Int Jr. of Mathematical Sciences & Applications
Vol. 5, No. 2, (July-December, 2015)

ISSN No: 2230-9888
www.journalshub.com

HYPERBOLIC HSU-STRUCTURE MANIFOLD EQUIPPED WITH QUARTER SYMMTRIC NON-METRIC CONNECTION

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ABSTRACT

In this paper, we defined a quarter symmetric non-metric connection in Hyperbolic Hsu- structure manifold and studied their properties. It has been also proved that a contravariant almost analytic vector field V with respect to the Riemannian connection ∇ is also contravariant almost analytic with respect to the quarter symmetric non-metric connection $\widetilde{\nabla}$ in a Hyperbolic Hsu-structure Kähler manifold.

Keywords: Hyperbolic Hsu- structure manifold, quarter symmetric non-metric connection, Hyperbolic Hsu-structure Kähler manifold, Contravariant almost analytic vector field.

1. INTRODUCTION

If on an even dimensional differentiable manifold V_n ,$n = 2m$ of differentiability class C^∞ , there exists a vector valued real linear function ϕ, satisfying

$\phi^2 X = -a^r X$. for arbitrary vector field X. (1.1)

Where r is an integer and a is a real or imaginary number. Then $\{\phi\}$ is said to give to V_n a Hyperbolic-Hsu structure defined by the equation (1.1) and the manifold V_n is called a Hyperbolic-Hsu structure manifold.

The equation (1.1) gives different structure for different value of ‘ a’ and ‘r’.

If $r = 0$, it is an almost complex structure.

If $a = 0$, it is an almost tangent structure.

If $r = \pm 1$ *and* $a = +1$, it is an almost complex structure.

If $r = \pm 1$ *and* $a = -1$, it is an almost product structure.

If $r = 2$ then it is a HGF-structure which includes π-structure for $a \neq 0$, An almost product structure for $a = \pm i$,

An almost complex structure for $a = \pm 1$,

An almost tangent structure for $a = 0$.

Let the Hyperbolic Hsu-Structure V_n be endowed with a metric $'g'$ such that

$$g(\phi X, \phi Y) = a^r g(X, Y) . \tag{1.2}$$

Then $\{\phi, g\}$ is said to give to metric Hyperbolic -Hsu-Structure and V_n is called a metric Hyperbolic-Hsu structure manifold.

In a sequel arbitrary vector fields are denoted by X, Y, Z ... . etc.

Let us consider on V_n, equipped with metic-Hyperbolic Hsu-Structure, then

$$\phi^*(X, Y) \stackrel{\text{def}}{=} g(\phi X, Y) = -g(X, \phi Y) . \tag{1.3}$$

Then it is easy to verify

$$\phi^*(\phi X, Y) = -\phi^*(X, \phi Y) . \tag{1.4}$$

$$\phi^*(\phi X, \phi Y) = a^r \phi^*(X, Y) . \tag{1.5}$$

And

$$\phi^*(X, Y) = -\phi^*(Y, X) . \tag{1.6}$$

The equation (1.6) shows that the 2-form ϕ^* is skew symmetric in V_n.

If Hyperbolic Hsu-structure manifold V_n satisfies a condition

$$(\nabla_X \phi)Y = 0. \tag{1.7}$$

Then V_n will be said to be Hyperbolic Hsu-structure K$\ddot{a}$hler manifold.

From the equation (1.7), it can be seen that

$$\nabla_X \phi Y = \phi(\nabla_X Y) \Leftrightarrow \phi(\nabla_X \phi Y) = -a^r(\nabla_X Y). \tag{1.8}$$

2. Quarter Symmetric non-metric connection:

A linear connection $\tilde{\nabla}$ defined as

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X. \tag{2.1}$$

For arbitrary vector field X and Y, is said to be quarter symmetric non-metric connection if the torsion tensor τ of the connection $\tilde{\nabla}$ and the metric tensor g are given by

$$\tau(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{2.2}$$

And

$$(\tilde{\nabla}_X g)(Y, Z) = -\eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y). \tag{2.3}$$

