Int Jr. of Mathematical Sciences & Applications
Vol.3, No.1, January-June 2013

ISSN No: 2230-9888
www.journalshub.com

ON b - δ - OPEN SETS IN TOPOLOGICAL SPACES

S.Padmanaban
Department of science and Humanities
Karpagam Institute of Technology
Coimbatore - 641105. Tamilnadu, India
ooty.sp@gmail.com

Abstract

In this paper, we introduce a new class of sets called b-δ-closed Sets in topological spaces. We study some of its basic properties and investigate properties of the collection of b-δ -closures of sets in a topological space.

Mathematical Subject Classification: 54A05

Key words and phrases: δ-closure, b-closure, b-δ-closure and b-δ closed Sets.

1 Introduction.

The notions of δ -open sets, δ -closed sets where introduced by Velicko [12] for the purpose of studying the important class of H-closed spaces. Dickman and Porter [3], [4], Joseph [6] and long and Herrington [7] continued the work of Velicko. Noiri [9] , Dontchev and Ganster [5] have also obtained several new and interesting results related to these sets.Andrijevic initiated the study of b-open [1] sets.This notion has been studied extensively in recent years by many topologists. In this paper, we introduce a new class of sets called b-δ -closed Sets in topological spaces. We study some of its basic properties and investigate properties of the collection of b-δ -closures of sets in a topological space.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, τ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space (X, τ). We denote closure and interior of A by cl(A) and int(A), respectively. The set A is said to be regular open (resp. regular closed) [11] if A = int(cl(A))(resp. A = cl(int(A))).The family of all regular open(resp. Regular closed) sets of (X, τ) is written by RO(X, τ) (resp. RC(X, τ)).This family is closed under the finite intersections (resp. Finite unions).The topology generated by the family RO(X, τ) is written by τ_s . Since the intersection of two regular open sets is regular open, the collection of regular open sets forms a base for a coarser topology τ_s than the original one τ .The family τ_s

is called semi-regularization of τ . A space (X, τ) is called semi-regular [8] if $\tau_s = \tau$. The δ –closure of A [12] is the set of all x in x such that the interior of every closed neighbourhood of x intersects A non trivally. The δ –closure of A is denoted by cl_δ(A) or δ-cl(A) .The δ –interior [12] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-int(A).The subset A is called δ-open[12] if A= δ-int(A). i.e, a set is δ-open if it is the union of regular open sets. The complement of δ-open set is δ-closed [12]. Alternatively, a set A $\subset$ (X, τ) is called δ-closed [12] if A= δ-cl(A),where δ-cl(A) = $\{x \in X$:int(cl(U))∩A≠$\phi, U \in \tau$ and $x \in U\}$. i.e, family of δ-open sets forms a topology on X and is denoted by τ_δ. It is well know that $\tau_s = \tau_\delta$ .A subset A is said to be b-open [1] if A $\subset$ cl(int(A)) $\cup$ int(cl(A)).The complement of b-open is said to be b-closed[1]. The intersection of b-closed sets of X containing A is called b-closure of A [1] and denoted by bcl(A).The union of all b-open sets of X contained in A is called b-interior of A [1] and is denoted by bint(A).The subset A is b-regular [10] if it is b-open and b-closed. The family of b-open (b-closed, b-regular) sets of X is denoted by BO(X) (resp. BC(X), BR(X)) and family of all b-open (b-regular) sets of X containing a point $x \in X$ is denoted by BO(X, x) (resp.BR(X, x)). A topological space (X, τ) is called $b-T_2$ [2] for any distinct pair of points x and y in X, there exists sets U and V in X containing x and y respectively such that $U \cap V = \phi$.

3. b - δ open sets.

Definition: 3.1 A point x is of X is called b-δ-cluster point of A if int (bcl(U)) ∩A ≠ ϕ for every $U \in BO(X, x)$. The set of all b-δ-cluster point of A is called b-δ-closure of A and is denoted by b-δ-cl(A). A subset A is said to be b-δ-closed if A = b-δ-cl(A).The complement of a b-δ-closed set is said to be b-δ-open set. The family of all b-δ-open subsets of X is denoted by $B_\delta O(X)$.

