Int Jr. of Mathematical Sciences & Applications
Vol. 4, No.1, January-June 2014
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ISSN No: 2230-9888
www.journalshub.com

Rough Semi Prime Ideals and Rough Bi-Ideals in Rings

Neelima C.A.*
Department of Mathematics and Statistics
SNM College Maliankara
Ernakulam (Dt.)-683 516, Kerala, India
e-mail: neelimaasokan@gmail.com

Paul Isaac
Centre for Post Graduate Studies and Research
Department of Mathematics
Bharata Mata College Thrikkakara
Kochi-682 021, Kerala, India
e-mail: pibmct@gmail.com

Abstract

In this paper, we shall introduce the concept of rough semi-prime ideal, rough bi-ideal and give some properties of anti-homomorphism on these rough ideals.

Key words: rough ring, rough ideal, rough semi-prime ideal, rough bi-ideal, rough ring anti-homomorphism
AMS (2010) Subject Classification: 13Axx, 08A72

1 Introduction

The theory of Rough set was proposed by Z. Pawlak in 1982 [8]. It is emerging as a powerful mathematical tool for imperfect data analysis. The rough set theory is an extension of set theory in which a subset of universe is approximated by a pair of ordinary sets, called upper and lower approximations. A key concept in Pawlak rough set model is an equivalence relation, which are the building blocks for the upper and lower approximations. Combining the theory of rough set with abstract algebra is one of the trends in the theory of rough set. Some authors studied the concept of rough algebraic structures. On the other hand, some authors substituted an algebraic structure for the universal set and studied the roughness in algebraic structure. Biswas and Nanda introduced

*All correspondence to this author only

the notion of rough subgroups. The concept of rough ideal in a semigroup was introduced by Kuroki in [7]. And then B. Davvaz [2] studied relationship between rough sets and ring theory and considered ring as a universal set and introduced the notion of rough ideals of a rings. Further studies in this direction is done by Osman Kazanci and B. Davaaz in [6]. The authors investigated the concepts of rough subrings and rough ideals in [5, 4]. Some rough anti-homomorphic properties of a rough group is available in [1]. Anti-homomorphic properties of rough prime ideals and rough primary ideals are studied in [3]. In this paper, we shall introduce the concept of rough semi-prime ideal and rough bi-ideal and explore some properties of homomorphism and anti-homomorphism on these rough ideals.

In section 2 we give the basic concepts of a rough ideal. Section 3 deals with the concepts of a rough semi-prime ideal. In section 4 we define rough bi-ideal and prove some related results.

2 Rough Ideal

Throughout this paper R is a ring and θ is an equivalence relation on R.

2.1 Definition. Let θ be an equivalence relation on R, then θ is called a full congruence relation if $(a, b) \in \theta$ implies$(a + x, b + x)$, (ax, bx), $(xa, xb) \in \theta$ for all $x \in R$.

A full congruence relation θ on R is called complete if $[ab]_\theta$ = $\{ xy \in R : x \in [a]_\theta, y \in [b]_\theta \}$ = $[a]_\theta[b]_\theta$

2.2 Definition. Let θ be a full congruence relation on R and A a subset of R. Then the sets

- $\theta_-(A)$ = $\{ x \in R : [x]_\theta \subseteq A\}$ and
- $\theta^-(A)$ = $\{x \in R : [x]_\theta \cap A \neq \phi\}$

are called, respectively , the θ - lower and θ - upper approximations of the set A.
$\theta(A)$ = $(\theta_-(A)$, $\theta^-(A))$ is called a rough set with respect to θ if $\theta_-(A) \neq \theta^-(A)$.