Where η is 1-form associated with the vector field ζsuch that

$$\eta(X) = g(X, \zeta). \tag{2.4}$$

And ∇ is the Riemannian connection

If we put (2.1) as

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y). \quad (2.5)$$

Then

$$H(X,Y) = \eta(Y)\phi X. \quad (2.6)$$

Let us define

$$H^*(X,Y)Z \stackrel{\text{def}}{=} g(H(X,Y),Z). \quad (2.7)$$

From the equation (2.6) and (2.7), we get

$$H^*(X,Y)Z = \eta(Y)g(\phi X,Z). \quad (2.8)$$

3. Hyperbolic Hsu-structure manifold V_n equipped with Quarter Symmetric non-metric connection:

Theorem 3.1. For a Hyperbolic Hsu-structure manifold V_n equipped with a quarter symmetric non-metric connection $\tilde{\nabla}$, the following results hold good.

(i) $H(\phi X,Y) = -a^r H(X,\phi Y)$ iff $\eta(Y)X = \eta(\phi Y)\phi X$,
(ii) $H(\phi X,Y) = H(\phi Y,X)$ iff $\eta(Y)X = \eta(X)Y$,
(iii) $H(\phi X,\phi Y) = -a^r H(X,Y)$ iff $\eta(\phi Y)X = \eta(Y)\phi X$. (3.1)

Proof: From the equation (2.6) and (1.1), we get

(i) $H(\phi X,Y) = -a^r\eta(Y)X$,
(ii) $H(X,\phi Y) = \eta(\phi Y)\phi X$. (3.2)

Clearly (3.1)(i) follows from (3.2)(i) and (3.2)(ii), interchanging X and Y in (2.6), we get

$$H(Y,X) = \eta(X)\phi Y. \quad (3.3)$$

Apply ϕ on Y in equation (3.3) and using equation (1.1), we get

$$H(\phi Y,X) = -a^r\eta(X)Y. \quad (3.4)$$

Hence the result of (3.1)(ii) we will get from (3.2)(i) and (3.4)

Again applying ϕ on X and Y in (2.6) and using (1.1), we get

$$H(\phi X,\phi Y) = -a^r\eta(\phi Y)X. \quad (3.5)$$

Hence the result (3.1)(iii), we will obtain with the help of (2.6) and (3.5).

Theorem 3.2. In a hyperbolic Hsu-structure manifold V_n with a quarter symmetric non-metric connection $\tilde{\nabla}$, we have

(i) $H^*(\phi X,\phi Y)Z = \phi^*(\phi X,Z)\eta(\phi Y) = -H^*(X,\phi Y)\phi Z$,
(ii) $H^*(\phi X,\phi Y)\phi Z = \eta(\phi Y)\phi^*(\phi X,\phi Z) = -H^*(\phi Z,\phi Y)\phi X$. (3.6)

Proof: Applying ϕ on X and Y in (2.8) and using the equations (1.3) and (1.5), we get

$$H^*(\phi X,\phi Y)Z = \eta(\phi Y)\phi^*(\phi X,Z). \quad (3.7)$$

Again applying ϕ on Y and Z in (2.8) and using equations (1.3) and (1.5), we get

$$H^*(X,\phi Y)\phi Z = -\eta(\phi Y)\phi^*(\phi X, Z). \quad (3.8)$$

Hence the equations (3.7) and (3.8) gives (3.6)(i).

Again applying ϕ on Z in (3.7), we get

$$H^*(\phi X,\phi Y)\phi Z = \eta(\phi Y)\phi^*(\phi X,\phi Z), \quad (3.9a)$$

Interchanging ϕX and ϕZ in (3.9a), we get

$$H^*(\phi Z,\phi Y)\phi X = -\eta(\phi Y)\phi^*(\phi X,\phi Z). \quad (3.9b)$$

Hence from equations (3.9a) and (3.9b), we will get (3.6)(ii).

