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Differential Quadrature Method for the General Singular Perturbation Problems

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Abstract

This paper extends the application of Differential Quadrature Method (DQM) for finding the numerical solution of general singularly perturbed two point boundary value problems with a boundary layer at right end point or both end point or at an internal point. The Differential Quadrature Method is an efficient descritization technique in solving initial and /or boundary value problems accurately using a considerably small number of grid points. To demonstrate the applicability of the method, we have solved model examples with right end boundary layer, internal layer and two boundary layers and presented the computational results. It is observed that the computed result approximates the exact solution with high accuracy and efficiency.

Keywords: *Boundary layer, Differential Quadrature Method, Ordinary Differential-Equation, Singular perturbation, Two point boundary value problem.*

1 Introduction

Differential equations with small parameter (the perturbation parameter ε) multiplying the highest derivatives are called singularly perturbed differential equations. These problems have received a significant amount of attention in past and recent years. The smoothness of the solutions of such singularly perturbed differential equations deteriorates when the parameter tends to zero. This type of problems arise in many fields such as, fluid mechanics, fluid dynamics, chemical reactor theory, elasticity, aerodynamics, and the other domain of the great world of fluid motion. A few notable examples are Boundary layer problems, WKB problems, the modelling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. Therefore, the numerical treatment of singularly perturbed problems presents some major computational difficulties. If we apply the existing standard numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behaviour. Thus, more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems.

The survey paper by Kadalbajoo and Reddy [1], gives an erudite outline of the singular perturbation problems and their treatment starting from Prandtl's paper [2] on fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on singular perturbation problems. For a detailed discussion on singular perturbation problems one may refer to the books and high level monographs: O'Malley [3,4], Nayfeh [5,6], Kevorkian and Cole [7], Bender and Orszag [8], Hemkar and Miller [9].

In this paper the DQM has been used for solving general singularly perturbed two-point boundary value problems with right end boundary layer, an internal layer and two boundary layers and presented the computational results. It is observed that the DQM approximates the result with high accuracy and efficiency with less number of sampling points.

2 Description of the Differential Quadrature Method

The Differential Quadrature Method(DQM) was introduced by Bellman et al.[10,11] in the early 1970s and, since then, the technique has been successfully employed in finding the solutions of many problems in applied and physical sciences (Shu[12]; Shu and Richards[13]; Yucel[14]).

The basic idea of differential quadrature method is that the derivative of a function with respect to a space variable at a given point is approximated as a weighted linear sum of the functional values at all discrete points in the domain of that variable.

In order to show the mathematical representation of the method, we consider a one dimensional field variable $f(x)$ prescribed in a field domain $a = x_1 \le x \le x_n = b$. Let $f_i = f(x_i)$ be the function values specified in a finite set of N discrete points $x_i, (i = 1,2,......,N)$ of the field domain in which the end points x_1 and x_N are included. Next, consider the value of the function derivative $d^r f / dx^r$ at some discrete points x_i, and let it be expressed as a linearly weighted some of the function values.

$$f^{(r)}(x_i) = \frac{d^r f(x_i)}{dx^r} = \sum_{J=1}^{N} A_{ij}^{(r)} f_j \left(i = 1,2,\ldots\ldots, N\right) \tag{1}$$

where $A_{ij}^{\ r}$ are the weighting coefficients of the r^th^-order derivative of the function associated with points x_i. Equation (1) the quadrature rule for a derivative is the essential basis of the Differential Quadrature Method. Thus using equation (1) for various order derivatives, one may write a given differential equation at each point of its solution domain and obtain the quadrature analog of the differential equation as a set of algebraic equations in terms of the N function values. These equations may be solved, in conjunction with the quadrature analog of the boundary conditions, to obtain the unknown function values provided that the weighting coefficients are known a priori.

The weighting coefficients may be determined by some appropriate functional approximations; and the approximate functions are referred to as test functions. The primary requirements for the choices of the test functions are of differentiability and smoothness. That is, the test function of the differential equation must be differentiable at least up to the nth derivative (here n is the highest order of the differential equation) and sufficiently smooth to be satisfied the condition of the differentiability.