2.3 Theorem. *[6] Let ϕ be a homomorphism of a ring R_1 onto a ring R_2 and let θ_2 be a full congruence relation on R_2. Let A be a subset of R_1. Then*

1. θ_1 = $\{(a, b) : (\phi(a), \phi(b)) \in \theta_2\}$ *is a full congruence relation on* R_1.
2. $\phi(\theta_1^-(A))$ = $\theta_2^-(\phi(A))$
3. $\phi(\theta_{1-}(A) \subseteq \theta_{2-}(\phi(A))$. *If* ϕ *is one to one* $\phi(\theta_{1-}(A)) = \theta_{2-}(\phi(A))$

2.4 Theorem. *Let ϕ be an anti-homomorphism of a ring R_1 onto a ring R_2 and let θ_2 be a full congruence relation on R_2. Let A be a subset of R_1. Then*

1. θ_1 = $\{(a, b) : (\phi(a), \phi(b)) \in \theta_2\}$ *is a full congruence relation on* R_1.

2. $\phi(\theta_1^-(A)) = \theta_2^-(\phi(A))$

3. $\phi(\theta_{1-}(A) \subseteq \theta_{2-}(\phi(A))$. *If* ϕ *is one to one* $\phi(\theta_{1-}(A)) = \theta_{2-}(\phi(A))$

Proof. The proof is similar to the proof of Theorem 2.3

2.5 Theorem. *[2] For every approximation space* (U, θ) *and every subsets* A, B *of* U*, we have, If* $A \subseteq B$*, then* $\theta^-(A) \subseteq \theta^-(B))$ *and* $\theta_-(A) \subseteq \theta_-(B)$

2.6 Theorem. *For every subsets* A, B *of* R *and a full congruence* θ *in* R*,* $\theta^-(A)\theta^-(B) \subseteq \theta^-(AB)$.

Proof. The proof is similar to that of Theorem 2.2 in [7].

2.7 Theorem. *For every subsets* A, B *of* R *and complete congruence* θ *in* R*,* $\theta_-(A)\theta_-(B) \subseteq \theta_-(AB)$.

Proof. The proof is similar to that of Theorem 2.3 in [7].

2.8 Theorem. *[6] Let* θ *be a full congruence relation on* R*. If* I *is an ideal of* R*, then* $\theta^-(I)$ *is an ideal of* R.

2.9 Theorem. *[6] Let* θ *be a full congruence relation on* R *and* I *be an ideal of* R*. If* $\theta_-(I) \neq \phi$*, then it is equal to* I.

Remark. *From here onwards* θ_1 *denotes a full congruence relation on* R_1 *and* θ_2 *a full congruence relation on* R_2.

2.10 Theorem. *[6] Let* $\phi : R_1 \to R_2$ *be a homomorphism from a ring* R_1 *onto a ring* R_2 *and let* A *be a subset of* R_1*. Then* $\theta_1^-(A)$ *is an ideal of* R_1 *if and only if* $\theta_2^-(\phi(A))$ *is an ideal of* R_2.

2.11 Theorem. *[6] Let* $\phi : R_1 \to R_2$ *be an isomorphism from ring* R_1 *to a ring* R_2 *and let* A *be a subset of* R_1*. Then* $\theta_{1-}(A)$ *is an ideal of* R_1 *if and only if* $\theta_{2-}(\phi(A))$ *is an ideal of* R_2.

3 Rough Semi-prime Ideal

3.1 Definition. An ideal $P \neq R$ is said to be semi-prime if for any ideal A in R
$A^n \subseteq P$ implies $A \subseteq P$, for some positive integer n.
A subset P of a ring R is an upper rough semi-prime ideal of R if $\theta^-(P)$ is a semi-prime ideal of R. A subset P of a ring R is a lower rough semi-prime ideal of R if $\theta_-(P)$ is a semi-prime ideal of R.
Let P be a subset of a ring R and $(\theta_-(P), \theta^-(P))$ a rough set. If $\theta_-(P)$ and $\theta^-(P)$ are semi-prime ideals of R, then we call $(\theta_-(P), \theta^-(P))$ a rough semi-prime ideal.

