Int Jr. of Mathematical Sciences & Applications
Vol. 5, No. 2, (July-December, 2015)

ISSN No: 2230-9888
www.journalshub.com

FUZZY SUBSEQUENCES AND LIMIT POINTS

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Abstract

In this paper We introduce the new concepts fuzzy subsequences and limit point of fuzzy sequences in a metric space.

Keywords : Fuzzy subsequence, Limit point of fuzzy sequence

I. Introduction

In a metric space sequence is an important tool to study many properties. The closure of a set A can be characterized using convergent sequences in A. The continuity of a function from one metric space to another can be characterized using convergent sequences. In the year 1965 Lotfi A.Zadeh [2] introduced the concept of fuzzy sets and fuzzy logic.In the year 1968 C.L.Chang [1] introduced Fuzzy topological spaces.

In the year 2014 We [3] introduced fuzzy sequences in a metric space and studied the properties of convergent fuzzy sequences.

Let X be a non empty set. A fuzzy set A on N X X is called a fuzzy sequence in X. i.e., A: N X X —> [0,1] is called a fuzzy sequence in X.

Let (M,d) be a metric space and let A be a fuzzy sequence on M. Let $\alpha \in (0,1]$. Let $a \in M$. A is said to converge to a at level α if

1. For each $n \in N$, there exists atleast one x in M where $A(n,x) \geq \alpha$

2.Given $\mathcal{E} > 0$, there exists $n_0 \in N$ such that $d(x,a) < \mathcal{E}$ for all $n \geq n_0$ and for all x with $A(n,x) \geq \alpha$

ie., given $\mathcal{E} > 0$, there exists $n_0 \in N$ such that $n \geq n_0$ and $A(n,x) \geq \alpha$ implies $d(x,a) < \mathcal{E}$.

We write A —> a.

In the year 2014 We [4] introduced fuzzy nets in a topological space and studied the properties of convergence.

In the same year 2014 We [5] introduced the concept of fuzzification of filters in topological space and studied the properties.

II. Fuzzy subsequences

Definition: 2.1

Let A be a fuzzy sequence in a non empty set X. Let $(n_k)_{k \epsilon N}$ be a strictly increasing sequence of natural numbers. Define a fuzzy sequence B in X as B(n,x) = A(n,x) if n = n_k for some k ϵ N and = 0 otherwise. B is called a fuzzy sub sequence of A.

B(n,x) = A(n,x) if n = n_k for some k ϵ N

= 0 otherwise

Definition:2.2

Let A be a fuzzy sequence in a non empty set X. Take $(2k)_{k \epsilon N}$. Define B in X as B(n,x) = A(n,x) if n = 2k for some k and B(n,x) = 0 otherwise. B is called even fuzzy sub sequence of A.

Definition:2.3

Let A be a fuzzy sequence in a non empty set X. Take $(2k-1)_{k \epsilon N}$. Define B in X as B(n,x) = A(n,k) if n = 2k-1 for some k ϵ N and B(n,x) = 0 otherwise. B is called odd fuzzy sub sequence of A.

Example:2.4

Let X = N. Consider fuzzy sequence A defined as A(n,x) = 1/n+x. Take (n_k) = (5k) consider B defined as B(n,x) = 1/n+x if n = 5k for some k

= 0 otherwise

B is a fuzzy subsequence of A.

Definition:2.5

Let A be a fuzzy sequence in a metric space M. Let B be a fuzzy subsequence given by (n_k). The subsequence B is said to converge to a at level α if given Ɛ > 0, there exists k_0 ϵ N, such that k ≥ k_0 and B(n_k , x) ≥ α implies d(x,a) < Ɛ.

Example:2.6

Let X = R. Consider the fuzzy sequence A given by A(n,x) = 1 if x = 1/n

= ½ otherwise

Let (n_k) = (2k) and α = 3/4 . Consider the subsequence B given by (2k).

Claim : B converges to 0. Let Ɛ > 0 be given. Choose k_0 ϵ N such that k_0 > 1/2Ɛ. Now let k ≥ k_0 and B(n_k,x) ≥ α. ie., A(n_k,x) ≥ α. Therefore A(n_k,x) = 1. Hence x = 1/n_k. Therefore x = 1/2k. now k ≥ k_0 implies k > 1/2Ɛ which implies 1/k > 2Ɛ . This gives 1/2k < Ɛ . Hence x < Ɛ. This gives ǀ x-0 ǀ < Ɛ. Therefore given Ɛ > 0, there exists k_0 ϵ N, such that k ≥ k_0 and B(n_k,x) ≥ α implies d(x,0) < Ɛ. Hence B converges to 0.

