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Some Common Fixed Point Theorems in L-Fuzzy Metric Spaces

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ABSTRACT :

In this paper we improve the result of R. Saadati [23] by proving the same theorem with more weaker condition of Occasionally Weakly Compatible Mapping .

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KEYWORDS : Occasionally Weakly Compatible Mapping, L-Fuzzy contractive mapping ,L-fuzzy metric space,Point of coincidence

1. INTRODUCTION :

The notion of fuzzy sets was introduced by Zadeh [25]. Various concepts of fuzzy metric spaces were considered in [7, 8, 14, 15]. Many authors have studied fixed point theory in fuzzy metric spaces; see for example [3, 4, 11, 12, 17, 18]. In the sequel, we shall adopt the usual terminology, notation and conventions of L-fuzzy metric spaces introduced by R.Saadati et al. [20] which are a generalization of fuzzy metric spaces [10] and intuitionistic fuzzy metric spaces [19, 21].

In this paper we improve the result of R. Saadati [23] by proving the same theorem with more weaker condition of Occasionally Weakly Compatible Mapping .

2. PRELIMINARIES :

Definition 2.1. [11] *Let $\mathcal{L} = (L,\leq_L)$ be a complete lattice, and U a non-empty set called a universe. An L-fuzzy set A on U is defined as a mapping $A : U \to L$. For each u in U, A(u) represents the degree (in L) to which u satisfies A.*

Lemma 2.2. [5,6] *Consider the set L^* and the operation $\leq_{L^*}$ defined by:*

$$L^* = \{ (x_1 ,x_2) : (x_1 ,x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1 \}$$

$(x_1 ,x_2) \leq_{L^*} (y_1 ,y_2) \leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1 ,x_2) , (y_1 ,y_2) \in L^*$. Then $(L^*,\leq_{L^*})$ *is a complete lattice.*

Classically, a triangular norm T on $([0, 1], \leq)$ is defined as an increasing, commutative, associative mapping T : $[0, 1]^2 \to [0,1]$ satisfying $T(1, x) = x$, for all $x \in [0,1]$

These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L ,\leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 2.3. [23] A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $T : L^2 \to L$ satisfying the following conditions: (1) $(\forall x \in L)$ $T(x, 1_{\mathcal{L}}) = x$; (boundary condition)

(2) $(\forall (x,y) \in L^2)$ $T(x,y) =T(y,x)$ (commutativity)

(3) $(\forall (x,y,z) \in L^3)$ $T (x, T(y,z))= T (T(x,y), z)$ (associativity)

(4) $(\forall\ (x,x',y,y') \in L^4)\ (x \leq_L x' \text{ and } y \leq_L y') \rightarrow T(x,y) \leq_L T(x',y')$ (monotonicity)

A t- norm T on $\mathcal{L}$ is said to be continous if for any x,y $\in \mathcal{L}$ and any sequence $\{x_n\}$ and $\{y_n\}$ which converge to x and y

$$\lim_{n\to\infty} T(x_n, y_n) = T(x,y)$$

For example T(x,y) = min(x,y) and T(x,y) = xy are two continuous t–norms on [0,1]. A t-norm can also be defined recursively as an (n + 1)-ary operation (n ∈ N) by $T^{-1} = T$ and

$$T^n = (x_1,x_2,\ldots\ldots x_{n+1}) = T(T^{-1}(x_1,x_2,\ldots\ldots x_n), x_{n+1})$$

Definition 2.4. *A negation on $\mathcal{L}$ is any decreasing mapping $N : L \rightarrow L$ satisfying $N(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $N(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $N(N(x)) = x$, for all $x \in L$, then N is called an involutive negation.*

Definition 2.5 *The 3-tuple $(X,M,\mathcal{T})$ is said to be an L-fuzzy metric space if X is an arbitrary (nonempty) set, T is a continuous t–norm on $\mathcal{L}$ and M is an $\mathcal{L}$-fuzzy set on $X^2 \times]0,+\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0,+\infty[$*
(a) $M(x, y, t) >_L 0_L$,
(b) $M(x, y, t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$,
(c) $M(x, y, t) = M(y, x, t)$
(d) $\mathcal{T}(M(x, y, t), M(y, z, s)) \leq_L M(x, z, t + s)$,
(e) $M(x, y, \cdot): [0, \infty] \rightarrow L$ is continuous and $\lim_{n\to\infty} M(x,y,t) = 1_{\mathcal{L}}$

Lemma 2.6. [10] *Let $(X,M,\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, M(x, y, t) is nondecreasing with respect to t, for all x, y in X.*

