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A FAMILY OF MEASURES OF FUZZY ENTROPY THROUGH FUNCTIONAL VARIANTS

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ABSTRACT

In the existing literature of information theory, there are several mathematical models providing the information content contained in a probabilistic/fuzzy experiment. In the present communication, we have discussed the importance and need for the development of non-probabilistic or fuzzy measures of entropy and consequently, obtained many existing measures of fuzzy entropy with the help of distance measure.

Keywords: Probability distribution, Information entropy, Differential entropy, Fuzzy set, Fuzzy information, Functional variant.

INTRODUCTION

Shannon's [20] mathematical theory of information entropy was introduced to analyze the information carrying capacity of communication channels, serving as a measure of the degree of uncertainty. Information entropy is an extremely important mathematical tool in data compression, signal processing, and communication processes. Despite the popularity of Shannon's information entropy in communication processes and signal processing. Shannon's [20] entropy is often not mathematically amenable to basic estimation methods and analytic manipulation in practice. This entropy is given by

$$H(P) = -\sum_{i=1}^{n} p_i \ \text{In} \ p_i \qquad (1.1)$$

Some work related with multivariate dynamic information has been done by Ebrahimi, Soofi and Kirmani [4]. The authors have covered three related topics. Firstly, it calculates joint, marginal, and conditional Shannon entropies for residual life distributions. Secondly, it studies monotonicity of residual entropy and transformations that preserve it. Thirdly, it provides an entropy characterization of the joint distribution of independent exponential random variables. Estimation of differential entropy from observations of a random variable is of great importance for a wide range of signal processing applications such as source coding, pattern recognition, hypothesis testing, and blind source separation. A new non-additive entropic model has been investigated and studied by Tsallis [23].

A measure of fuzziness often used and citied in the literature of information theory, known as fuzzy entropy, was first introduced by Zadeh [24]. The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon's [20] entropy. However, the two functions measure fundamentally different types of uncertainty.

Basically, the Shannon's [20] entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment whereas fuzzy entropy is the quantitative description of fuzziness in fuzzy sets. De Luca and Termini [2] introduced some requirements which capture our intuition for the degree of fuzziness. Kaufmann [11] proposed to measure the degree of fuzziness of any fuzzy set A by a metric distance between its membership function and the characteristic function of its nearest crisp set. Keeping in view the existing probabilistic measures, Parkash and Sharma [18] introduced the measure of fuzzy entropy, given by

$$K_a(A)=\sum_{i=1}^{n}\left[\log(1+a\mu_A(x_i))+\log(1+a(1-\mu_A(x_i)))-\log(1+a)\right];a\geq 0 \tag{1.2}$$

Some other measures of fuzzy entropy and their generalizations have been studied by Ebanks [3], Kapur [10], Klir and Folger [12], Kosko [13], Kandel [8], Hu, Yu [7], Liu and Kao [14], Lowen [15], Pal and Bezdek [16] etc.

2. A VARIETY OF MEASURES OF FUZZY ENTROPY THROUGH FUNCTIONAL VARIANTS

In this section, we shall deduce many existing measures of fuzzy entropy with the help of distance measure.
We first define the distance between two fuzzy sets A and B as:

$$D(A:B)=\sum_{i=1}^{n}[\mu_A(x_i)-\mu_B(x_i)] \tag{2.1}$$

and the functional variant of this distance can be taken as

$$D[(f(A):f(B)]=\sum_{i=1}^{n}\left[f(\mu_A(x_i))+f(1-\mu_A(x_i))-f(\mu_B(x_i))-f(1-\mu_B(x_i))\right] \tag{2.2}$$