It is supposed that the solution of a one–dimensional differential equation is approximated by a (N-1)th degree polynomial:

$$f(x) = \sum_{k=1}^{N} c_k . x^{k-1} \tag{2}$$

where c_k is a constant.

Although there can be many choices of the test functions, a convenient and most commonly used choice in one-dimensional problems is the Lagrangian interpolation shape functions $l_j(x)$, where

$$f(x) = \sum_{j=1}^{N} l_j(x) f_j \tag{3}$$

where $l_j(x)$ are the monomials of the (N-1)th order polynomials.

Note that the number of test functions is equal to the number of the sampling points. For completeness, the number of the sampling points should at least be equal to one plus the order of the highest derivatives.

Substituting $l_j(x)$ of equation (3) into equation (1), it may be seen that the weighting coefficients can be easily obtained. The detailed procedures can be found in references (Shu and Rechards [13], Quan and Chang [15]).

2.1 The polynomial test function-based weighting coefficients

The accuracy of differential quadrature solution depends on the accuracy of the weighting coefficients. To obtain accurate weighting coefficients, Quan and Chang [15] derived explicit formulae of the Lagrangian-interpolation-function-based weighting coefficients for the first and second-order derivatives. Shu and Rechards [13] gave a general relationship for any higher order derivatives. These formulae were obtained by considering the test function in the Lagrangian interpolation process as in eq.(1) and (3). These explicit formulae's merit is that highly accurate weighting coefficients may be determined for any number of arbitrarily spaced sampling points.

Villadsen and Michelsen [16] and Quan and Chang [15] have shown that the weighting coefficients of rth-order derivatives of the Lagrangian interpolation test functions are

$$A_{ij}^{(r)} = \frac{d^r}{dx^r} l_j(x_i) \qquad (i, j = 1,2,.......N) \tag{4}$$

where

$$l_j(x) = \frac{\phi(x)}{(x - x_j)\phi^{(1)}(x_j)}; \quad \phi(x) = \prod_{m=1}^{N}(x - x_m);$$

$$\phi^{(1)}(x_j) = \frac{d\phi(x_j)}{dx} = \prod_{m=1; m \neq j}^{N}(x_j - x_m)$$

and x_i 's are the locations of the grid points. N is the number of sampling points. Note that the eq. (4) is valid as long as linearly independent polynomials are used as a trial functions and, thus, the values of the coefficients are affected only by the distribution of the grid points.

Note that the lagrangian interpolation shape functions $l_j(x)$ have following

properties:

$$l_j(x_i) = \delta_{ij} = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases} \tag{5}$$

Using Eqs. (1), (3), and (4) based on Lagrangian interpolation shape functions, Quan and Chang [15] and Shu and Rechards [13] obtained the following weighting coefficients:

$$A_{ij}^{(1)} = \frac{dl_j(x_i)}{dx} = \frac{\phi^{(1)}(x_i)}{(x_i - x_j)\phi^{(1)}(x_j)}, \qquad (i, j = 1,2,.......N; \quad i \neq j)$$

$$A_{ij}^{(r)} = \frac{d^r l_j(x_i)}{dx^r} = r\left(A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{(x_i - x_j)}\right), \qquad (i, j = 1,2,......N;\ i \neq j; r \geq 2)$$

$$A_{ii}^{(r)} = \frac{d^r l_i(x_i)}{dx^r} = -\sum_{j=1; i\neq j}^{N} A_{ij}^{(r)} \quad ,(i = 1,2,......N;\ r \geq 1) \tag{6}$$

2.2 Choice of sampling points

A convenient and natural choice for the sampling points is that of the equally spaced points. But the Differential Quadrature solutions usually deliver more accurate results with unequally spaced sampling points. A rational basis for the sampling points is provided by the zeros of the orthogonal polynomials. A well accepted kind of sampling points in the DQM is the so called Gauss-Lobatto-Chebyshev sampling points. For a domain specified by $a \leq x \leq b$ and discretised by a set of unequally spaced points (non-uniform grid), then the coordinate of any point i can be evaluated by:

$$x_i = a + \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right)(b-a) \tag{7}$$