3.2 Theorem. *Let* θ *be a complete congruence relation on* R*. If* P *is a semi-prime ideal of* R *such that* $\theta^-(P) \neq R$ *, then it is an upper rough semi-prime ideal of* R.

Proof. First of all note that P is an ideal by definition of a semi-prime ideal. By Theorem 2.8, we have $\theta^-(P)$ is an ideal of R. For any ideal A of R, assume $A^n \subseteq \theta^-(P)$. Let $x \in A$. Then $x^n \in A^n$. This implies, $x^n \in \theta^-(P)$. Then $[x^n] \cap P = [x]^n \cap P \neq \phi$. So there exists a $y \in P$ which is of the form $y = a^n$, $a \in [x]$. Since P is a semi-prime ideal of R, we have $a \in P$. Thus $[x] \cap P \neq \phi$. So $x \in \theta^-(P)$. This implies $A \subseteq \theta^-(P)$. Thus $\theta^-(P)$ is a semi-prime ideal of R. Therefore P is an upper rough semi-prime ideal of R.

3.3 Theorem. *Let θ be a full congruence relation on R and P be a semi-prime ideal of R. If $\theta_-(P) \neq \phi$, then $\theta_-(P)$ is a semi-prime ideal of R.*

Proof. The proof is straightforward by theorem 2.9.

3.4 Corollary. *If P is a semi-prime ideal of R and θ be a complete congruence relation on R, then P is a rough semi-prime ideal of R.*

Proof. This follows from Theorems 3.2 and 3.3.

3.5 Theorem. *Let $\phi : R_1 \to R_2$ be an isomorphism from ring R_1 onto a ring R_2 and let P be a subset of R_1. If $\theta_1^-(P)$ is a semi-prime ideal of R_1, then $\phi(\theta_1^-(P))$ is a semi-prime ideal of R_2.*

Proof. By theorems 2.10 and 2.3, $\phi(\theta_1^-(P))$ is an ideal of R_2. Let A be an ideal in R_2 such that $A^n \subseteq \phi(\theta_1^-(P))$. Then

$$\begin{aligned}
&\Rightarrow \phi^{-1}(A^n) \subseteq \theta_1^-(P) \\
&\Rightarrow (\phi^{-1}(A))^n \subseteq \theta_1^-(P) \qquad (\because \phi^{-1} \text{ is an homomorphism}) \\
&\Rightarrow \phi^{-1}(A) \subseteq \theta_1^-(P) \qquad (\because \theta_1^-(P) \text{ is a semi-prime ideal}) \\
&\Rightarrow A \subseteq \phi(\theta_1^-(P))
\end{aligned}$$

Therefore, $\phi(\theta_1^-(P))$ is a semi-prime ideal of R_2.

3.6 Theorem. *Let $\phi : R_1 \to R_2$ be an isomorphism from ring R_1 onto a ring R_2 and let P be a subset of R_1. If $\theta_{1-}(P)$ is a semi-prime ideal of R_1, then $\phi(\theta_{1-}(P))$ is a semi-prime ideal of R_2.*

Proof. The proof is similar to that of Theorem 3.5.

3.7 Theorem. *Isomorphic image of an upper rough semi-prime ideal is an upper rough semi-prime ideal.*

Proof. Let $\phi : R_1 \to R_2$ be an isomorphism from ring R_1 onto a ring R_2 and let P be an upper rough semi-prime ideal of R_1. Then by definition, $\theta_1^-(P)$ is a semi-prime ideal of R_1. By Theorem (3.5), $\phi(\theta_1^-(P))$ is a semi-prime ideal of R_2. By Theorem (2.3) , ${\theta_2}^-(\phi(P))$ is a semi-prime ideal of R_2. Therefore, $\phi(P)$ is an upper rough semi-prime ideal of R_2. Hence the theorem is proved.

3.8 Theorem. *Isomorphic image of a lower rough semi-prime ideal is a lower rough semi-prime ideal.*

Proof. The proof is similar to that of Theorem 3.7.