The weighted distance between the two sets A and B can be taken as:

$$D\ (A:\ B/W)=\sum_{i=1}^{n}W_i\left[\mu_A(x_i)-\mu_B(x_i)\right] \tag{2.3}$$

Now, we maximize

$$D\,[f\ (1):\ f\ (A)]\ =\ \sum_{i=1}^{n}\left[f(1)+f(0)-f(\mu_A(x_i))-f(1-\mu_A(x_i))\right] \tag{2.4}$$

under the following two conditions

$$\sum_{i=1}^{n}\mu_A(x_i)=K \tag{2.5}$$

and

$$\sum_{i=1}^{n}a_i\mu_A(x_i)=K' \tag{2.6}$$

The corresponding Lagrangian is

$$L=\sum_{i=1}^{n}[f(1)+f(0)-f(\mu_A(x_i))-f(1-\mu_A(x_i))]-\nu\left(\sum_{i=1}^{n}\mu_A(x_i)-K\right)-\delta\left(\sum_{i=1}^{n}a_i\mu_A(x_i)-K'\right) \tag{2.7}$$

Differentiating (2.7) with respect to $\mu_A(x_i)$; ($i=1,\ 2,...,\ n$) , ν and δ, we have

$$f'(\mu_A(x_i))-f'(1-\mu_A(x_i))=-\nu-a_i\delta \tag{2.8}$$

$$\sum_{i=1}^{n}\mu_A(x_i)=K \tag{2.9}$$

$$\sum_{i=1}^{n} a_i \mu_A(x_i) = K' \qquad (2.10)$$

Taking $g(\mu_A(x_i)) = f'(\mu_A(x_i)) - f'(1-\mu_A(x_i))$, equation (2.8) becomes

$$g(\mu_A(x_i)) = K'' \qquad (2.11)$$

where $K'' = -\nu - a_i\delta$

A unique value of $\mu_A(x_i)$ will exist if the function g is strictly monotonic in (0,1) and this happens if f is twice differentiable on (0,1) and the derivative

$$g'(\mu_A(x_i)) = f''(\mu_A(x_i)) - f''(1-\mu_A(x_i)) \qquad (2.12)$$

is either strictly positive or strictly negative on (0,1). Thus equation (2.11) gives

$\mu_A(x_i) = g^{-1}(K'')$ which is the required solution.

Below we discuss the concavity of the functional variant $D[f(1): f(A)]$:

We have

$$D[f(1): f(A)] = \sum_{i=1}^{n} \psi(\mu_A(x_i)) \qquad (2.13)$$

where

$$\psi(\mu_A(x_i)) = f(1) + f(0) - f(\mu_A(x_i)) - f(1-\mu_A(x_i))$$

$$\psi''(\mu_A(x_i)) = -f''(\mu_A(x_i)) - f''(1-\mu_A(x_i))$$

$$= -g'(\mu_A(x_i)) \qquad (2.14)$$

Obviously, $\psi(\mu_A(x_i))$ is a concave function of $\mu_A(x_i)$ if $g'(\mu_A(x_i)) > 0$ on (0,1). Thus equation (2.13) implies that $D[f(1): f(A)]$ is a concave function of $\mu_A(x_i)$; $i = 1, 2, 3, ..., n$. Hence any solution of (2.4) found by the method of differential calculus yields an absolute maximum rather than relative maximum.

Particular Cases:

I. If we take $f(t) = t\log t,\ t \in (0,1)$, equation (2.4) becomes

$$D[f(1): f(A)] = -\sum_{i=1}^{n}\left[1\log 1 + 0\log 0 - \mu_A(x_i)\log\mu_A(x_i) - (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

$$= -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

which is a fuzzy entropy corresponding to Shannon's [20] entropy

II. If we take $f(t) = -\dfrac{t - t^{\alpha}}{\alpha - 1}$, equation (2.4) becomes

$$D[f(1): f(A)] = \sum_{i=1}^{n}\left[-\frac{\mu_A^{\alpha}(x_i) - \mu_A(x_i)}{\alpha - 1} - \frac{(1-\mu_A(x_i))^{\alpha} - (1-\mu_A(x_i))}{\alpha = 1}\right]$$

$$= \frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - 1\right]$$

which is fuzzy entropy corresponding to Havrada and Charvat’s [6] probabilistic entropy.