Theorem 3.2 (see [10]):Let A be a subset of a topological space X. Then

(i) A $\in$ BO(X) if and only if bcl(A) $\in$ BR(X).

(ii) A $\in$ BC(X) if and only if bint(A) $\in$ BR(X).

Theorem : 3.3 For any subset A of a space X, the following hold:

b-δ-cl(A) = $\bigcap\{V : A \subset V$ and V is b-δ-closed $\}$

= $\bigcap\{V : A \subset V \text{ and } V \in BR(X)\}$.

Proof. First, suppose that $x \notin$ b-δ-cl(A). Then there exists $V \in BO(X, x)$ such that int (bcl(V)) ∩ A ≠ φ. By Theorem 3.2, X \ bcl(V) is b-regular and hence X \ bcl(V) is b- δ-closed set containing A and $x \notin$ X \ bcl(V). Therefore we have $x \notin \cap\{V : A \subset V$ and V is b-δ-closed $\}$

Conversely, suppose that $x \notin \cap\{V : \subset V$ and V is b-δ-closed $\}$ There exists a b-δ-closed V such that $A \subset V$ and $x \notin V$. There exists $U \in BO(X)$ such that $x \in U \subset bcl(U) \subset X \setminus V$. Therefore, we have int (bcl(U)) ∩A $\subset$ int (bcl(U)) ∩ V = φ. This shows that $x \notin$ b-δ-cl(A).Similarly we can prove the second equality.

Theorem: 3.4 Let A and B be any subsets of a space X. Then the following properties hold:

(a) $x \in$ b-δ-cl(A) if and only if (int$V$)∩A =ϕ for each $V \in BR(X, x)$,

(b) If $A \subset B$, then b-δ-cl(A) $\subset$ b-δ-cl(B),

(c) b-δ-cl(b-δ-cl(A)) = b-δ-cl(A) ,a

(d) if A_α is b-δ-closed in X for each $\alpha \in \Lambda$,then $\bigcap_{\alpha\in\wedge} A_\alpha$ is b-δ closed in X.

Proof: The proofs of (a) and (b) are obvious.

(c) Generally we have b-δ-cl(b-δ-cl (A)) $\supset$ b-δ-cl(A). Suppose that $x \notin$ b-δ-cl(A). There exists $V \in BR(X, x)$ such that (int$V$)∩A = ϕ . Since $V \in BR(X)$, we have (int$V$)∩ b-δ-cl(A) = φ. This shows that $x \notin$ b-δ-cl(b-δ-cl(A)). Therefore, we obtain b-δ-cl (b-δ-cl (A)) ⊂ b-δ-cl(A).

(d) Let A_α be b-δ-closed in X for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, A_α = b-δ-cl(A_α), Hence we have

b-δ-cl $\left(\bigcap_{\alpha\in\wedge} A_\alpha\right) \subset \left(\bigcap_{\alpha\in\wedge}\right)$ b-δ-cl (A_α) = $\left(\bigcap_{\alpha\in\wedge} A_\alpha\right)$ (A_α) ⊂ b-δ-cl $\left(\bigcap_{\alpha\in\wedge} A_\alpha\right)$

Therefore, we obtain b-δ-cl $\left(\bigcap_{\alpha\in\wedge} A_\alpha\right)$ = $\bigcap_{\alpha\in\wedge} A_\alpha$ .This shows that $\bigcap_{\alpha\in\wedge} A_\alpha$ is b-δ closed.

Remark: 3.5 The union of two b-δ closed sets is not necessarily b-δ closed as shown by the following example.

Example: 3.6. Let $X = \{a, b, c\}\, and\, T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. the

Corollary: 3.7. Let A and A_α ($\alpha \in \Lambda$) be any subset of a space X. Then the following properties hold:

(a) if A is b-δ-open in X if and only if for each x ∈ A there exists $V \in BR(X, x)$ such that x ∈ V ⊂ A.
(b) b-δ-cl(A) is b-δ-closed
(c) if A_α is b-δ-open in X for each $\alpha \in \Lambda$,then $\bigcup_{\alpha\in\wedge} A_\alpha$ is b-δ-open in X .

Theorem: 3.8. For a subset A of a space X, the following properties hold:

(a) if A ∈ BO(X), then b-cl(A) = b-δ-cl(A).
(b) A ∈ BR(X) if and only if A is b-δ-open and b-δ-closed.

