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NON – EXTENDABLE SPECIAL RATIONAL DIO TRIPLES

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ABSTRACT

In this paper, we present three non – extendable special rational Dio triples with suitable property.

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1. INTRODUCTION

A set of positive integers $(a_1, a_2, \dots, a_m)$ is said to have the property $D(n), n \in z - \{0\}$, if $a_i a_j + n$ is a perfect for all $1 \leq i < j \leq m$ and such a set is called a Diophantine m-tuple with property $D(n)$.Many mathematicians considered the problem of existence of Diophantine triples with the property $D(n)$ for any arbitrary integer *n* [1] and also for any linear polynomial in *n*. In this context, one may refer [2-21] for an extensive review of various problem on Diophantine triples.

These results motivated us to search for non-extendable special rational Dio triples with suitable property, where the special mention is provided because it differs from the earlier one and special Dio triple is constructed where the product of any two members of the triple with the addition of their sum and increased by the given property is a perfect square.

2. METHOD OF ANALYSIS

Section A: Non- extendable $D\left(\frac{k^2+2k-1}{n^2}\right)$ – special rational Dio triple.

Let $a = \frac{1}{n^2}$ and $b = \frac{1}{n^2+1}$ be two rational numbers such that $ab + a + b + \left(\frac{k^2+2k-1}{n^2}\right)$ is a perfect square.

Let *c*be any rational number such that

$$\frac{1}{n^2}c + \frac{1}{n^2} + c + \left(\frac{k^2+2k-1}{n^2}\right) = \alpha^2 \qquad (1)$$

$$\frac{1}{n^2+1}c + \frac{1}{n^2+1} + c + \left(\frac{k^2+2k-1}{n^2}\right) = \beta^2 \qquad (2)$$

Eliminating *c*from (1) and (2), we obtain

$$-\frac{1}{n^2(n^2+1)}\left(\frac{k^2+2k-n^2-1}{n^2}\right) = \left(\frac{n^2+2}{n^2+1}\right)\alpha^2 - \left(\frac{n^2+1}{n^2}\right)\beta^2 \qquad (3)$$

Using the linear transformation

$$\alpha = X + \left(\frac{n^2+1}{n^2}\right)T \qquad (4)$$

$$\beta = X + \left(\frac{n^2+2}{n^2+1}\right)T$$

in (3),it leads to the Pell equation

$$X^2 = \left(\frac{n^2+1}{n^2}\right)\left(\frac{n^2+2}{n^2+1}\right)T^2 + \left(\frac{k^2-n^2+2k-1}{n^2}\right) \qquad (5)$$

Let $T_0 = 1$; $X_0 = \frac{k+1}{n}$ be the initial solution of (5).Thus (4) yields

$$\alpha = \frac{n^2+(k+1)n+1}{n^2}$$

and using (1), we get

$$c = \frac{n^2+2(k+1)n+1}{n^2}$$

Hence $(a,b,c)=\left(\frac{1}{n^2},\frac{1}{n^2+1},\frac{n^2+2(k+1)n+1}{n^2}\right)$ is a special rational Dio triple with property $D\left(\frac{k^2+2k-1}{n^2}\right)$.

We show that the above triple cannot be extended to a quadruple.

Let d be any rational number such that

$$\frac{1}{n^2}d+\frac{1}{n^2}+d+\left(\frac{k^2+2k-1}{n^2}\right)=p^2 \tag{6}$$

$$\frac{1}{n^2+1}d+\frac{1}{n^2+1}+d+\left(\frac{k^2+2k-1}{n^2}\right)=q^2 \tag{7}$$

$$\left(\frac{n^2+2(k+1)n+1}{n^2}\right)d+\left(\frac{n^2+2(k+1)n+1}{n^2}\right)+d+\left(\frac{k^2+2k-1}{n^2}\right)=r^2 \tag{8}$$

Eliminating d from (7) and (8), we obtain

$$\frac{n^2-(n^2+1)(n^2+2(k+1)n+1)}{n^2(n^2+1)}=\left(\frac{2n^2+2(k+1)n+1}{n^2}\right)q^2-\left(\frac{n^2+2}{n^2+1}\right)r^2 \tag{9}$$

Using the linear transformations

$$q=X+\left(\frac{n^2+2}{n^2+1}\right)T \tag{10}$$

$$r=X+\left(\frac{2n^2+2(k+1)n+1}{n^2}\right)T$$

in (9),it leads to the Pell equation

$$X^2=\left(\frac{n^2+2}{n^2+1}\right)\left(\frac{2n^2+2(k+1)n+1}{n^2}\right)T^2-\left(\frac{n^2-k^2-2k+1}{n^2}\right) \tag{11}$$