Result:2.7

In the above example B converges at level α= ¾. At the same time A also converges at level α = ¾. We see another example to show that this need not be true always.

Example:2.8

consider R with usual metric.

Consider the fuzzy sequence A defined as $A(n,x) = 1$ if $n=2k-1$ and $x=n^2$

$= \frac{1}{2}$ if $n = 2k$ and $x = 1/n^2$

$= 0$ otherwise

Consider the subsequence B given by (2k) $B(n,x) = \frac{1}{2}$ if $n = 2k$ and $x = 1/n^2$

$= 0$ otherwise

Claim : B converges to 0 at level ½.

Let $\varepsilon > 0$ be given. Choose $k_0 \in N$ such that $k_0 > 1/2(\varepsilon)^{1/2}$. Now let $k \geq k_0$ and $B(2k,x) \geq \frac{1}{2}$. $B(2k,x) \geq \frac{1}{2}$ gives $x = 1/4k^2$. Now $k \geq k_0$ implies $k > 1/2(\varepsilon)^{1/2}$. This implies $1/4k^2 < \varepsilon$. Hence $x < \varepsilon$. Therefore $d(x,0) < \varepsilon$. Hence given $\varepsilon > 0$, there exists $k_0 \in N$ such that $k \geq k_0$ and $B(n_k,x) \geq \frac{1}{2}$ implies $d(x,0) < \varepsilon$.Therefore B converges to 0 at level ½. But A does not converge to 0 at level ½.

Theorem:2.9

Let A be a fuzzy sequence in a metric space M. Let B be a fuzzy subsequence of A. If A converges to a at level α then B converges to a at level α.

Proof :

Fuzzy sequence A converges to a at level α. Let B be the fuzzy subsequence of A given by (n_k).

Now we prove that B converges to a at level α. Let $\varepsilon > 0$ be given. Since A converges to a at

level α, there exists $n_0 \in N$, such that $n \geq n_0$ and $A(n,x) \geq \alpha$ implies $d(x,a) < \varepsilon$. Now choose $k_0 \in N$ such that $n_{k0} \geq n_0$. Now let $k \geq k_0$ and $B(n_k,x) \geq \alpha$. Now $k \geq k_0$ implies $n_k \geq n_{k0}$ and hence $n_k \geq n_0$. $B(n_k,x) \geq \alpha$ implies $A(n_k,x) \geq \alpha$. Now $n_k \geq n_0$ and $A(n_k,x) \geq \alpha$ implies $d(x,a) < \varepsilon$. Hence given $\varepsilon > 0$, there exists $k_0 \in N$ such that $k \geq k_0$ and $B(n_k,x) \geq \alpha$ implies $d(x,a) < \varepsilon$. Hence B converges to a at level α.

Result:2.10

Converse is not true. This is established by the following example.

Example :2.11

Consider R with usual metric. Consider the fuzzy sequence A given by

$A(n,x) = 1$ if $n = 2k$ and $x = 1/n$

$= \frac{1}{2}$ if $n = 2k-1$, $x = n^2$

$= 0$ otherwise

Consider the subsequence B given by (2k). Now $B(n,x) = 1$ if $n = 2k$, $x = 1/n$

$= 0$ otherwise

Take $\alpha = \frac{1}{2}$. Clearly B converges to 0 at level α. But A does not converge to 0 at level α .

Theorem:2.12

Let A be a fuzzy sequence in a metric space M. Let B and C be the even and odd subsequences of A. If B and C converge to the same limit a at level α, then A also converges to a at level α.

Proof :

A is a fuzzy sequence in metric space M. B and C are even and odd subsequences. Hence B(n, x) =A(n, x) if n is even and B(n, x) = 0 if n is odd. Also C(n,x) = A(n,x) if n is odd and C(n,x) = 0 if n is even. B and C converge to the same limit a at level α. Let $\mathcal{E} > 0$ be given. Since B converges to a at level α, there exists $n_1 \in N$ such that $n \geq n_1$ and n is even and B(n, x) $\geq \alpha$ implies d(x, a) $< \mathcal{E}$. Since C converges to a at level α, there exists $n_2 \in N$ such that $n \geq n_2$ and n is odd and C(n,x) $\geq \alpha$ implies d(x,a) $< \mathcal{E}$. Let $n_0 = \max \{n_1, n_2\}$. Take any $n \geq n_0$. Let A(n, x) $\geq \alpha$. If n is even then A(n,x) = B(n,x). Hence if n is even then $n \geq n_1$ and B(n,x) $\geq \alpha$. This implies d(x,a) $< \mathcal{E}$

If n is odd then A(n, x) = C(n, x). Hence if n is odd then $n \geq n_2$ and C(n,x) $\geq \alpha$. This implies d(x,a) $< \mathcal{E}$. Hence given $\mathcal{E} > 0$, there exists $n_0 \in N$, such that $n \geq n_0$ and A(n,x) $\geq \alpha$ implies d(x,a) $< \mathcal{E}$. Therefore A converges to a at level α .