Proof . (a) Generally we have b-cl(B) ⊂ b-δ-cl(B) for every subset B of X. $Let\ A \in BO(X)$ and suppose that x ∉ b cl(A). Then there exists $V \in BO(X, x)$ such that $V \cap A = \phi$. Since $A \in BO(X)$, we have int(b-cl(V)) ∩ A = φ. This shows that x ∉ b-δ-cl(A).

Hence we obtain b-cl(A) = b-δ-cl(A).

(b) Let $A \in BR(X)$. Then $A \in BO(X)$ and by (a), A = b-cl(A) = b-δ-cl(A).Therefore, A is b-δ-closed .Since X \ A $\in BR(X)$, by the argument above X \ A is b-δ-closed and hence b-δ-open. The converse is obvious.

Definition: 3.9. A subset B of a topological space (X, τ), is said to be δ-complement-b-open (briefly δ -c-b-open) provided there exists a subset A of X for which X – B = b-δ-cl(A). We call a set δ-complement-b-closed if its complement is δ -c-b-open.

Remark 3.10. It should be mentioned that by Corollary3.7, X – B = b-δ-cl(A) is b-δ-closed and B is b-δ-open. Therefore, the equivalence of δ-c-b-open and b-δ-open is obvious from the definition.

Theorem 3.11. If **A** $\subseteq$ X is b-open, then b-int(b-δ-cl(A)) is b-δ-open.

Proof. Since X –bint (bcl(A) = bcl (X-bcl(A)) then by complements bint (bcl(A) =(X –bcl(X-bcl(A))). Since X - (bcl(A) (=B,say) is b-open , b-cl(B) = (b-δ-cl(B) from Theorem 3.8. Therefore, there exists a subset B = X – bcl(A) for which X – b int(b-cl(A)) = b-δ-cl(B). Hence bint(bcl(A)) is b-δ-open.

Corollary 3.12. If $A \subseteq X$ is b- regular, then A is b-δ-open.

Proof. Obvious by Theorem 3.8, since A is b-regular if and only if A = b int(bcl(A)).

Theorem 3.13. b-δ-open is equivalent to b-regular if and only if b-δ-cl(A) is b-regular form every set $A \subseteq X$.

Proof. Let X be topological space. Assume b-δ-open is equivalent to b-regular and let $A \subseteq X$. Then by Corollary3.7, X – b-δ-cl(A) is b-δ-open which implies that b-δ-cl(A) is b-regular.

Conversely, assume b-δ-cl(A) is b-regular for every set A. Suppose U is b-δ-open and let $A \subseteq X$ such that X – U = b-δ-cl(A). That is U = X - b-δ-cl(A). Then b-δ-cl(A) is b- regular and U is b-regular.Therefore b – δ – open is equivalent to b-regular.

Theorem 3.14. If $B \subseteq X$ is b-δ-open, then B is an union of b-regular sets.

Proof. Let B be b-δ-open and let $x \in B$. Since B is b-δ-open, then there exists a set $A \subseteq X$ such that B = X – (b-δ-cl(A)). Because $x \notin$ b-δ-cl(A), there exists a b-open set W for which $x \in W$ and int (bcl(W)) ∩A ≠ ϕ . Hence $x \in$ bint (bcl(W)) $\subseteq$ X – (b-δ-cl(A)) ,where b-int (b-cl(W) (= V(say)) $\in$ BR(X), that is, $B = \bigcup\{V : V \subseteq B, V \in BR(X)\}$.

Corollary 3.15. If B is b-δ-closed then B is the intersection of b-regular sets.

4. On b-δ-D_i closed (resp. b-δ-T_i) topological spaces:

Now, we study some classes of topological spaces in terms of the concept of b-δ-open sets. The relations with other notions, directly or indirectly connected with these classes are investigated.

Definition: 4.1. A subset A of a topological space (X,τ) is called a b-δ-D set if there are two sets $U, V \in B_\delta O(X)$ such that $U \neq X$ and A = U – V.

It is true that every b-δ-open set U different from X is a b-δ-D set if A = U and $V = \phi$.

Definition:4.2. A topological space (X,τ) is called b-δ-D_0 if for any distinct pair of points x and y of X, there exists a b-δ-D set of X containing one of the points but not the other.

