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SEMI MAGIC CONSTANTS AS A FIELD

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Abstract

Some advanced mathematical properties of semi magic constants are discussed in this paper.
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1. *Introduction* : A magic square of order 'n' is an n^{th} order matrix such that the sum of elements in every row/column/diagonal remains the same. The common sum is known as 'magic constant' or 'magic number'. If the above condition is valid only for the sum of elements of rows and columns and not for the diagonal elements, then that array is known as a semi magic square. All magic squares are semi magic squares. In real life situations, some problems relating to division of objects equal in numbers and value can be easily solved by constructing a semi magic square in accordance with the given conditions.

Apart from the recreational aspects of semi magic squares, it is found that they posses several advanced mathematical properties. Here more concentration is given to some properties of magic constants of semi magic squares.

2. *Notations and Mathematical Preliminaries*

2.1 *Magic Square* : A magic square of order 'n' is an n^{th} order matrix $[a_{ij}]$ such that

$$\sum_{j=1}^{n} a_{ij} = k, \text{ for } i = 1,2,3,\dots\dots\dots,n \longrightarrow (1)$$

$$\sum_{j=1}^{n} a_{ji} = k, \text{ for } i = 1,2,3,\dots\dots\dots n \longrightarrow (2)$$

$$\sum_{i=1}^{n} a_{ii} = k \text{ and } \sum_{i=1}^{n} a_{i,n-i+1} = k, \longrightarrow (3)$$

Where $'k'$ is a constant and the above mentioned a_{ij}'s and a_{ji}'s are the row and column elements and a_{ii}'s & $a_{i,n-i+1}$'s are the left and right diagonal elements of the magic square respectively.

2.2 *Magic Constant* : The constant $'k'$ in the above definition is known as the magic constant or magic number. Magic constant of the magic square A is denoted as $\rho(A)$.

For example, the below given magic squares A and A′ are of order 3 and

$\boldsymbol{\rho(A) = 21}$ & $\boldsymbol{\rho(A') = 15}$

$A =$

4	9	8
11	7	3
6	5	10

$A' =$

4	3	8
9	5	1
2	7	6

$\boldsymbol{2.3}$ ***Semi magic square***: In definition 2.1, if only conditions (1) and (2) are satisfied, then that square array is known as a semi magic square.
$\boldsymbol{2.4}$ ***Semi magic constant*** **:** The magic constant of a semi magic square is known as semi magic constant. If A is a semi magic square, then the semi magic constant of A is also denoted as $\rho(A)$.
For example, the below given array B is a semi magic square of order $\mathbf{3}$ ***with*** $\boldsymbol{\rho(B) = 30}$

$\boldsymbol{B} =$

8	18	4
16	2	12
6	10	14

Here sum of elements of each row/column = 30 .
$\boldsymbol{2.5}$ ***Group*** **:** A nonempty set G together with an operation * is known as a group if the following properties are satisfied.
(i) G is closed with respect to *. i.e., $a * b \epsilon G, \forall\ a, b \in G$.
(ii) * is associative in G. i.e., $a * (b * c) = (a * b) * c\ \ \forall\, a, b, c \in G$.
(iii) $\exists\ e \epsilon G$, such that $e * a = a * e = a$, $\forall\ a \in G$. Here e is called the 'identity element' in G with respect to *.
(iv) $\forall\ a \in G, \exists\ b \in G$ such that $a * b = b * a = e$, where $'e'$ is the identity element. Here b is called the 'inverse of a' and similarly vice versa. The inverse of the element a is denoted as a^{-1}.
Note **:** If G is a group with respect to *, it is denoted as $\langle G,* \rangle$
$\boldsymbol{2.6}$ ***Ring*** **:** A non – empty set R together with two binary operations called 'addition' and 'multiplication' denoted by '+' and '**.**' respectively is called a Ring, if the following postulates are satisfied.
(i) $\langle R, + \rangle$ is an abelian group.(If $\langle G,* \rangle$ is an abelian group, then $a * b = b * a, \forall a, b \in G$).
(ii) Multiplication is associative, i.e., $a.(b.c) = (a.b).c\ \ \ \forall a, b, c \in R$.
(iii) Multiplication is distributive with respect to addition, i.e., $\forall a, b, c \in R$,
$a.(b + c) = a.b + a.c$ (Left distributive law) and $(b + c).a = b.a + c.a$ (Right distributive law).
$\boldsymbol{2.7}$ ***Ring with unity*** **:** If in a ring $R, \exists$ an element denoted by 1, such that $1.a = a.1 = a$, $\forall a \in R$, then R is called a ring with unit element. Here $'1'$ is called the unit element of the ring and is obviously the multiplicative identity. If a ring possesses multiplicative identity, then it is called a ring with unity.
$\boldsymbol{2.8}$ ***Commutative ring*** **:** If in a ring R, the multiplication is also commutative, i.e., if $a.b = b.a, \forall a, b \in R$, then R is called a commutative ring.
$\boldsymbol{2.9}$ ***Field*** **:** A ring R with at least two elements is called a field if it,
(i) is commutative (ii) has unity (iii) is such that each non zero element possesses multiplicative inverse .
$\boldsymbol{2.10}$ **:** We use
$(i)\Re$ to denote the set of all real numbers.
(ii) $'\boldsymbol{S_a}'$ to denote the set of all n^{th} order semi magic squares of the form $[a_{ij}]$, where $a_{ij} = \begin{cases} a, \text{ for } i = j, & i, j = 1,2, \dots \dots, n \\ \quad 0, \ otherwise \end{cases}$ $a \in \Re$
(Here we are not excluding the case $a = 0$. $\rho([a_{ij}]) = a$).
(iii) $'\boldsymbol{\rho_{(S_a)}}'$ to denote the set of all semi magic constants of the elements in $\boldsymbol{S_a}$. i.e., if $A \in \boldsymbol{S_a}$, then $\rho(A) \in \boldsymbol{\rho_{(S_a)}}$. $(iv) I_n$ to denote the identity matrix of order n .