III. If we take $f(t) = \dfrac{\log[t^{\alpha} + (1-t)^{\alpha}]}{2(\alpha - 1)}$, equation (2.4.4) becomes

$$D[f(1):f(A)] = \sum_{i=1}^{n}\left[-\frac{1}{2(\alpha-1)}\log\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}\right.$$
$$\left.-\frac{1}{2(\alpha-1)}\log\left\{(1-\mu_A(x_i))^{\alpha}+(1-(1-\mu_A(x_i)))^{\alpha}\right\}\right]$$

$= \dfrac{1}{1-\alpha}\sum_{i=1}^{n}\log\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}; \alpha>0, \alpha\neq 1$ which is fuzzy entropy corresponding to Renyi's [19] entropy

IV If we take $f(t)=\dfrac{t^{\beta}+(1-t)^{\beta}-t^{\alpha}-(1-t)^{\alpha}]}{2(\beta-\alpha)}, \alpha\neq\beta$ equation (2.4) gives the following measure of fuzzy entropy:

$$D[f(1):f(A)] = \sum_{i=1}^{n}\left[-\frac{\mu_A{}^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}-\mu_A{}^{\alpha}(x_i)-(1-\mu_A(x_i))^{\alpha}}{2(\beta-\alpha)}\right.$$
$$\left.-\frac{(1-\mu_A(x_i))^{\beta}+\mu_A{}^{\beta}(x_i)-(1-\mu_A(x_i))^{\alpha}-\mu_A{}^{\alpha}(x_i)}{2(\beta-\alpha)}\right]$$
$$=\frac{1}{(\beta-\alpha)}\sum_{i=1}^{n}\left[\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}-\mu_A{}^{\beta}(x_i)-(1-\mu_A(x_i))^{\beta}\right]$$

which is fuzzy entropy corresponding to Sharma and Taneja's [22] probabilistic entropy.

V. If we take $f(t)=\dfrac{2-t^{\alpha}-(1-t)^{\alpha}-t^{\beta}-(1-t)^{\beta}]}{2(\beta+\alpha-2)},$ equation (2.4.4) becomes

$$D\left[f(1):f(A)\right]=\sum_{i=1}^{n}\left[-\frac{2-\mu_A{}^{\alpha}(x_i)-(1-\mu_A(x_i))^{\alpha}-\mu_A{}^{\beta}(x_i)-(1-\mu_A(x_i))^{\beta}}{2(\alpha+\beta-2)}\right.$$
$$\left.-\frac{2-(1-\mu_A(x_i))^{\alpha}-\mu_A{}^{\alpha}(x_i)-(1-\mu_A(x_i))^{\beta}-\mu_A{}^{\beta}(x_i)}{2(\alpha+\beta-2)}\right]$$
$$=\frac{1}{\alpha+\beta-2}\sum_{i=1}^{n}\left[\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}+\mu_A{}^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}-2\right]$$

which is fuzzy entropy corresponding to Kapur's [9] probabilistic entropy

VI. If we take $f(t)=\dfrac{1+\lambda t}{\lambda}\log\dfrac{1+\lambda t}{\lambda}, \lambda>0$ equation (2.4) becomes

$$D[f(1):f(A)] = \sum_{i=1}^{n}\left[\frac{1+\lambda}{\lambda}\log 1+\frac{1}{\lambda}\log\frac{1}{1+\lambda}-\frac{1+\lambda\mu_A(x_i)}{\lambda}\log\frac{1+\lambda\mu_A(x_i)}{1+\lambda}\right.$$
$$\left.-\frac{1+\lambda(1-\mu_A(x_i))}{\lambda}\log\frac{1+\lambda(1-\mu_A(x_i))}{1+\lambda}\right]$$

$$=-\frac{1}{\lambda}\sum_{i=1}^{n}\Big[\big(1+\lambda\mu_A(x_i)\big)\log\left[\frac{1+\lambda\mu_A(x_i)}{1+\lambda}\right]+\Big[\big(1+\lambda\big(1-\mu_A(x_i)\big)\log\left[\frac{1+\lambda\big(1-\mu_A(x_i)\big)}{1+\lambda}\right]-\log\frac{1}{1+\lambda}$$

which is fuzzy entropy corresponding to Ferreri's [5] probabilistic entropy.