3.9 Theorem. *Isomorphic image of a rough semi-prime ideal is a rough semi-prime ideal.*

Proof. This follows from Theorems 3.7 and 3.8.

3.10 Theorem. *Let $\phi : R_1 \to R_2$ be an isomorphism from ring R_1 onto a ring R_2 and let P' be a subset of R_2 . If $\theta_2^-(P')$ is a semi-prime ideal of R_2, then $\phi^{-1}(\theta_2^-(P'))$ is a semi-prime ideal of R_1.*

Proof. By theorems 2.9 and 2.3, $\phi^{-1}(\theta_2^-(P'))$ is an ideal of R_1. Let A be an ideal in R_1 such that $A^n \subseteq \phi^{-1}(\theta_2^-(P'))$. Then

$$
\begin{aligned}
&\Rightarrow \phi(A^n) \subseteq \theta_2^-(P') \\
&\Rightarrow (\phi(A))^n \subseteq \theta_2^-(P') \qquad (\because \phi \text{ is a homomorphism}) \\
&\Rightarrow \phi(A) \subseteq \theta_2^-(P') \qquad (\because \theta_2^-(P') \text{ is a semi-prime ideal}) \\
&\Rightarrow A \subseteq \phi^{-1}(\theta_2^-(P'))
\end{aligned}
$$

Therefore, $\phi^{-1}(\theta_2^-(P'))$ is a semi-prime ideal of R_1.

3.11 Theorem. *Let $\phi : R_1 \to R_2$ be an isomorphism from a ring R_1 onto a ring R_2 and let P' be a subset of R_2. If $\theta_{2-}(P')$ is a semi-prime ideal of R_2, then $\phi^{-1}(\theta_{2-}(P'))$ is a semi-prime ideal of R_1.*

Proof. The proof is similar to the proof of Theorem 3.10.

3.12 Theorem. *Isomorphic pre-image of an upper rough semi-prime ideal is an upper rough semi-prime ideal.*

Proof. Let $\phi : R_1 \to R_2$ be an isomorphism from ring R_1 onto a ring R_2 and let P' be an upper rough semi-prime ideal of R_2. Then by definition, $\theta_2^-(P'))$ is a semi-prime ideal of R_2. By Theorem 3.10 , $\phi^{-1}(\theta_2^-(P'))$ is a semi-prime ideal of R_1. By Theorem (2.3) , $\theta_1^-(\phi^{-1}(P'))$ is a semi-prime ideal of R_1. Therefore, $\phi^{-1}(P')$ is an upper rough semi-prime ideal of R_1. Hence the theorem is proved.

3.13 Theorem. *Isomorphic pre-image of a lower rough semi-prime ideal is a lower rough semi-prime ideal.*

Proof. The proof is similar to the proof of Theorem 3.12.

3.14 Theorem. *Isomorphic pre-image of a rough semi-prime ideal is a rough semi-prime ideal.*

Proof. This follows from Theorems 3.13 and 3.14.

The following theorems can be proved in a similar way as the corresponding theorems with homomorphism.