Definition 2.7. *A sequence $\{x_n\}$ $n \in N$ in an $\mathcal{L}$-fuzzy metric space $(X,M,\mathcal{T})$ is called a Cauchy sequence, if for each $\varepsilon \in L \setminus \{0_L\}$ and $t > 0$, there exists $n_0 \in N$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$,*

$$M(x_m, x_n, t) >_L N(\varepsilon).$$

A sequence $\{x_n\}$ $n \in N$ is said to be convergent to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X,M,\mathcal{T})$ if $M(x_n, x, t) = M(x, x_n, t) \rightarrow 1_{\mathcal{L}}$ whenever $n \rightarrow +\infty$ for every $t > 0$. A L -fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Definition 2.8. *Let $(X,M,\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. M is said to be continuous on $X \times X \times]0,+\infty[$,if*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times]0,+\infty[$ converge to a point $(x,y,t) \in X \times X \times]0,+\infty[$ i.e $\lim_{n\to\infty} M(x_n, x, t_n) = \lim_{n\to\infty} M(y_n, y, t) = 1_{\mathcal{L}}$ and $\lim_{n\to\infty} M(x, y, t_n) = M(x, y, t)$

Lemma 2.9. *Let $(X,M,\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then M is a continuous function on $X \times X \times]0,+\infty[$*

Proof : *The proof is the same as that for fuzzy spaces (see Proposition 1 of [16]).*

Definition 2.10. *Let A and S be mappings from an $\mathcal{L}$-fuzzy metric space $(X,M,\mathcal{T})$ into itself .Then the mapping are said to be weak compatible if they commute at their coincidence point, that is, Ax = Sx implies that ASx = SAx.*

Definition 2.11. *Let A and S be mappings from an $\mathcal{L}$-fuzzy metric space $(X,M,\mathcal{T})$ into itself. .Then the mapping are said to be compatible if*

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1_{\mathcal{L}}\ \forall\ t > 0$$

Whenever *$\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$$

Definition 2.12. *Let A and S be mappings from an $\mathcal{L}$ -fuzzy metric space $(X,M, \mathcal{T})$ into itself are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.*

:

Definition 2.13. [9] *We say that the $\mathcal{L}$ -fuzzy metric space $(X,M, \mathcal{T})$ has property* (C), *if it satisfies the following condition:*

$$M(x, y, t) = C , \quad \text{for all } t > 0 \text{ implies } C = 1_{\mathcal{L}}$$

Lemma 2.14.[13]: *Let X be a set, f , g owc self maps of X. If f and g have unique point of coincidence,*

$w = f x = g x$, then w is the unique common fixed point of f and g .

Our objective is to prove common fixed point theorem in $\mathcal{L}$ -fuzzy metric space by

1. *By Relaxing weak compatiblity to Occasionally Weakly Compatible Mapping.*
2. *Replacing Completness of the space .*
3. *Removing the assumtion of closeness of one or more range.*
4. *Relaxing the required containment of range of one mapping into the range of other.*

3. Main Result :

Theorem 3.1 *Let A, B, S and T be four self maps of $\mathcal{L}$ -fuzzy metric space $(X,M, \mathcal{T})$ which has property (C), satisfying :*

(i) *The pair {A, S} and {B ,T} be owc.*

(ii) $M(Ax, By, t) \geq_L M(Sx,Ty, kt)$, *for every x, y in X and some $k > 1$.*

then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = T z = z$.

Moreover, z = w, so that there is a unique common fixed point of A, B, S and T

Proof : Let the pairs {A, S} and {B, T } be owc, so there are points x, y $\in$ X such that Ax = Sx and By = T y. We claim that Ax = By. If not, by inequality (ii)

$$\begin{aligned} M(Ax, By, t) &\geq_L M(Ax ,By, kt) \\ &\geq_L M(Sx,Ty, k^2t), \\ &= M(Ax,By, k^2t), \\ &\geq_L M(Sx,Ty, k^3t), \\ &\vdots \\ &\geq_L M(Ax,By, k^nt), \end{aligned}$$

On the other hand from lemma 2.6 we have that

$$M(Ax, By, t) \leq_L M(Ax ,By, k^nt),$$

Hence $M(Ax, By, t) = C$ for all t >0 .Since $(X,M, \mathcal{T})$ has property (C), it follows that C = $1_{\mathcal{L}}$ i.e. $Ax = By$ therefore $Ax = Sx = By = T y$.

Suppose that there is a another point z such that Az = Sz then by (i) we have $Az = Sz = By = T y$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of

A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Uniqueness : Let w be another common fixed point of A, B, S and T.