2.3 Application to differential equation

The basic key procedure in the DQM is to approximate the derivatives in a differential equation by equation (1). Substituting the equation (1) into the governing equations and equating both sides of the governing equations, we obtain simultaneous equations which can be solved by use of Gauss elimination or other methods. That is, DQM is composed of the following procedure:

(a) The function to be determined is replaced by a group of function values at a group of selected sampling points. Gauss-Lobatto-Chebyshev sampling points are strongly recommended for numerical stability.
(b) Approximate derivatives in a differential equation by these N unknown function values.
(c) Form a system of linear equations and
(d) Solving the system of linear equation yields the desired unknowns.

2.4 Implementation of boundary condition

The proper implementation of boundary condition is very important for the accurate numerical solution of differential equation. Essential and natural boundary condition can be approximated by DQM. Using the technique in solving differential equation, the governing equations are actually satisfied at each sampling point of the domain, so one has one equation for each point, for each unknown. In the resulting system of algebraic equation from the DQM, each boundary condition replaces the corresponding field equation. This procedure is straightforward when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary.

3 Application to singular perturbation problems

To show the applicability of DQM, we consider the general singularly perturbed two point boundary value problems of the form:

$$\varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad p \le x \le q \tag{8}$$

With $y(p) = \alpha$ and $y(q) = \beta$ (9)

Where ε is a small parameter $0 < \varepsilon \le 1; \alpha, \beta, p, q$ are given constants; $a(x), b(x),$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[p,q]$. The interval $[p,q]$ is either the interval $[0,1]$ or the interval $[-1,1]$.

For finding the solution of the equation (8) with the boundary condition (9) by DQM, we first discritise the interval $[p,q]$, such that $p = x_1 < x_2 < x_3 < \ldots\ldots < x_N = q$ where N is the number of sampling points. We denote $y_i = y(x_i),\ f_i = f(x_i),$ etc. Then we apply the DQM to approximate the derivatives in the equation (8), that leads to the following discritised form of the equation:

$$\varepsilon \sum_{k=1}^{N} A_{ik}^{(2)} y_k + a_i \sum_{k=1}^{N} A_{ik}^{(1)} y_k + b_i y_i - f_i = 0. \quad (i = 1,2,\ldots\ldots, N) \tag{10}$$

Now we apply the equation (10) at all interior points $x_i, i = 2,3,\ldots\ldots\ldots\ldots, N-1$. This application leads to a system of (N-2) equations with N unknowns. By applying boundary conditions (9), we get a system of (N-2) equations with (N-2) unknowns, which can be solved by Gaussian elimination or other methods. We have applied the Gaussian elimination method with partial pivoting and employed the double precision Fortran, to solve the obtained system of linear equations, for the unknowns $y_2, y_3, \ldots\ldots., y_{N-1}$.

Note that the DQM results are given at uniform grids which have been interpolated through the use of natural cubic spline interpolation polynomial. For the derivation of natural cubic spline interpolation polynomial, we have used the DQM results (x_i, y_i), $i = 1,2,\ldots\ldots, N$, where $y_i, i = 1,2,\ldots\ldots, N$ are the value of y at non-uniform grid points(Gauss - Lobatto - Chebyshev points) $x_i, i = 1,2,\ldots\ldots\ldots, N$. We have compared the DQM results at uniform grid points with the exact/asymptotic solution for different values of N and ε.

4 Numerical illustrations

To demonstrate the applicability of the **DQM**, we have applied it to

(a) two singular perturbation problems with right end boundary layer,

(b) one singular perturbation problem with an internal layer and

(c) two singular perturbation problem with two boundary layers.