3.15 Theorem. *Anti-isomorphic image of an upper rough semi-prime ideal is an upper rough semi-prime ideal.*

3.16 Theorem. *Anti-isomorphic image of a lower rough semi-prime ideal is a lower rough semi-prime ideal.*

3.17 Theorem. *Anti-isomorphic image of a rough semi-prime ideal is a rough semi-prime ideal.*

3.18 Theorem. *Anti-isomorphic pre-image of an upper rough semi-prime ideal is an upper rough semi-prime ideal.*

3.19 Theorem. *Anti-isomorphic pre-image of a lower rough semi-prime ideal is a lower rough semi-prime ideal.*

3.20 Theorem. *Anti-isomorphic pre-image of a rough semi-prime ideal is a rough semi-prime ideal.*

4 Rough Bi-Ideal

4.1 Definition. A subring A of a ring R is called a bi-ideal of R if $ARA \subseteq A$.
A subset A of a ring R is an upper rough bi-ideal of R if $\theta^-(A)$ is a bi-ideal of R. A subset A of a ring R is an lower rough bi-ideal of R if $\theta_-(A)$ is a bi-ideal of R.
Let A be a subset of a ring R and $(\theta_-(A), \theta^-(A))$ a rough set. If $\theta_-(A)$ and $\theta^-(A)$ are bi-ideals of R, then we call $(\theta_-(A), \theta^-(A))$ a rough bi-ideal.

4.2 Theorem. *Let θ be a full congruence relation on R. If A is a bi-ideal of R, then it is an upper rough bi-ideal of R.*

Proof. Let A be a bi-ideal of R. Then $ARA \subseteq A$. By Theorem 2.5 , we have $\theta^-(ARA) \subseteq \theta^-(A)$. Now $\theta^-(A)R\theta^-(A) = \theta^-(A)\theta^-(R)\theta^-(A)$. By theorem 2.6, we get $\theta^-(A)R\theta^-(A) \subseteq \theta^-(ARA)$. This implies $\theta^-(A)R\theta^-(A) \subseteq \theta^-(A)$. Thus $\theta^-(A)$ is a bi-ideal of R. Therefore A is an upper rough bi-ideal of R.

4.3 Theorem. *Let θ be a complete congruence relation on R. If A is a bi-ideal of R, then $\theta_-(A)$ is a bi-ideal of R.*

Proof. Let A be a bi-ideal of R. Then $ARA \subseteq A$. By Theorem 2.5, we have $\theta_-(ARA) \subseteq \theta_-(A)$. Now $\theta_-(A)R\theta_-(A) = \theta_-(A)\theta_-(R)\theta_-(A)$. By theorem 2.7, we get $\theta_-(A)R\theta_-(A) \subseteq \theta_-(ARA)$. This implies $\theta_-(A)R\theta_-(A) \subseteq \theta_-(A)$. Thus $\theta_-(A)$ is a bi-ideal of R.

4.4 Corollary. *If A is a bi-ideal of R and θ a complete congruence relation on R, then A is a rough bi-ideal of R.*

Proof. This follows from Theorems 4.2 and 4.3.

4.5 Theorem. *Let $\phi : R_1 \to R_2$ be a homomorphism from ring R_1 onto a ring R_2 and let A be a subset of R_1. If $\theta_1^-(A)$ is a bi-ideal of R_1, then $\phi(\theta_1^-(A))$ is a bi- ideal of R_2.*

Proof. Let $\theta_1^-(A)$ be a bi-ideal of R_1. Then

$$\begin{aligned}
&\Rightarrow \quad \theta_1^-(A)R_1\theta_1^-(A) \subseteq \theta_1^-(A) \\
&\Rightarrow \quad \phi(\theta_1^-(A)R_1\theta_1^-(A)) \subseteq \phi(\theta_1^-(A)) \\
&\Rightarrow \quad \phi(\theta_1^-(A))\phi(R_1)\phi(\theta_1^-(A)) \subseteq \phi(\theta_1^-(A)) \qquad (\because \phi \text{ is a homomorphism}) \\
&\Rightarrow \quad \phi(\theta_1^-(A))R_2\phi(\theta_1^-(A)) \subseteq \phi(\theta_1^-(A)) \qquad (\because \phi \text{ is a onto})
\end{aligned}$$

Therefore, $\phi(\theta_1^-(A))$ is a bi- ideal of R_2.

4.6 Theorem. *Let $\phi : R_1 \to R_2$ be a homomorphism from ring R_1 onto a ring R_2 and let A be a subset of R_1. If ${\theta_1}_-(A)$ is a bi-ideal of R_1, then $\phi({\theta_1}_-(A))$ is a bi- ideal of R_2.*

Proof. The proof is similar to that of 4.5.