Let $T_0=1$ and $X_0=\frac{n^3+(k+1)n^2+2n+(k+1)}{n(n^2+1)}$ be the initial solution of (11).Thus (10) yields

$$q=\frac{2n^3+(k+1)n^2+4n+(k+1)}{n(n^2+1)}$$

and using (7),we get

$$d=\frac{4n^6+4(k+1)n^5+17n^4+12(k+1)n^3+19n^2+8(k+1)n+2}{n^2(n^2+1)(n^2+2)}$$

Verification for non-extendabiltiy of quadruple from the above triple is given below:

Substituting the values of a and d in LHS of (6),we have

$$\text{LHS of (6)} = \frac{\left(2n^2+(k+1)n+2\right)^2-3}{n^4}$$

Note that the RHS is not a perfect square.

Section B: Non- extendable $D\left(\frac{k^2+2k}{n^2}\right)$ – special rational Dio triple.

Let $a = \frac{1}{2n^2}$ and $b = \frac{1}{2n^2+1}$ be two rational numbers such that $ab + a + b + \left(\frac{k^2+2k}{n^2}\right)$ is a perfect square.

Let $c$be any rational number such that

$$\frac{1}{2n^2}c + \frac{1}{2n^2} + c + \left(\frac{k^2+2k}{n^2}\right) = \alpha^2 \qquad (12)$$

$$\frac{1}{2n^2+1}c + \frac{1}{2n^2+1} + c + \left(\frac{k^2+2k}{n^2}\right) = \beta^2 \qquad (13)$$

Eliminating $c$from (12) and (13), we obtain

$$-\frac{1}{2n^2(2n^2+1)}\left(\frac{k^2+2k-n^2}{n^2}\right) = \left(\frac{2n^2+2}{2n^2+1}\right)\alpha^2 - \left(\frac{2n^2+1}{2n^2}\right)\beta^2 \qquad (14)$$

Using the linear transformation

$$\alpha = X + \left(\frac{2n^2+1}{2n^2}\right)T \qquad (15)$$

$$\beta = X + \left(\frac{2n^2+2}{2n^2+1}\right)T$$

in (14),it leads to the Pell equation

$$X^2 = \left(\frac{2n^2+1}{2n^2}\right)\left(\frac{2n^2+2}{2n^2+1}\right)T^2 + \left(\frac{k^2+2k-n^2}{n^2}\right) \qquad (16)$$

Let $T_0 = 1$; $X_0 = \frac{k+1}{n}$ be the initial solution of (16).Thus (15) yields

$$\alpha = \frac{2n^2+2(k+1)n+1}{2n^2}$$

and using (12), we get

$$c = \frac{4n^4+8(k+1)n^3+6n^2+4(k+1)n+1}{2n^2(2n^2+1)}$$

Hence $(a,b,c) = \left(\frac{1}{2n^2}, \frac{1}{2n^2+1}, \frac{4n^4+8(k+1)n^3+6n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right)$ is a special rational Dio triple with property $D\left(\frac{k^2+2k}{n^2}\right)$.

We show that the above triple cannot be extended to a quadruple.

Let *d* be any rational number such that

$$\frac{1}{2n^2}d + \frac{1}{2n^2} + d + \left(\frac{k^2+2k}{n^2}\right) = p^2 \quad (17)$$

$$\frac{1}{2n^2+1}d + \frac{1}{2n^2+1} + d + \left(\frac{k^2+2k}{n^2}\right) = q^2 \quad (18)$$

$$\left(\frac{4n^4+8(k+1)n^3+6n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right)d + \left(\frac{4n^4+8(k+1)n^3+6n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right) + d + \left(\frac{k^2+2k}{n^2}\right) = r^2 \quad (19)$$

Eliminating d from (18) and (19), we obtain

$$\left(\frac{4n^4+8(k+1)n^3+6n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right)\left(\frac{k^2+2k-n^2}{n^2}\right) = \left(\frac{8n^4+8(k+1)n^3+8n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right)q^2 - \left(\frac{2n^2+2}{2n^2+1}\right)r^2 \quad (20)$$

Using the linear transformations

$$q = X + \left(\frac{2n^2+2}{2n^2+1}\right)T \quad (21)$$

$$r = X + \left(\frac{8n^4+8(k+1)n^3+8n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right)T$$

in (20) leads to the Pell equation

$$X^2 = \left(\frac{2n^2+2}{2n^2+1}\right)\left(\frac{8n^4+8(k+1)n^3+8n^2+4(k+1)n+1}{2n^2(2n^2+1)}\right)T^2 + \left(\frac{k^2+2k-n^2}{n^2}\right) \quad (22)$$