Theorem 3.3. A Hyperbolic Hsu-structure manifold V_n with a quarter symmetric non-metric connection $\widetilde{\nabla}$, satisfies the following relation.

$$(\widetilde{\nabla}_{\phi X}\phi)\phi Y = a^r(\widetilde{\nabla}_X\phi)Y \text{ iff } (\nabla_{\phi X}\phi)\phi Y = a^r(\nabla_X\phi)Y. \quad (3.10)$$

Proof: Applying ϕ on Y in equation (2.1), we get

$$\widetilde{\nabla}_X\phi Y = \nabla_X\phi Y + \eta(\phi Y)\phi X. \quad (3.11)$$

The above equation can also be modified as

$$(\widetilde{\nabla}_X\phi)Y = \nabla_X\phi Y + \eta(\phi Y)\phi X - \phi(\widetilde{\nabla}_X Y). \quad (3.12)$$

Applying ϕ on both side of equation (2.1), we get

$$\phi(\widetilde{\nabla}_X Y) = \phi(\nabla_X Y) - a^r\eta(Y)X. \quad (3.13)$$

From the equations (3.12) and (3.13), we get

$$(\widetilde{\nabla}_X\phi)Y = (\nabla_X\phi)Y + \eta(\phi Y)\phi X + a^r\eta(Y)X. \quad (3.14)$$

Applying ϕ on X and Y, we get

$$(\widetilde{\nabla}_{\phi X}\phi)\phi Y = (\nabla_{\phi X}\phi)\phi Y + a^r\eta(\phi Y)\phi X + a^{2r}\eta(Y)X. \quad (3.15)$$

Subtraction of the equation (3.14) and (3.15), we get

$$(\widetilde{\nabla}_{\phi X}\phi)\phi Y - (\widetilde{\nabla}_X\phi)Y = (\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y + (a^r - 1)(\eta(\phi Y)\phi X + a^r\eta(Y)X). \quad (3.16)$$

Now from (3.14) and (3.16), we get

$$(\widetilde{\nabla}_{\phi X}\phi)\phi Y + a^r(\nabla_X\phi)Y = (\nabla_{\phi X}\phi)\phi Y + a^r(\widetilde{\nabla}_X\phi)Y. \quad (3.17)$$

Hence the equation (3.17) proves the statement.

Theorem 3.4. If a Hyperbolic Hsu-structure manifold V_n admits a quarter symmetric non-metric connection $\widetilde{\nabla}$, then the Nijenhuis tensor of the Riemannian connection ∇ and $\widetilde{\nabla}$ coincide.

Proof : From equation (3.14), we get

$$(\widetilde{\nabla}_X\phi)Y = (\nabla_X\phi)Y + \eta(\phi Y)\phi X + a^r\eta(Y)X. \quad (3.18)$$

Applying ϕ on X in (3.18), we get

$$(\widetilde{\nabla}_{\phi X}\phi)Y = (\nabla_{\phi X}\phi)Y - a^r\eta(\phi Y)X + a^r\eta(Y)\phi X. \quad (3.19)$$

Interchanging X and Y in (3.19), we get

$$(\tilde{\nabla}_{\phi Y}\phi)X = (\nabla_{\phi Y}\phi)X - a^r\eta(\phi X)Y + a^r\eta(X)\phi Y. \quad (3.20)$$

Applying ϕ on both side of equation (3.18), we get

$$\phi(\tilde{\nabla}_X\phi)Y = \phi(\nabla_X\phi)Y + a^r\eta(\phi Y)X - a^r\eta(Y)\phi X. \quad (3.21)$$

Again interchanging X and Y in the above equation, we get

$$\phi(\tilde{\nabla}_Y\phi)X = \phi(\nabla_Y\phi)X + a^r\eta(\phi X)Y - a^r\eta(X)\phi Y. \quad (3.22)$$

The Nijenhuis tensor with respect to ϕ is a vector valued bilinear function, defined as

$$N(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] - a^r[X,Y]$$

The Nijenhuis tensor with respect to the Riemannian connection ∇ is given as

$$N(X,Y) = (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X - \phi(\nabla_X\phi)Y + \phi(\nabla_Y\phi)X \quad . \quad (3.23)$$

By using equations (3.19), (3.20), (3.21), (3.22) and (3.23), we get

$$N(X,Y) = (\tilde{\nabla}_{\phi X}\phi)Y - (\tilde{\nabla}_{\phi Y}\phi)X - \phi(\tilde{\nabla}_X\phi)Y + \phi(\tilde{\nabla}_Y\phi)X$$

$$N(X,Y) = N^*(X,Y)$$

Where $N^*(X,Y)$ denotes the Nijenhuis tensor with respect to the quarter symmetric non- metric connection $\tilde{\nabla}$. Hence the theorem.