VII If we take $f(t)=t\log t-\frac{1+at}{a}\log(1+at),$ equation (2.4) becomes

$$D[f(1):f(A)] = \sum_{i=1}^{n}\left[\begin{array}{l}-\frac{1+a}{a}\log(1+a)-\mu_A(x_i)\log\mu_A(x_i)+\frac{1+a\mu_A(x_i)}{a}\log\big(1+a\mu_A(x_i)\big)\\ -\big(1-\mu_A(x_i)\big)\log\big(1-\mu_A(x_i)\big)+\frac{1+a\big(1-\mu_A(x_i)\big)}{a}\log\Big(1+a\big(1-\mu_A(x_i)\big)\Big)\end{array}\right]$$

$$\sum_{i=1}^{n}\Big[-\mu_A(x_i)\log\mu_A(x_i)-\big(1-\mu_A(x_i)\big)\log\big(1-\mu_A(x_i)\big)\Big]+$$

$$=\frac{1}{a}\sum_{i=1}^{n}\Big[\big(1+a\mu_A(x_i)\big)\log\big(1+a\mu_A(x_i)\big)+\Big(1+a\big(1-\mu_A(x_i)\big)\Big)\log\Big(1+a\big(1-\mu_A(x_i)\big)\Big)\Big]$$

$$-\frac{n(1+a)}{a}\log(1+a)$$

which is fuzzy entropy corresponding to Kapur’s [9] probabilistic entropy.

VIII If we take $f(t)=\frac{\left(t^{1/\alpha}+(1-t)^{1/\alpha}\right)^{\alpha}-1}{2(\alpha-1)}$, equation (2.4) becomes

$$D[f(1):f(A)]=\sum_{i=1}^{n}\left[-\frac{\left\{\mu_A^{1/\alpha}(x_i)+\big(1-\mu_A(x_i)\big)^{1/\alpha}\right\}^{\alpha}-1}{2(\alpha-1)}-\frac{\left\{\big(1-\mu_A(x_i)\big)^{1/\alpha}+\mu_A^{1/\alpha}(x_i)\right\}^{\alpha}-1}{2(\alpha-1)}\right]$$

$$=\frac{1}{2(1-\alpha)}\sum_{i=1}^{n}\left[-2\left\{\mu_A^{1/\alpha}(x_i)+(1-\mu_A(x_i))^{1/\alpha}\right\}^{\alpha}+2\right]$$

$$=\frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\left\{\mu_A^{1/\alpha}(x_i)+(1-\mu_A(x_i))^{1/\alpha}\right\}^{\alpha}-1\right]$$

which is fuzzy entropy corresponding to Arimoto's [1] entropy.

IX. If we take $f(t)=\frac{\left(t^{\alpha}+(1-t)^{\alpha}\right)^{\beta}-1}{2(\alpha-1)\beta}$, equation (2.4) becomes

$$D[f(1):f(A)] = \sum_{i=1}^{n}\left[-\frac{\left\{\mu_A^{\alpha}(x_i)+\big(1-\mu_A(x_i)\big)^{\alpha}\right\}^{\beta}-1}{2(\alpha-1)\beta}-\frac{\left\{\mu_A^{\alpha}(x_i)+\big(1-\mu_A(x_i)\big)^{\alpha}\right\}^{\beta}-1}{2(\alpha-1)\beta}\right]$$

$$=\frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}\left[\left\{\mu_A^{\alpha}(x_i)+\big(1-\mu_A(x_i)\big)^{\alpha}\right\}^{\beta}-1\right]$$

which is fuzzy entropy developed by Parkash [17].