These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

4.1 Right-end boundary layer problems

To show the applicability of the DQM for solving problems with the boundary layer at the right end of the underlying interval , we consider the following singular perturbation problem:

$$\varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad 0 \le x \le 1 \qquad (11)$$

$$\text{with } y(0) = \alpha \quad \text{and} \quad y(1) = \beta \qquad (12)$$

where ε is a small positive parameter $0 < \varepsilon << 1$; α, β are given constants; $a(x), b(x),$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \le M < 0$ throughout the interval $[0,1]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of the point $x = 1$.

Example 4.1.1: Consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) = 0;\ x \in [0,1]$,
with $y(0) = 1$ and $y(1) = 0$.

For this example we have $a(x) = -1, b(x) = 0$ and $f(x) = 0$.
Further, we have a boundary layer of width $O(\varepsilon)$ at $x = 1$.

The exact solution is given by:
$$y(x) = \frac{1 - \exp((x-1)/\varepsilon)}{1 - \exp(-1/\varepsilon)}$$

The computational results are presented in Table 4.1.1(a) and 4.1.1(b), for different values of N and ε.

Table 4.1.1(a): Computational results for example-4.1.1

x	Exact solution- $y(x)$	DQ Solution- $y(x)$ $N = 72$, $\varepsilon = 0.001$	DQ Solution- $y(x)$, $N = 101$, $\varepsilon = 0.001$
0.00	1.0000000	1.0000000	1.0000000
0.20	1.0000000	0.9987986	0.9999996
0.40	1.0000000	0.9988069	0.9999984
0.60	1.0000000	1.0017060	0.9999984
0.80	1.0000000	1.0017140	0.9999995
0.90	1.0000000	1.0003732	0.9999993
0.92	1.0000000	1.0016751	0.9999961
0.94	1.0000000	1.0012183	0.9999962
0.96	1.0000000	1.0014203	1.0000042
0.98	1.0000000	1.0000278	1.0000051
1.00	0.0000000	0.0000190	0.0000206

Table 4.1.1(b): Computational results for example-4.1.1

x	Exact solution- $y(x)$	DQ Solution- $y(x)$ $N = 251$, $\varepsilon = .0001$	DQ Solution- $y(x)$ $N = 264$, $\varepsilon = .0001$
0.00	1.0000000	1.0000000	1.0000000
0.20	1.0000000	0.9996943	1.0000740
0.40	1.0000000	1.0003622	0.9999588
0.60	1.0000000	1.0003626	1.0000804
0.80	1.0000000	0.9996945	0.9999650
0.90	1.0000000	1.0002018	0.9998627
0.92	1.0000000	0.9998296	0.9998132
0.94	1.0000000	1.0000487	1.0000483
0.96	1.0000000	0.9995581	0.9999309
0.98	1.0000000	1.0000148	0.9999144
1.00	0.0000000	0.0001958	1.0001985

Example 4.1.2: Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0;\ x \in [0,1],$$

with $y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$; and $y(1) = 1 + 1/e$.

For this example we have $a(x) = -1, b(x) = -(1+\varepsilon)$ and $f(x) = 0$.
Further we have a boundary layer of width $O(\varepsilon)$ at $x = 1$.

The exact solution is given by:
$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$$

The computational results are presented in Table 4.1.2(a) and 4.1.2(b), for different values of N and ε.

Table 4.1.2(a): Computational results for example-4.1.2

x	Exact solution- $y(x)$	DQ Solution- $y(x)$, $N=86$, $\varepsilon=0.001$	DQ Solution- $y(x)$, $N=101$, $\varepsilon=0.001$
0.00	1.0000000	1.0000000	1.0000000
0.20	0.8187308	0.8188494	0.8187472
0.40	0.6703201	0.6704523	0.6703402
0.60	0.5488116	0.5486716	0.5488289
0.80	0.4493290	0.4492024	0.4493375
0.90	0.4065697	0.4065426	0.4065744
0.92	0.3985190	0.3985219	0.3985257
0.94	0.3906278	0.3905943	0.3906339
0.96	0.3828929	0.3827839	0.3828901
0.98	0.3753111	0.3753479	0.3753064
1.00	1.3678794	1.3678595	1.3678588