4.7 Theorem. *Homomorphic image of an upper rough bi-ideal is an upper rough bi-ideal.*

Proof. Let $\phi : R_1 \to R_2$ be a homomorphism from a ring R_1 onto a ring R_2 and let A be an upper rough bi-ideal of R_1. Then by definition, $\theta_1^-(A)$ is a bi-ideal of R_1. By Theorem 4.5, $\phi(\theta_1^-(A))$ is a bi-ideal of R_2. By Theorem 2.3, ${\theta_2}^-(\phi(A))$ is a bi- ideal of R_2. Therefore, $\phi(A)$ is an upper rough bi-ideal of R_2, proving the theorem.

4.8 Theorem. *Isomorphic image of a lower rough bi-ideal is a lower rough bi-ideal.*

Proof. The proof is similar to that of 4.7.

4.9 Corollary. *Isomorphic image of a rough bi-ideal is a rough bi-ideal.*

Proof. This follows from Theorems 4.7 and 4.8.

4.10 Theorem. *Homomorphic pre-image of an upper rough bi-ideal is an upper rough bi-ideal.*

Proof. Let $\phi : R_1 \to R_2$ be a homomorphism from ring R_1 onto a ring R_2 and let A' be an upper rough bi-ideal of R_2. Then $\theta_2^-(A')$ is a bi-ideal of R_2. Then by definition, $\theta_2^-(A') \;\; R_2 \;\; \theta_2^-(A')$ $\subseteq \;\; \theta_2^-(A')$.
Now $\phi^{-1}(\theta_2^-(A'))R_1\phi^{-1}(\theta_2^-(A')) = \phi^{-1}(\theta_2^-(A')R_2\theta_2^-(A'))$. This implies $\phi^{-1}(\theta_2^-(A'))R_1\phi^{-1}(\theta_2^-(A'))$ $\subseteq \;\phi^{-1}(\theta_2^-(A'))$. Thus, $\phi^{-1}(\theta_2^-(A'))$ is a bi-ideal of R_1. Hence the theorem is proved.

4.11 Theorem. *Isomorphic pre-image of a lower rough bi-ideal is a lower rough bi-ideal.*

Proof. The proof is similar to that of 4.10.

4.12 Theorem. *Isomorphic pre-image of a rough bi-ideal is a rough bi-ideal.*

Proof. This follows from Theorems 4.10 and 4.11.

The following theorems can be proved in similar way as the corresponding theorems in homomorphism.

4.13 Theorem. *Anti-homomorphic image of an upper rough bi-ideal is an upper rough bi-ideal.*

4.14 Theorem. *Anti-isomorphic image of a lower rough bi-ideal is a lower rough bi-ideal.*

4.15 Theorem. *Anti-isomorphic image of a rough bi-ideal is a rough bi-ideal.*

4.16 Theorem. *Anti-homomorphic pre-image of an upper rough bi-ideal is an upper rough bi-ideal.*

4.17 Theorem. *Anti-isomorphic pre-image of a lower rough bi-ideal is a lower rough bi-ideal.*

4.18 Corollary. *Anti-isomorphic pre-image of a rough bi-ideal is a rough bi-ideal.*

4.19 Theorem. *Every rough left (right) ideal is a rough bi-ideal.*

Proof. Let A be a rough left ideal of R. Then $\theta_{-}(A)$ and $\theta^{-}(A)$ are left ideals of R. Now $\theta^{-}(A)$ $R\, \theta^{-}(A) \subseteq \theta^{-}(A).\theta^{-}(A)$. This implies $\theta^{-}(A)\; R\, \theta^{-}(A) \subseteq \theta^{-}(A)$. Thus, $\theta^{-}(A)$ is a bi-ideal of R. Similarly, $\theta_{-}(A)$ is a bi-ideal of R. Therefore, A is a rough bi-ideal of R. The proof of the other part is similar.

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