X If we take $f(t)=\frac{\left[t^{\alpha}+(1-t)^{\alpha}-1\right]^{\beta}}{2(\alpha-1)\beta}$, equation (2.4) becomes

$$D[f(1):f(A)]=\sum_{i=1}^{n}\left[-\frac{\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}^{\frac{\beta-1}{\alpha-1}}-1}{2(\alpha-1)\beta}-\frac{\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right\}^{\frac{\beta-1}{\alpha-1}}-1}{2(\alpha-1)\beta}\right]$$

$$=\frac{1}{(1-\alpha)\beta}\left[\sum_{i=1}^{n}\left[\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right]-1\right]^{\beta},\alpha>0,\alpha\neq1,\beta\neq0$$

which is fuzzy entropy corresponding to Sharma and Mittal's [21] entropy.

XI If we take $f(t)=\frac{\left[t^{\alpha}+(1-t)^{\alpha}-1\right]^{1/\alpha}}{2(\alpha-1)}$, equation (2.4) becomes

$$D[f(1):f(A)]=\sum_{i=1}^{n}\left[-\frac{\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}-1\right\}^{1/\alpha}}{2(\alpha-1)}-\frac{\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}-1\right\}^{1/\alpha}}{2(\alpha-1)}\right]$$

$$=\frac{1}{2(\alpha-1)}\sum_{i=1}^{n}\left[-2\left\{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}-1\right\}^{1/\alpha}\right]$$

$=\frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}-1\right];\alpha>0,\alpha\neq1$ which is fuzzy entropy developed by Kapur [10].

XII If we take $f(t)=\frac{1}{2(\alpha-\beta)}\log\left[\frac{t^{\alpha}+(1-t)^{\alpha}}{t^{\beta}+(1-t)^{\beta}}\right]$, equation (2.4) becomes

$$D[f(1):f(A)]=\sum_{i=1}^{n}\left[-\frac{1}{2(\alpha-\beta)}\log\left\{\frac{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}}{\mu_A{}^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}}\right\}-\frac{1}{2(\alpha-\beta)}\log\left\{\frac{(1-\mu_A(x_i))^{\alpha}+\mu_A{}^{\alpha}(x_i)}{(1-\mu_A(x_i))^{\beta}+\mu_A{}^{\beta}(x_i)}\right\}\right]$$

$=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\log\left[\frac{\mu_A{}^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}}{\mu_A{}^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}}\right]$, which is fuzzy entropy again introduced by Kapur [10] and it corresponds to probabilistic entropy due to Kapur [9].

Next, we obtain analytical expressions for the fuzzy values $\mu_A(x_i)$ corresponding to the various functions discussed above:

1. The solution of the equation (2.4) corresponding to $f(t)=t\log t$, $t\in(0,1)$ is given by $\mu_A(x_i)=\frac{1}{1+\lambda}$ where $\lambda=e^{\nu+a_i\delta}$

2. The solution of the equation (2.4) corresponding to $f(t)=\dfrac{t-t^{\alpha}}{\alpha-1}$ is given by, $\mu_A(x_i)=\dfrac{1}{1+\lambda_1}$

where $\lambda_1=\dfrac{(1-\alpha)(\nu-\delta a_i)}{\alpha}\geq 0$

3. The solution of the equation (2.4) corresponding to

$$f(t)=\frac{1+\lambda t}{\lambda}\log\left[\frac{1+\lambda t}{1+\lambda}\right];\lambda>0;\ \text{is given by}\ \mu_A(x_i)=\frac{1+\lambda-K}{\lambda(K+1)}$$

where $\lambda=e^{\nu+a_i\delta}$ and the solution exists for $K\geq\dfrac{1}{1+\lambda}$ or $K\leq 1+\lambda$

Since, the analytical expressions for the fuzzy values corresponding to the other functions discussed above involve parameters, solutions of (2.4) for these functions, though exist, are in complicated form and can be obtained under certain suitable conditions.