Table 4.1.2(b): Computational results for example-4.1.2

x	Exact solution- $y(x)$	DQ Solution— $y(x)$ $N=272$, $\varepsilon=.0001$	DQ Solution- $y(x)$ $N=301$, $\varepsilon=.0001$
0.00	1.0000000	1.0000000	1.0000000
0.20	0.8187308	0.8185887	0.8187294
0.40	0.6703201	0.6702133	0.6703083
0.60	0.5488116	0.5489048	0.5487999
0.80	0.4493290	0.4494549	0.4493274
0.90	0.4065697	0.4065670	0.4065675
0.92	0.3985190	0.3985064	0.3985069
0.94	0.3906278	0.3905718	0.3906168
0.96	0.3828929	0.3828253	0.3828949
0.98	0.3753111	0.3753123	0.3752930
1.00	1.3678794	1.3676792	1.3676750

4.2 Internal layer problems

We will now demonstrate the applicability of the DQM technique for solving singular perturbation problems with an internal layer of the underlying interval. In this case a(x) changes sign in the domain of interest. Without loss of generality, we can take $a(0) = 0$, and the interval to be $[-1,1]$.
To describe the method we shall consider a class of linear singularly perturbed two point boundary value problems of the form:

$$\varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad -1 \le x \le 1 \tag{13}$$

with

$$y(-1) = \alpha \tag{14}$$

and

$$y(1) = \beta \tag{15}$$

where ε is a small parameter $0 < \varepsilon << 1; \alpha, \beta$ are given constants; $a(x), b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[-1,1]$. Furthermore, we assume that $a(x) \le M < 0$ throughout the interval $[-1,0]$ where M is some negative constant and $a(x) \ge M > 0$ throughout the interval $[0,1]$ where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

We now proceed as follows:

Step 1: We first find the approximate solution at $x = 0$. Without loss of generality we can take a(0)=0.

At $x = 0$ equation (13) becomes

$$\varepsilon\, y''(0) + b(0)y(0) = f(0) \tag{16}$$

The reduced problem of (16) gives us an approximation to $y(0)$.

$$\therefore\ y(0) = \frac{f(0)}{b(0)} = \mu \text{ (say)} \tag{17}$$

Step 2: We now divide the interval $[-1,1]$ into two sub intervals $[-1,0]$ and $[0,1]$ so that equation (13) has a right layer in $[-1,0]$ and a left layer in $[0,1]$. In this way we have divided the original problem into two problems:

$$\varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad -1 \le x \le 0 \tag{18}$$

with

$$y(-1) = \alpha \quad \text{and} \quad y(0) = \mu\ . \tag{19}$$

and

$$\varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad 0 \le x \le 1 \tag{20}$$

with

$$y(0) = \mu \quad \text{and} \quad y(1) = \beta \tag{21}$$

Step 3: We now apply DQM to solve these problems to obtain solutions over the intervals $[-1,0]$ and $[0,1]$ respectively.

Example 4.2.1: Consider the following singular perturbation problem

$\varepsilon y''(x) + xy'(x) - y(x) = 0;\ x \in [-1,1],$

with

$y(-1) = 1;$ and $y(1) = 2.$

For this example we have $a(x) = x, b(x) = -1$ and $f(x) = 0.$

Further we have an internal layer of width $O(\sqrt{\varepsilon})$ at $x = 0.$ (For details, see O'Malley [3, pp.168-172, Eq.(8.1),case(i)], and Kevorkian and Cole [7, pp. 41-43, Eqs. (2.3.76) and (2.3.77)]).