Let $T_0 = 1$ and $X_0 = \frac{2n^3+2(k+1)n^2+2n+(k+1)}{n(2n^2+1)}$ be the initial solution of (22).Thus (21) yields $q = \frac{4n^3+2(k+1)n^2+4n+(k+1)}{n(2n^2+1)}$

and using (18),we get

$$d = \frac{16n^6 + 16(k+1)n^5 + 34n^4 + 24(k+1)n^3 + 19n^2 + 8(k+1)n + 1}{2n^2(2n^2+1)(n^2+1)}$$

Verification for non-extendabiltiy of quadruple from the above triple is illustrated below:

Substituting the values of a and d in LHS of (17),we have

$$\text{LHS of (17)} = \frac{\left(4n^2+2(k+1)n+2\right)^2-3}{4n^4}$$

Note that the RHS is not a perfect square.

Section C: Non- extendable $D\left(1-\frac{1}{4n}\right)$ – special rational Dio triple.

Let $a=\frac{1}{n}$ and $b=\frac{1}{4n}$ be two rational numbers such that $ab+a+b+\left(1-\frac{1}{4n}\right)$ is a perfect square.

Let *c*be any rational number such that

$$\frac{1}{n}c+\frac{1}{n}+c+\left(1-\frac{1}{4n}\right)=\alpha^2 \qquad (23)$$

$$\frac{1}{4n}c+\frac{1}{4n}+c+\left(1-\frac{1}{4n}\right)=\beta^2 \qquad (24)$$

Eliminating *c*from (23) and (24), we obtain

$$\frac{3}{16n^2}=\left(\frac{4n+1}{4n}\right)\alpha^2-\left(\frac{n+1}{n}\right)\beta^2 \qquad (25)$$

Using the linear transformation

$$\alpha = X+\left(\frac{n+1}{n}\right)T \qquad (26)$$

$$\beta = X+\left(\frac{4n+1}{4n}\right)T$$

in (25),it leads to the Pell equation

$$X^2 = \left(\frac{n+1}{n}\right)\left(\frac{4n+1}{4n}\right)T^2 - \frac{1}{4n} \qquad (27)$$

Let $T_0 = 1$; $X_0 = \frac{2n+1}{2n}$ be the initial solution of (27).Thus (26) yields

$$\alpha = \frac{4n+3}{2n}$$

and using (23), we get

$$c = \frac{12n+9}{4n}$$

Hence $(a, b, c) = \left(\frac{1}{n}, \frac{1}{4n}, \frac{12n+9}{4n}\right)$ is a special rational Dio triple with property $D\left(1 - \frac{1}{4n}\right)$.

We show that the above triple cannot be extended to a quadruple.

Let d be any rational number such that

$$\frac{1}{n}d + \frac{1}{n} + d + \left(1 - \frac{1}{4n}\right) = p^2 \qquad (28)$$

$$\frac{1}{4n}d + \frac{1}{4n} + d + \left(1 - \frac{1}{4n}\right) = q^2 \qquad (29)$$

$$\left(\frac{12n+9}{4n}\right)d + \left(\frac{12n+9}{4n}\right) + d + \left(1 - \frac{1}{4n}\right) = r^2 \qquad (30)$$

Eliminating d from (29) and (30), we obtain

$$\left(\frac{3n+2}{n}\right)\left(\frac{-1}{4n}\right) = \left(\frac{16n+9}{4n}\right)q^2 - \left(\frac{4n+1}{4n}\right)r^2 \qquad (31)$$

Using the linear transformations

$$q = X + \left(\frac{4n+1}{4n}\right)T \qquad (32)$$

$$r = X + \left(\frac{16n+9}{4n}\right)T$$

in (31),it leads to the Pell equation

$$X^2 = \left(\frac{4n+1}{4n}\right)\left(\frac{16n+9}{4n}\right)T^2 - \frac{1}{4n} \qquad (33)$$

Let $T_0 = 1$ and $X_0 = \frac{8n+3}{4n}$ be the initial solution of (33).Thus (32) yields

$$q = \frac{3n+1}{n}$$

and using (29),we get

$$d = \frac{8n+4}{n}$$

Verification for non-extendabiltiy of quadruple from the above triple is given below:

Substituting the values of a and d in LHS of (28),we have

LHS of (28) = $\frac{(6n+4)^2+3}{4n^2}$

Note that the RHS is not a perfect square.

3. CONCLUSION

To conclude, one may search for other non-extendable special rational Dio triples with suitable properties.

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