Then $Aw = Bw = Sw = Tw = w$. Assume $w \neq z$. by (ii) we have

$$
\begin{aligned}
M(w,z,t) &=. M(Aw,Bz\ t) \\
&\geq_L M(Sw,Tz,kt), \\
&= M(Aw,Bz,kt), \\
&\geq_L M(Sw,Tz,k^2t), \\
&\ \vdots \\
&\geq_L M(w,z,k^nt),
\end{aligned}
$$

On the other hand from lemma 2.6 we have that

$$M(w,z,t) \leq_L M(w,z,k^nt),$$

Hence $M(w,z,t) = C$ for all $t > 0$ .Since $(X,M,\mathcal{T})$ has property (C), it follows that $C = 1_{\mathcal{L}}$ i.e. $w = z$
By Lemma 2.14 and z is a common fixed point of A, B, S and T . The uniqueness of the fixed point holds from (ii)

Theorem 3.2 *Let A, B, S and T be four self maps of $\mathcal{L}$-fuzzy metric space $(X,M,\mathcal{T})$ which has property (C), satisfying :*

(i) *The pair {A, S} and {B ,T} be owc.*

(b) $M(Ax,By,t) \geq_L . \emptyset\{M(Sx,Ty,kt), M(Ax,Sx,kt), M(Ax,Ty,kt), M(Sx,By,kt)\}$ *for every x, y in X and some* $k > 1$.

For all $x,y \in X$, $t > 0$ and $\emptyset: [0,1]^4 \rightarrow [0,1]$ *such that* $\emptyset(t,1,t,t) \geq_L t$ *for all* $0 < t < 1$ *then there exists a unique point* $w \in X$ *such that* $Aw = Sw = w$ *and a unique point* $z \in X$ *such that* $Bz = Tz = z$. *Moreover,* $z = w$, *so that there is a unique common fixed point of A, B, S and T.*

Proof : Let the pairs {A, S} and {B, T } be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$.
We claim that $Ax = By$. If not, by inequality (ii)

$$
\begin{aligned}
M(Ax,By,t) &\geq_L \emptyset\{M(Ax,By,kt), M(Ax,Ax,kt), M(Ax,By,kt), M(Ax,By,kt)\} \\
&= \emptyset\{M(Ax,By,kt), 1, M(Ax,By,kt), M(Ax,By,kt)\} \\
&\geq_L M(Ax,By,kt) \\
&\geq_L \emptyset\{M(Sx,Ty,k^2t), M(Ax,Sx,k^2t), M(Ax,Ty,k^2t), M(Sx,By,k^2t)\} \\
&= \emptyset\{M(Ax,By,k^2t), M(Ax,Ax,k^2t), M(Ax,By,k^2t), M(Ax,By,k^2t)\} \\
&= \emptyset\{M(Ax,By,k^2t), 1, M(Ax,By,k^2t), M(Ax,By,k^2t)\} \\
&\geq_L M(Ax,By,k^2t)
\end{aligned}
$$

$$\ldots\ldots\ldots\ldots\ldots \geq_L M(Ax,By,k^n t)$$

On the other hand from lemma 2.6 we have that

$$M(Ax,By,t) \leq_L M(Ax,By,k^n t),$$

Hence $M(Ax, By, t) = C$ for all t >0 .Since $(X,M,\mathcal{T})$ has property (C), it follows that C = $1_{\mathcal{L}}$
i.e. $Ax = By$ therefore $Ax = Sx = By = Ty$.

Suppose that there is a another point z such that Az = Sz then by (i) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S. Similarly there is a unique point z ∈ X such that z = Bz = T z.

Uniqueness : : Let w be another common fixed point of A, B, S and T.

Then Aw = Bw = Sw = Tw = w. Assume w≠ z . by (ii) we have

$$M(w,z,t) =. M(Aw,Bz\ t)$$

$$\geq_L \emptyset\{M(Sw,Tz,kt), M(Aw,Sw,kt), M(Aw,Tz,kt), M(Sw,Bz\ kt)\}$$

$$= \emptyset\{M(w,z,kt), M(w,w,kt), M(w,z,kt), M(w,z,kt)\}$$

$$= \emptyset\{M(w,z,kt), 1, M(w,z,kt), M(w,z\ kt)\}$$

$$\geq_L M(w,z,kt)$$

$$=. M(Aw,Bz\ kt)$$

$$\geq_L \emptyset\{M(Sw,Tz,k^2t), M(Aw,Sw,k^2t), M(Aw,Tz,k^2t), M(Sw,Bz\ k^2t)\}$$

$$= \emptyset\{M(w,z,k^2t), M(w,w,k^2t), M(w,z,k^2t), M(w,z,k^2t)\}$$

$$= \emptyset\{M(w,z,k^2t), 1, M(w,z,k^2t), M(w,z,k^2t)\}$$

$$\geq_L M(w,z,k^2t)\ldots\ldots\ldots\ldots\ldots \geq_L M(w,z,k^n t)$$

On the other hand from lemma 2.6 we have that

$$M(w,z,t) \leq_L M(w,z,k^n t)$$

Hence $M(w, z, t) = C$ for all t >0 .Since $(X,M,\mathcal{T})$ has property (C), it follows that C = $1_{\mathcal{L}}$ i.e. $w = z$
By Lemma 2.14 and z is a common fixed point of A, B, S and T . The uniqueness of the fixed point holds from (ii)

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