We see that the function

$a(x) = x < 0$ for $-1 \le x < 0,$

$a(x) = x = 0$ for $x = 0,$

$a(x) = x > 0$ for $0 < x \le 1.$

The asymptotic solution is given by:

$y(x) = -x,$ for $x < 0$

$y(x) = 3\sqrt{\varepsilon / 2\pi}$, for $x = 0$

$y(x) = 2x$, for $x > 0$

Step 1: $y(0) = \dfrac{f(0)}{b(0)} = 0 = \mu$

Step 2: We now divide the interval $[-1,1]$ into two sub intervals $[-1,0]$ and $[0,1]$ so that equation (13) has a right layer in $[-1,0]$ and a left layer in $[0,1]$. In this way we have divided the original problem into two problems:

In the interval $[-1,0]$ we have right layer. The problem is

$$\varepsilon\, y''(x) + xy'(x) - y(x) = 0; \qquad -1 \le x \le 0$$

with $y(-1) = 1$ and $y(0) = 0$.

And in the interval $[0,1]$ we have left layer. The problem is

$$\varepsilon\, y''(x) + xy'(x) - y(x) = 0; \qquad 0 \le x \le 1$$

With $y(0) = 0$ and $y(1) = 2$

Step 3: We now apply DQM to solve these problems to obtain solutions over the intervals $[-1,0]$ and $[0,1]$ respectively.

The computational results are presented in Table 4.2.1(a), 4.2.1(b) , 4.2.1(c), 4.2.1(d), 4.2.1(e) and 4.2.1(f) for different values of N and ε.

Table 4.2.1(a): Computational results for example-4.2.1

x	Asymptotic solution.- $y(x)$	DQ solution- $y(x)$, over[-1,0] $N = 32, \varepsilon = 0.01$	DQ solution- $y(x)$, over[-1,0] $N = 51, \varepsilon = 0.01$
-1.00	1.0000000	1.0000000	1.0000000
-.50	.5000000	.4999998	.5000005
-.10	.1000000	.0999999	.1000001
-.08	.0800000	.0799999	.0800000
-.06	.0600000	.0600000	.0600000
-.04	.0400000	.0400000	.0400000
-.02	.0200000	.0200000	.0200000
.00	.0000000	.0000000	.0000000

Table 4.2.1(b): Computational results for example-4.2.1

x	Asymptotic solution.- $y(x)$	DQ solution- $y(x)$, over[0,1] $N = 32, \varepsilon = 0.01$	DQ solution- $y(x)$, over[0,1] $N = 51, \varepsilon = 0.01$
.00	.0000000	.0000000	.0000000
.02	.0400000	.0400000	.0400001
.04	.0800000	.0800000	.0800001
.06	.1200000	.1200001	.1200002
.08	.1600000	.1600001	.1600002
.10	.2000000	.2000001	.2000002
.50	1.0000000	.9999999	1.0000008
1.00	2.0000000	2.0000000	2.0000000

Table 4.2.1(c): Computational results for example-4.2.1

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[-1,0] $N = 64, \varepsilon = 0.001$	DQ solution- $y(x)$, over[-1,0] $N = 82, \varepsilon = 0.001$
-1.00	1.0000000	1.0000000	1.0000000
-.50	.5000000	.4999995	.4999998
-.10	.1000000	.0999999	.1000000
-.08	.0800000	.0799999	.0800000
-.06	.0600000	.0600000	.0600000
-.04	.0400000	.0400000	.0400000
-.02	.0200000	.0200000	.0200000
.00	.0000000	.0000000	.0000000

Table 4.2.1(d): Computational results for example-4.2.1

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[0,1] $N = 64, \varepsilon = 0.001$	DQ solution- $y(x)$, over[0,1] $N = 82, \varepsilon = 0.001$
.00	.0000000	.0000000	.0000000
.02	.0400000	.0399999	.0399998
.04	.0800000	.0799999	.0799998
.06	.1200000	.1199999	.1199999
.08	.1600000	.1599999	.1600000
.10	.2000000	.1999998	.1999999
.50	1.0000000	1.0000000	.9999993
1.00	2.0000000	2.0000000	2.0000000

Table 4.2.1(e): Computational results for example-4.2.1

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[-1,0] $N = 51, \varepsilon = 0.0001$	DQ solution- $y(x)$, over[-1,0] $N = 82, \varepsilon = 0.0001$
-1.00	1.0000000	1.0000000	1.0000000
-.50	.5000000	.5000000	.5000000
-.10	.1000000	.1000000	.1000000
-.08	.0800000	.0800000	.0800000
-.06	.0600000	.0600000	.0600000
-.04	.0400000	.0400000	.0400000
-.02	.0200000	.0200000	.0200000
.00	.0000000	.0000000	.0000000