REFERENCES

[1] Arimoto, S. (1971). Information theoretical consideration on estimation problems. *Information and Control* **19**: 181-194.

[2] De Luca, A. and Termini, S. (1972). A definition of non-probabilistic entropy in setting of fuzzy set theory. *Information and Control* **20**: 301-312.

[3] Ebanks, B.R. (1983). On measures of fuzziness and their representations. *Journal of Mathematical Analysis and Applications* **94:** 24-37.

[4] Ebrahimi, N., Soofi, E.S. and Zahedi, H. (2004). Information properties of order statistics and spacings. *IEEE Transactions on Information Theory* **50**: 177-183.

[5] Ferreri, C. (1980). Hypoentropy and related heterogeneity divergence measures. *Statistic*a **40**: 155-168.

[6] Havrada, J.H. and Charvat, F. (1967). Quantification methods of classification process:Concept of structural α-entropy. *Kybernetika* **3**: 30-35.

[7] Hu, Q. and Yu, D. (2004). Entropies of fuzzy indiscernibility relation and its operations. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **12**: 575-589.

[8] Kandel, A. (1986). *Fuzzy Mathematical Techniques with Applications*. Addison-wesley.

[9] Kapur, J.N. (1994). *Measures of Information and their Applications*. Wiley Eastern, New Delhi.

[10] Kapur, J.N. (1997). *Measures of Fuzzy Information*. Mathematical Sciences Trust Society, New Delhi.

[11] Kaufmann, A. (1975). *Introduction to Theory of Fuzzy Subsets.* Academic Press, New York.

[12] Klir, G.J. and Folger, T.A. (1988). *Fuzzy Sets, Uncertainty and Indetermination*. Prentice Hall, New York.

[13] Kosko, B. (1991). Fuzzy entropy and conditioning. *Information Sciences* **42**: 165-174.

[14] Liu, S. T. and Kao, C. (2002). Fuzzy measures for correlation coefficient of fuzzy numbers. *Fuzzy Sets and Systems* **128**: 267-275.

[15] Lowen, R. (1996). *Fuzzy Set Theory-Basic Concepts, Techniques and Bibliography*. Kluwer Academic Publishers, Boston.

[16] Pal, N. R. and Bezdek, J. C. (1994). Measuring fuzzy uncertainty. *IEEE Transaction on Fuzzy Systems* **2**: 107-118.

[17] Parkash, O. (1998). A new parametric measure of fuzzy entropy. *Information Processing and Management of Uncertainty* **2**:1732-1737.

[18] Parkash, O. and Sharma, P. K. (2004). Measures of fuzzy entropy and their relations. *Inernationa. Journal of Management & Systems* **20**: 65-72.

[19] Renyi, A. (1961). On measures of entropy and information. *Proceedings 4th Berkeley Symposium on Mathematical Statistics and Probability* **1**: 547-561.

[20] Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal* **27**: 379-423, 623-659.

[21] Sharma, B. D.and Mittal, D. P. (1975). New non-additive measures of entropy for a discrete probability distributions. *Journal of Mathematical Sciences* **10**: 122-133.

[22] Sharma, B. D. and Taneja, I. J. (1975). Entropies of type (α,β) and other generalized measures of information theory. *Metrica* **22**: 202-215.

[23] Tsallis, C. (2009). Non-additive entropy: The concept and its use. *Eur. Phys. J. A* **40**: 257-266.

[24] Zadeh, L. A. (1968). Probability measures of fuzzy events. *Journal of Mathematical Analysis and Applications* **23**: 421-427.