4. Hyperbolic Hsu-structure K$\ddot{a}$hler manifold with Quarter Symmetric non-metric connection:

As discussed earlier that if the Hyperbolic Hsu-structure manifold V_n satisfies the condition (1.7), then V_n is called a hyperbolic Hsu-structure K$\ddot{a}$hler manifold. In this section we have some following theorem.

Theorem 4.1. If V_n be a Hyperbolic Hsu-structure K$\ddot{a}$hler manifold admitting a quarter symmetric non-metric connection, then in V_n

(i) $(\tilde{\nabla}_{\phi X}\phi)\phi Y = a^r(\tilde{\nabla}_X\phi)Y$,

(ii) $(\tilde{\nabla}_X\phi)Y = 0$ iff $\eta(\phi Y)\phi X = -a^r\eta(Y)X$. (4.1)

Proof: In a Hyperbolic Hsu-structure K$\ddot{a}$hler manifold V_n, the equation (3.11) can be written as

$$(\tilde{\nabla}_X\phi)Y = \phi(\nabla_X Y) + \eta(\phi Y)\phi X - \phi(\tilde{\nabla}_X Y). \quad (4.2)$$

From the equation (4.2) and (3.13), we get

$$(\tilde{\nabla}_X\phi)Y = \eta(\phi Y)\phi X + a^r\eta(Y)X. \quad (4.3)$$

Applying ϕ on X and Y in (4.3) and using (1.1), we get

$$(\tilde{\nabla}_{\phi X}\phi)\phi Y = a^r[\eta(\phi Y)\phi X + a^r\eta(Y)X]. \quad (4.4)$$

We will get (4.1)(i) and (ii) by using the equation (4.3) and (4.4).

Theorem 4.2. If the Nijenhuis tensor with respect to the quarter symmetric non-metric connection $\tilde{\nabla}$ in a Hyperbolic Hsu-structure K$\ddot{a}$hler manifold V_n vanishes i.e the manifold V_n is integrable.

Proof : The Nijenhuis tensor with respect to the quarter symmetric non-metric connection $\tilde{\nabla}$ is defined as

$$N^*(X,Y) = (\tilde{\nabla}_{\phi X}\phi)Y - (\tilde{\nabla}_{\phi Y}\phi)X - \phi(\tilde{\nabla}_X\phi)Y + \phi(\tilde{\nabla}_Y\phi)X. \quad (4.5)$$

Applying ϕ on X in (4.3), we get

$$(\widetilde{\nabla}_{\phi X}\phi)Y = a^r[\eta(Y)\phi X - \eta(\phi Y)X] \quad . \tag{4.6}$$

Interchanging X and Y in (4.6), we obtain

$$(\widetilde{\nabla}_{\phi Y}\phi)X = a^r[\eta(X)\phi Y - \eta(\phi X)Y] \quad . \tag{4.7}$$

Applying ϕ on both side of the equation (4.3)

$$\phi(\widetilde{\nabla}_X\phi)Y = a^r[\eta(Y)\phi X - \eta(\phi Y)X] \quad . \tag{4.8}$$

Interchanging X and Y in (4.8), we obtain

$$\phi(\widetilde{\nabla}_Y\phi)X = a^r[\eta(X)\phi Y - \eta(\phi X)Y]. \tag{4.9}$$

By using equations (4.6), (4.7), (4.8) and (4.9) in the equation (4.5), we get

$N^*(X,Y) = 0$.

hence the theorem.