Table 4.2.1(f): Computational results for example-4.2.1

x	Asymptotic Solution- $y(x)$	DQ solution- $y(x)$, over[0,1] $N = 51, \varepsilon = 0.0001$	DQ solution- $y(x)$, over[0,1] $N = 82, \varepsilon = 0.0001$
.00	.0000000	.0000000	.0000000
.02	.0400000	.0400000	.0400000
.04	.0800000	.0800000	.0800000
.06	.1200000	.1200000	.1200000
.08	.1600000	.1599999	.1600000
.10	.2000000	.1999999	.2000000
.50	1.0000000	.9999998	.9999999
1.00	2.0000000	2.0000000	2.0000000

4.3 Problems with two boundary layers

The suggestions given for internal layer problems can be extended mutatis mutandis to problems with two boundary layers.
To describe the method we shall consider a class of linear singularly perturbed two point boundary value problems of the form:

$$\varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad -1 \le x \le 1 \tag{22}$$

with

$$y(-1) = \alpha \tag{23}$$

and

$$y(1)=\beta \tag{24}$$

where ε is a small parameter $0<\varepsilon<<1; \alpha, \beta$ are given constants; $a(x), b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[-1,1]$. Furthermore, we assume that $a(x) \geq M>0$ throughout the interval $[-1,0]$ where M is some positive constant and $a(x) \leq M<0$ throughout the interval $[0,1]$ where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x=-1$ and 1. Without loss of generality, we can take $a(x)=0$ at $x=0$ since it changes sign in the domain of interest.

We now proceed as follows:

Step 1: We first find the approximate solution at $x=0$. Without loss of generality we can take $a(0)=0$. At $x=0$ equation (22) becomes

$$\varepsilon y''(0)+b(0) y(0)=f(0) \tag{25}$$

The reduced problem of (25) gives us an approximation to $y(0)$.

$$\therefore\ y(0)=\frac{f(0)}{b(0)}=\mu \text{ (say)} \tag{26}$$

Step 2: We now divide the interval $[-1,1]$ into two sub intervals $[-1,0]$ and $[0,1]$ so that equation (13) has a right layer in $[-1,0]$ and a left layer in $[0,1]$.

In this way we have divided the original problem into two problems:

$$\varepsilon y''(x)+a(x) y'(x)+b(x) y(x)=f(x); \qquad -1 \leq x \leq 0 \tag{27}$$

with

$$y(-1)=\alpha \quad \text{and} \quad y(0)=\mu\ . \tag{28}$$

and

$$\varepsilon y''(x)+a(x) y'(x)+b(x) y(x)=f(x); \qquad 0 \leq x \leq 1 \tag{29}$$

with

$$y(0)=\mu \quad \text{and} \quad y(1)=\beta \tag{30}$$

Step 3: We now apply DQM to solve these problems to obtain solutions over the intervals $[-1,0]$ and $[0,1]$ respectively.

Example 4.3.1: Consider the following singular perturbation problem

$$\varepsilon y''(x) - xy'(x) - y(x) = 0;\ x \in [-1,1],$$

with $y(-1) = 1$; and $y(1) = 2$.

For this example we have $a(x) = -x, b(x) = -1$ and $f(x) = 0$.

Further we have two boundary layers one at $x = -1$ and one at $x = 1$. (For details, see O'Malley [3, pp.168-173, Eq.(8.1),case(ii)].