Definition:4.3. A topological space (X,τ) is called b-δ-D_1 if for any distinct pair of points x and y of X, there exists a b-δ-D set F of X containing x but not y and a b-δ-D set G of X containing y but not x.

Definition: 4.4. A topological space (X,τ) is called b-δ-D_2 if for any distinct pair of points x and y of X, there exists a disjoint b-δ-D sets G and E of X containing x and y respectively.

Definition: 4.5. A topological space (X,τ) is called b-δ-T_0 if for any distinct pair of points in X, there exists a b-δ-open set containing one of the points but not the other.

Definition: 4.6. A topological space (X,τ) is called b-δ-T_1 if for any distinct pair of points x and y of X, there exists a b-δ-open U in X containing x but not y and a b-δ-open set V in X containing y but not x.

Definition: 4.7. A topological space (X,τ) is called b-δ-T_2 if for any distinct pair of points x and y of X, there exists a b-δ-open sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

Remark:4.8. From Definition 4.1 to 4.7, we obtain the following diagram:

$$\begin{tikzcd}
\text{b-}\delta\text{-}T_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{b-}\delta\text{-}T_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{b-}\delta\text{-}T_0 \arrow[d, Rightarrow] \\
\text{b-}\delta\text{-}D_2 \arrow[r, Rightarrow] & \text{b-}\delta\text{-}D_1 \arrow[r, Rightarrow] & \text{b-}\delta\text{-}D_0
\end{tikzcd}$$

Theorem: 4.9. If a topological space (X,τ) is b-δ-T_0 then it is b-δ-T_2.

Proof. For any points $x \neq y$, let V be a b-δ-open such that $x \in V$ and $y \notin V$ .Then, there exists $U \in BO(X)$ such that x $\in$ U $\subseteq$ b-cl(U) $\subseteq$ V. By Theorem 3.8, bcl(U) $\in$ BR(X).Then b-cl(U) is b-δ-open and also X - bcl(U) is a b-δ-open set containing y. Therefore, X is b-δ-T_2

Theorem: 4.10. For a topological space (X,τ), the six properties in the diagram are equivalent.

Proof. By Theorem 4.9, we have that b-δ-T_0 implies b-δ-T_2. Now we prove that b-δ-D_0 implies b-δ-T_2. Let (X,τ) be b-δ-D_0 so that for any distinct pair of points x and y of X, one of them belongs to a b-δ-D set A. Therefore, we choose $x \in A$ and $y \notin A$. Suppose A = U – V for which U≠X and $U, V \in B_\delta O(X)$. This implies that $x \in U$. For the case that $y \notin A$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$ For (i), the space X is b-δ-T_0 since $x \in U$ but $y \notin U$. For (ii), the space X is also b-δ-T_0 since $y \in V$ but $x \notin V$.

Definition:4.11. Let x be point of X and V a subset of X. The set V is called a b-δ-neighborhood of x in X if there exists a b-δ-open set A of X such that $x \in A \subseteq V$.

Definition:4.12. A point $x \in X$ which has only X as the b-δ-neighborhood is called a point common to all b-δ-closed sets (b-δ-cc).

Theorem :4.13. If a topological space (X,τ) is b-δ-D_1 then (X,τ) has no b-δ-cc -point

Proof**.** Since is (X,τ) is b-δ-D_1 so each point x of X is contained in a b-δ-D set, $A = U - V$ and thus in U. By definition $U \neq X$ and this implies that x is not a b-δ-cc-point.

Definition:4.14. A subset A of topological space (X,τ) is called a quasi b-δ-closed (briefly qbd -closed) set if b-δ-cl(A) $\subseteq$ U whenever $(A) \subseteq U$ and U is b-δ-open in (X,τ).

Theorem 4.15. For a topological space (X,τ) the following properties hold:

(i) For each pair of points x and y in X, $x \in$ b-δ-cl $(\{y\})$ implies $y \in$ b-δ-cl $(\{x\})$

(ii) For each $x \in X$, the singleton {x} is qbd-closed in (X,τ).