Theorem:2.13

A fuzzy sequence in a metric space converges at level α if and only if both the odd subsequence and the even subsequence converge to the same limit at level α.

Proof : Follows from the previous two theorems.

3.Limit point

Definition:3.1

Let A be a fuzzy sequence in a metric space M. An element $a \in M$ is called a limit point of A at level α if given $\mathcal{E} > 0$ and $n_0 \in N$ there exists $n \in N$ such that $n \geq n_0$ and

1. A(n,x) $\geq \alpha$ for atleast one x in M
2. d(x,a) $< \mathcal{E}$ for all x with A(n,x) $\geq \alpha$.

Example:3.2

Consider R with usual metric. Consider the fuzzy sequence A defined as

A(n,x) = 1 if n = 2k, x = 1/n

1 if n = 2k-1, x=n/n+1

0 otherwise.

Take $\alpha \in (0,1]$. We claim that 0 is a limit point of A at level α. Let $\mathcal{E} > 0$ be given and let $n_0 \in N$ be given.Choose $k \in N$ such that $k>1/2\mathcal{E}$ and $k>n_0$. Take n = 2k. Now A(n,1/n) = 1. Therefore A(n,1/n) $\geq \alpha$. Hence A(n,x) $\geq \alpha$ for some x in R. Now A(n,x) $\geq \alpha$ implies A(n,x) = 1. This implies x =1/n.Now $k >1/2\mathcal{E}$ implies $2k > 1/\mathcal{E}$ which implies $1/2k < \mathcal{E}$ i.e., $1/n < \mathcal{E}$.Hence $x < \mathcal{E}$. Therefore $|x-0| < \mathcal{E}$. $k>n_0$ implies $2k>n_0$ i.e., $n > n_0$.Therefore given $\mathcal{E} > 0$ and $n_0 \in N$, there exists $n \in N$ such that $n \geq n_0$, A(n,x) $\geq \alpha$ for atleast one x in R and A(n,x) $\geq \alpha$ implies d(x,0) $< \mathcal{E}$ therefore 0 is a limit point of A at level α.

Theorem:3.3

Let A be a fuzzy sequence in a metric space M. $a \in M$ is a limit point of A iff for every open ball $B(a,\mathcal{E})$ there exists (n_k) a strictly increasing sequence of natural numbers such that for each n_k. $A(n_k,x) \geq \alpha$ for atleast one x in M and $A(n_k,x) \geq \alpha$ implies $x \in B(a,\mathcal{E})$.

Proof :

A is a fuzzy sequence in M. Let $\alpha \in (0,1]$. Let $a \in M$ be a limit point of A. Take r > 0 and $1 \in N$.

By definition of limit point of A, there exists $n_1 \in N$ such that $n_1 \geq 1$ and A(n,x) $\geq \alpha$ for atleast one x in M and $A(n,x) \geq \alpha$ implies $d(x,a) < r$. ie., $x \in B(a,\mathcal{E})$. Consider $r > 0$ and $n_1 \in N$. By definition of limit point, there exists $n_2 \in N$ such $n_2 > n_1$, $A(n_2,x) \geq \alpha$ for atleast one x in M and $A(n_2,x) \geq \alpha$ implies $d(x,a) < r$. i.e., $x \in B(a,\mathcal{E})$. Now considering $r > 0$ and $n_2 \in N$ by definition of limit point we get n_3. Thus we get (n_k) a strictly increasing sequence of natural numbers such that for each n_k, $A(n_k,x) \geq \alpha$ for atleast one x in M and $A(n_k,x) \geq \alpha$ implies $x \in B(a,\mathcal{E})$.

Conversely, suppose the given condition is satisfied we claim that a is a limit point of A at level α. Let $\mathcal{E} > 0$ be given and $n_0 \in N$ is given. Choose $n_k \geq n_0$. Let $n = n_k$. Now $n \geq n_0$. $A(n,k)$ implies $A(n_k,x) \geq \alpha$ and hence this is true for atleast one x in M. $A(n,x) \geq \alpha$ implies $A(n_k,x) \geq \alpha$. This implies $x \in B(a,\mathcal{E})$. Hence $d(x,a) < \mathcal{E}$. Hence given $\mathcal{E} > 0$ and $n_0 \in N$, there exists $n \in N$ such that $n \geq n_0$ and 1. $A(n,x) \geq \alpha$ for atleast one x in M and 2. $d(x,a) < \mathcal{E}$ for all x with $A(n,x) \geq \alpha$. Therefore a is a limit point of A at level α.

