole situation recalls the elliptic geometry [16] by Riemann [23].

6. COMPARISON AMONG THE RESULTS

The diversity of approach among the *SPT, GM* and *NPAP* can be evidenced by showing simultaneously their respective probability graphs:

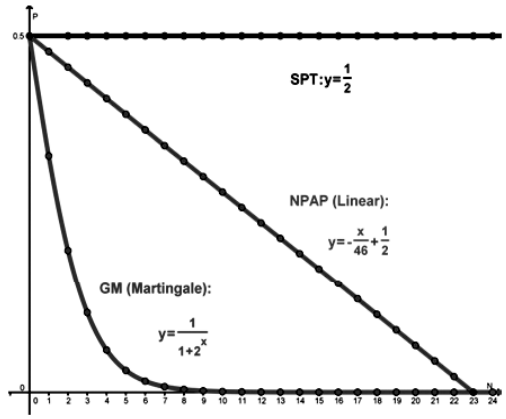

Figure 7: Graph Union of the Figs. 1, 2 and 4.

Analogously, the graphs of the *Figs*.1, 2 and 6 can be compared after having labeled each curve with the corresponding "geometric" probability:

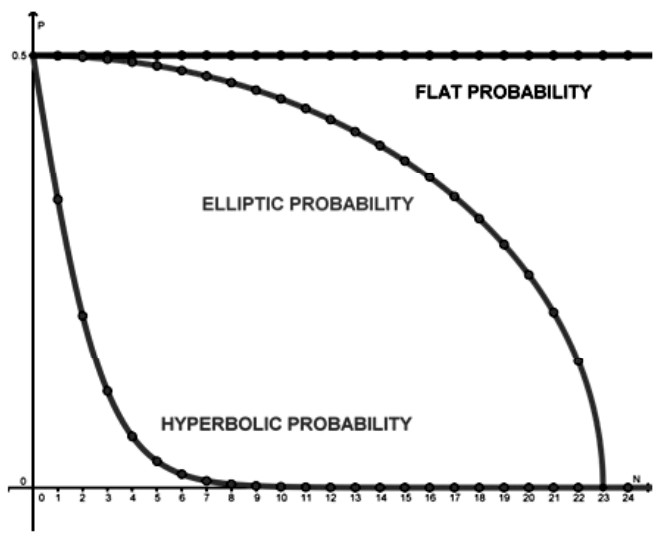

Figure 7: Comparison among the FP, HP and EP.

7. CONCLUSIONS

We have expounded the three-fold classification into *flat*, *hyperbolic* and *elliptic* probability.

We have considered the *SPT* as *FP* because its probability distribution is on a line "unique" and "parallel" to the null probability axis (Figure 1); it refers directly to that "uniqueness of the parallel line" characterizing the Euclidean geometry.

We have included the *GM* in the *HP* because its probability distributions are on "infinitely many" curves "never intersecting" the *x*-axis (Figure 3); they seem linked to the "infinitely many parallel lines" of Lobachevsky's geometry. Let us point out that the adjective "hyperbolic" is not at all related to the nature of the curves.

Similarly, it does not matter whether the *NPAP's* probability distribution is on a line (Figure 5), on an ellipse (Figure 6) or on a higher degree curve, because it "always" intersects $y = 0$; it invokes the "absence of parallel

lines" of the Elliptic Geometry. The fact that in the NPAP's range of probability distributions there is also an arch of ellipse is just a fortuitous coincidence.

The title of this paper is motivated by the awareness that the only original element in the hypothetical triad *FP*, *HP*, *EP* is the last one.

We remark that this note does not solve the problem of "how" to establish a connection between Probability and Geometry, an open question worthy to be explored.

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