Proof. (i): Let $y \notin$ b-δ-cl $(\{x\})$ This implies that there exists $V \in BO(X,y)$ such that int(bcl(V)) $\cap$ $\{x\} = \phi$ and X –bcl(V) $\in BR(X,x)$ which means that $x \notin$ b-δ-cl $(\{y\})$ (ii): Suppose that $U \in BO_\delta(X)$. This implies that there exists $V \in BO(X)$ such that $x \in V \subseteq bcl(V) \subseteq U$. Now we have b-δ-cl $(\{x\})$ $\subseteq$ b-δ-cl(V) = bcl(V) $\subseteq$ U.

Definition: 4.16. A topological space (X,τ) is said to be b-δ-$T_{½}$ if every qbd-closed set is b-δ-closed.

Theorem:4.17. For a topological space (X,τ) the following are equivalent:

(i) (X,τ) is b-δ-$T_{½}$;

(ii) (X,τ) is b-δ-T_1 .

Proof. $(i) \Rightarrow (ii)$: For distinct points x,y of X, {x} is qbd-closed by Theorem 4.15. By hypothesis, X – {x} is b-δ-open and $y \in X - \{x\}$. By the same token, $y \in X - \{y\}$ and X – {y} is b-δ-open. Therefore, (X,τ) is b-δ-T_1

$(ii) \Rightarrow (i)$ **:** Suppose that A is a qbd-closed set which is not b-δ-closed. There exists $x \in$ [b-δ-cl (A)] – A. For each $a \in A$, there exists a b-δ-open set V_a such that $a \in V_a$ and $x \notin V_a$. Since $A \subseteq \bigcup_{a \in V_a} V_a$ and $\bigcup_{a \in V_a} V_a$ is b-δ-open, we have b-δ-cl(A) $\subseteq \bigcup_{a \in V_a} V_a$. Since $x \in$ b-δ-cl(A) ,there exists $a_0 \in A$ such that $x \in V_{a0}$. But this is a contradiction.

Theorem 4.18. For a topological space (X,τ), the following are equivalent:

(i) (X,τ) is b-δ-T_2 ;

(ii) (X,τ) is b-T_2

Proof. $(ii) \Rightarrow (i)$ This is obvious since every b-δ-open set is b-δ-open.

$(ii) \Rightarrow (i)$ **:** Let x and y be distinct points of X. There exist b-open sets U and V such that $x \in U$, $y \in V$ and bcl(U) ∩ bcl(V) = ϕ .Since bcl (U) and bcl(V) are b-regular, then they b-δ-open and hence (X,τ) is b-δ-T_2.

References

[1] Andrijevic D, On b-open sets, Math.Vesnik, (1996)48, 5 9-64.

[2]Caldas.M,jafari.S and Noiri.T, On $\wedge_b$ - sets and the associated topology $\tau^{\wedge_b}$, Acta Math. Hungar(2006)110,337 - 345.

[3] Dickman.R.F and Porter.J.R, θ-closed subsets of Hausdorff spaces, Pacific J.Math. , (1975)59, 407 – 415.

[4] Dickman.R.F and Porter.J.R, θ-perfect and θ-absolutely closed functions, Ilinois J. Math. ,(1977)21, 42 – 60.

[5] Dontchev.J and Ganster.M , On δ – generalized closed set and $T_{3/4}$ - spaces, Mem fac Kochi Univ Ser A (Math), (1996) 17,15 – 31.

[6] Joseph.J.E, θ-closure and θ-subclosed graphs, Math, Chronicle, (1979)8 ,99 – 117.

[7] Long.P.E and Herrington.L.L, The τ_θ - topology and faintly continuous functions, Kyungpook Math. J.,(1982)22, 7 – 14.

[8] Mrsevic.M, Reilly.I.L and Vamanamurthy.M.K, On semi-regularization properties,J Austral Math Soc. ,(1985)38,40 – 54.

[9] Noiri.T, On δ - continuous functions, J. Korean Math. Soc.,(1980)16, 161 – 166.

[10] Park.J.H,Strongly θ –b-continuous functions,Acta math.Hunger.,(2006)110,347-359.

[11] Stone.M ,Application of the theory of Boolean ring to general topology,trans. Amer. Math. Soc. ,(1937)41,374 – 481.

[12] Velicˇko .N.V, H-closed topological spaces, Amer Math Soc Transl, (1968)78, 103–18.