5. Contravariant almost analytic vector field :

A vector field V is said to be contravariant almost analytic if the Lie-derivative of the tensor field with respect to V vanishes identically

$$(L_V\phi)X = 0. \qquad \text{for all X} \tag{5.1}$$

The equation (5.1) can also be written as

$$[V,\phi X] = \phi[V,X]. \tag{5.2}$$

In Hyperbolic Hsu-structure manifold, the equation (5.2) becomes

$$(\nabla_V\phi)X - \nabla_{\phi X}V + \phi(\nabla_X V) = 0. \tag{5.3}$$

In a Hyperbolic Hsu-structure K$\ddot{a}$hler manifold, the equation (5.2) becomes

$$\nabla_{\phi X}V - \phi(\nabla_X V) = 0. \tag{5.4}$$

Applying ϕ on both side of equation (5.4) and using equation (1.1),we get

$$\phi(\nabla_{\phi X}V) = -a^r(\nabla_X V).$$

Theorem 5.1. On a Hyperbolic Hsu-structure manifold, a contravariant almost analytic vector field V with respect to the Riemannian connection ∇ is also contravariant almost analytic with respect to the quarter symmetric non- metric connection $\widetilde{\nabla}$ if

$$\eta(\phi X)\phi V = -a^r\eta(X)V.$$

Proof : For any vector field V, the equation (2.1) can be written as

$$\widetilde{\nabla}_X V = \nabla_X V + \eta(V)\phi X. \tag{5.5}$$

Operating ϕ on X in (5.5), we get

$$\widetilde{\nabla}_{\phi X}V = \nabla_{\phi X}V - a^r\eta(V)X. \tag{5.6}$$

Applying ϕ on both side of equation (5.5), we get

$\phi(\widetilde{\nabla}_X V) = \phi(\nabla_X V) - a^r \eta(V)X.$ (5.7)

Interchanging X and V in (5.5), we get

$\widetilde{\nabla}_V X = \nabla_V X + \eta(X)\phi V.$ (5.8)

From (5.8) we can write

(i) $\widetilde{\nabla}_V \phi X = \nabla_V \phi X + \eta(\phi X)\phi V,$ (5.9)
(ii) $\phi(\widetilde{\nabla}_V X) = \phi(\nabla_V X) - a^r \eta(X)V.$

As we know that

$(\widetilde{\nabla}_V \phi)X = \widetilde{\nabla}_V \phi X - \phi(\widetilde{\nabla}_V X).$

From (5.9), we will get

$$(\widetilde{\nabla}_V \phi)X = \nabla_V \phi X + \eta(\phi X)\phi V - \phi(\nabla_V X) + a^r \eta(X)V. \quad (5.10)$$

From (5.7) , (5.8) and (5.10), we get

$$(\widetilde{\nabla}_V \phi)X - (\widetilde{\nabla}_{\phi X} V) + \phi(\widetilde{\nabla}_X V) = (\nabla_V \phi)X - \nabla_{\phi X} V + \eta(\phi X)\phi V + \phi(\nabla_X V) + a^r \eta(X)V. \quad (5.11)$$

Since V is contravariant almost analytic vector field with respect to the Riemannian connection ∇, so using equation (5.3) in the equation (5.11), we get

$$(\widetilde{\nabla}_V \phi)X - (\widetilde{\nabla}_{\phi X} V) + \phi(\widetilde{\nabla}_X V) = \eta(\phi X)\phi V + a^r \eta(X)V. \quad (5.12)$$

If the vector field V is also contravariant almost analytic with respect to the quarter symmetric non-metric connection $\widetilde{\nabla}$, then it will satisfy

$(\widetilde{\nabla}_V \phi)X - (\widetilde{\nabla}_{\phi X} V) + \phi(\widetilde{\nabla}_X V) = 0.$

Hence the theorem.

Theorem 5.2. On a Hyperbolic Hsu-structure K$\ddot{a}$hler manifold, a contravariant almost analytic vector field V with respect to the Riemannian connection ∇ is also contravariant almost analytic with respect to the quarter symmetric non-metric connection $\widetilde{\nabla}$.

Proof : From (5.6) and (5.7), we get

$\widetilde{\nabla}_{\phi X} V - \phi(\widetilde{\nabla}_X V) = \nabla_{\phi X} V - \phi(\nabla_X V).$ (5.13)

Since V is contravariant almost analytic vector field with respect to the Riemannian connection in the Hyperbolic Hsu-structure K$\ddot{a}$hler manifold, so by equations (5.4) and (5.13), we get

$\widetilde{\nabla}_{\phi X} V - \phi(\widetilde{\nabla}_X V) = 0.$

Which implies that V is also contravariant almost analytic vector field with respect to the quarter symmetric non-metric connection ∇.

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