$\boldsymbol{3.}$ ***Propositions and Theorems***

Theorem* 3.1 :** $\boldsymbol{\rho}_{(S_a)}$ ***forms an abelian group with respect to addition of real numbers.

Proof : We know that $\Re$ is an abelian group under addition and $\boldsymbol{\rho}_{(S_a)} \subset \Re$. Then, it is enough to prove that $\boldsymbol{\rho}_{(S_a)}$ is a subgroup of $\Re$ under addition. For that, we need only to show that if $a, b \in \boldsymbol{\rho}_{(S_a)}$, then $a - b \in \boldsymbol{\rho}_{(S_a)}$.

Let $a, b \in \boldsymbol{\rho}_{(S_a)}$. i.e., $\exists$ $A, B \in \boldsymbol{S_a}$, such that $\rho(A) = a$ & $\rho(B) = b$. We can show that $A - B \in \boldsymbol{S_a}$, with $\rho(A - B) = a - b$. Hence $a - b \in \boldsymbol{\rho}_{(S_a)}$ and this completes the proof.

Proposition* 3.1: *If $\boldsymbol{a, b} \in \boldsymbol{\rho}_{(S_a)}$, ***then*** $\boldsymbol{a.b} \in \boldsymbol{\rho}_{(S_a)}$ ***and*** $\boldsymbol{a.b = b.a}$; ***where*** $'.'$ ***denotes multiplication of real numbers.***

Proof : Given that $a, b \in \boldsymbol{\rho}_{(S_a)}$. We have to show that their product also belongs to $\boldsymbol{\rho}_{(S_a)}$. Since $a, b \in \boldsymbol{\rho}_{(S_a)}$, $\exists$ $A, B \in \boldsymbol{S_a}$, such that $\rho(A) = a$ &
$\rho(B) = b$. We can show that their product under matrix multiplication $A.B \in \boldsymbol{S_a}$, with $\rho(A.B) = a.b$. Hence $a.b \in \boldsymbol{\rho}_{(S_a)}$. Multiplication of real numbers is commutative and this completes the proof.

Theorem* 3.2 :** $\langle \boldsymbol{\rho}_{(S_a)}, +, . \rangle$ ***forms a commutative ring with unity, where $'+'$ ***and*** $'.'$ ***denote the addition and multiplication of real numbers respectively.***

Proof : Multiplication of real numbers is associative, commutative and distributive over addition. $I_n \in \boldsymbol{S_a}$ with $\rho(I_n) = 1$. Hence $1 \in \boldsymbol{\rho}_{(S_a)}$ and will act as the unity element. The rest of the proof will follow from Theorem 3.1 and Proposition 3.1.

Proposition* 3.2 : *If $\boldsymbol{a} \neq 0 \in \boldsymbol{\rho}_{(S_a)}$, ***then*** $\boldsymbol{1/a = a^{-1}} \in \boldsymbol{\rho}_{(S_a)}$.

Proof : $a \neq 0 \in \boldsymbol{\rho}_{(S_a)}$. Then $\exists$ $A = [a_{ij}] \in \boldsymbol{S_a}$; where

$$a_{ij} = \begin{cases} a, \text{ for } i = j, \quad i, j = 1, 2, \dots\dots, n \\ \quad 0, \ otherwise \end{cases}.$$

The ij^{th} element of A^{-1}, the matrix inverse of $A = \begin{cases} 1/a, \text{ for } i = j, \quad i, j = 1, 2, \dots\dots, n \\ \quad 0, \ otherwise \end{cases}$.

Clearly, $A^{-1} \in \boldsymbol{S_a}$. Then $\rho(A^{-1}) = 1/a$ and hence $1/a \in \boldsymbol{\rho}_{(S_a)}$.

Theorem* 3.3 :** $\langle \boldsymbol{\rho}_{(S_a)}, +, . \rangle$ ***forms a field, where $'+'$ ***and*** $'.'$ ***denote the addition and multiplication of real numbers respectively.***

Proof : It immediately follows from Theorem 3.2 and Proposition 3.2. Proposition 3.2 shows that $\forall a \in \boldsymbol{\rho}_{(S_a)}$, $\exists$ an inverse in $\boldsymbol{\rho}_{(S_a)}$ under multiplication of real numbers.

4. *Conclusion*

We have seen that the set $\boldsymbol{\rho}_{(S_a)}$ of all semi magic constants of the n^{th} order semi magic squares of the form $A = [a_{ij}]$, where $= \begin{cases} a, \text{ for } i = j, \quad i, j = 1, 2, \dots\dots, n \\ \quad 0, \ otherwise \end{cases}$, $a \in \Re$ forms a field under addition and multiplication of real numbers.

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