Theorem:3.4

Let A be a fuzzy sequence in a metric space M. $a \in M$ is a limit point of M iff there exists a subsequence converging to a.

Proof :

A is a fuzzy sequence in metric space M. $a \in M$ is a limit point of A. Take $\mathcal{E} = 1 > 0$ and consider $1 \in N$. By definition of limit point, there exists $n_1 \in N$ such that $n_1 \geq 1$ and $A(n_1,x) \geq \alpha$ for atleast one x in M and $A(n_1,x) \geq \alpha$ implies $d(x,a) < 1$.aNow consider $\mathcal{E} = ½ > 0$ and $n_1 \in N$. By definition of limit point there exists $n_2 \in N$ such that $n_2 > n_1$ and 1. $A(n_2,x) \geq \alpha$ for atleast one x in M. 2. $A(n_2,x) \geq \alpha$ implies $d(x,a) < ½$. Proceeding we get a sequence (nk). Now (n_k) is a strictly increasing sequence of natural numbers with

1. For each n_k, $A(n_k,x) \geq \alpha$ for atleast one x in M and
2. $A(n_k,x) \geq \alpha$ implies $d(x,a) < 1/k$. Now consider the subsequence B given by (n_k). B is defined as $B(n,x) = A(n,x)$ if $n = n_k$ for some k and 0 otherwise.

Claim : B converges to a. Let $\mathcal{E} > 0$ be given. Choose $k_0 \in N$ such that $k_0 > 1/\mathcal{E}$. Take any $k \geq k_0$. $B(n_k,x) = A(n_k,x)$. Therefore $B(n_k,x) \geq \alpha$ for atleast one x in M. $B(n_k,x) \geq \alpha$ implies $A(n_k,x) \geq \alpha$. This implies $d(x,a) < 1/k \leq 1/k_0 < \mathcal{E}$. Hence given $\mathcal{E} > 0$ there exists $k_0 \in N$ such that for each $k \geq k_0$. 1. $B(n_k,x) \geq \alpha$ for atleast one x in M and 2. $B(n_k,x) \geq \alpha$ implies $d(x,a) < \mathcal{E}$. Hence B converges to a. Therefore we get a subsequence B converging to a.

Conversely, suppose A has a subsequence B converging to a at level α. We claim that a is a limit

point of A at level α. Let $\mathcal{E} > 0$ be given and $n_0 \in N$. Since B converges to a, given $\mathcal{E} > 0$, there

exists $k_0 \in N$ such that for any $k \geq k_0$ we have 1. $B(n_k,x) \geq \alpha$ for atleast one x in M and 2. $B(nk,x) \geq \alpha$ implies $d(x,a) < \mathcal{E}$. Choose k such that $n_k \geq n_0$. Put $n = n_k$. Then $A(n,x) = B(n_k,x)$. Hence we have

1. $A(n,x) \geq \alpha$ implies $B(n_k,x) \geq \alpha$ and hence this is true for atleast one x in M.
2. $A(n,x) \geq \alpha$ implies $B(n_k,x) \geq \alpha$. Hence $d(x,a) < \mathcal{E}$.
   Therefore a is a limit point of A.

Theorem:3.5

Let A be a fuzzy sequence in a metric space M. If A converges to a at level α then a is a limit point of A at level α. i.e., Every limit is a limit point.

Proof :

A is a fuzzy sequence in M. A converges to a at level α. We claim that a is a limit point of A.Let $\mathcal{E} > 0$ be given and let $n_0 \in N$. Since A converges to a, there exists $m \in N$ such that for each $n \geq m$

1. $A(n,x) \geq \alpha$ for atleast one x in M 2. $A(n,x) \geq \alpha$ implies $d(x,a) < \mathcal{E}$.

Choose n ϵ N such that $n \geq n_0$ and $n \geq m$ then for this n we have 1. $A(n,x) \geq \alpha$ for atleast one x in M. 2. $A(n,x) \geq \alpha$ implies $d(x,a) < \varepsilon$. Hence l is a limit point of A.

Result:3.6

Converse of the theorem is not true. This is established by the following example.

Example:3.7

Consider X = R with usual metric. Consider the fuzzy sequence A defined as

A(n,x) = 1 if n = 2k, x = n and =1 if n = 2k-1, x = 1/n and = 0 otherwise.

Take $\alpha \in (0,1]$ The odd subsequence converges to 0. Hence 0 is a limit point of A. The even subsequence is not convergent. Hence A is not convergent. Hence 0 is not the limit of A. Hence a limit point need not be a limit.

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