We see that the function

$a(x) = -x > 0$ for $-1 \le x < 0,$

$a(x) = -x = 0$ for $x = 0,$

$a(x) = -x < 0$ for $0 < x \le 1,$

The asymptotic solution is given by:

$y(x) = 1,$ for $x = -1$

$y(x) = 0,$ for $-1 < x < 1$

$y(x) = 2,$ for $x = 1$

Step 1: $y(0) = \dfrac{f(0)}{b(0)} = 0 = \mu$

Step 2:

In the interval $[-1,0]$ we have left layer. The problem is

$\varepsilon\, y''(x) - xy'(x) - y(x) = 0; \qquad -1 \le x \le 0$

with $y(-1) = 1$ and $y(0) = 0$.

and in the interval $[0,1]$ we have right layer. The problem is

$\varepsilon\, y''(x) - x\, y'(x) - y(x) = 0; \qquad 0 \le x \le 1$

with $y(0) = 0$ and $y(1) = 2$

Step 3: We now apply DQM to solve these problems to obtain solutions over the intervals $[-1,0]$ and $[0,1]$ respectively.

The computational results are presented in Table 4.3.1(a), 4.3.1(b) , 4.3.1(c) and 4.3.1(d) for different values of N and ε.

Table 4.3.1(a), Computational results for example-4.3.1

x	DQ solution- $y(x)$, over [-1,0] $N = 64, \varepsilon = 0.01$	DQ solution- $y(x)$, over [-1,0] $N = 96, \varepsilon = 0.01$
-1.00	1.0000000	1.0000000
-.98	.1454163	.1416148
-.96	.0203409	.0203505
-.94	.0031414	.0030004
-.92	.0005549	.0004954
-.90	.0000835	.0000823
-.70	.0000000	.0000000
-.50	.0000000	.0000000
-.10	.0000000	.0000000
.00	.0000000	.0000000

Table 4.3.1(b), Computational results for example-4.3.1

x	DQ solution- $y(x)$, over[0,1] $N = 64, \varepsilon = 0.01$	DQ solution- $y(x)$, over[0,1] $N = 96, \varepsilon = 0.01$
.00	.0000000	.0000000
.50	.0000000	.0000000
.70	.0000000	.0000000
.90	.0001671	.0001646
.92	.0011097	.0009909
.94	.0062827	.0060007
.96	.0406818	.0407012
.98	.2908318	.2832308
1.00	1.9999956	1.9999956

Table 4.3.1(c), Computational results for example-4.3.1

x	DQ solution- $y(x)$, over[-1,0] $N = 151, \varepsilon = 0.001$	DQ solution- $y(x)$, over[-1,0] $N = 164, \varepsilon = 0.001$
-1.00	1.0000000	1.0000000
-.98	.0000000	.0000000
-.96	.0000000	.0000000
-.94	.0000000	.0000000
-.92	.0000000	.0000000
-.90	.0000000	.0000000
-.70	.0000000	.0000000
-.50	.0000000	.0000000
-.10	.0000000	.0000000
.00	.0000000	.0000000

Table 4.3.1(d), Computational results for example-4.3.1

x	DQ solution- $y(x)$, over[0,1] $N = 151, \varepsilon = 0.001$	DQ solution- $y(x)$, over[0,1] $N = 164, \varepsilon = 0.001$
.00	.0000000	.0000000
.50	.0000000	.0000000
.70	.0000000	.0000000
.90	.0000000	.0000000
.92	.0000000	.0000000
.94	.0000000	.0000000
.96	.0000000	.0000000
.98	.0000000	.0000000
1.00	1.9999568	1.9999566

Example 4.3.2: Consider the following singular perturbation problem

$\varepsilon y''(x) - 2xy'(x) + (1 + x^2)y(x) = 0;\ x \in [-1,1],$
with $y(-1) = 2$; and $y(1) = 1$.

For this example we have $a(x) = -2x, b(x) = (1 + x^2)$ and $f(x) = 0.$
Further we have two boundary layers one at $x = -1$ and one at $x = 1$. (For details, see C.M. Bender, S.A. Orszag [8, pp.458-461 Eq.(9.6.20),case(II)].

We see that the function
$a(x) = -2x > 0$ for $-1 \le x < 0$,
$a(x) = -2x = 0$ for $x = 0$,
$a(x) = -2x < 0$ for $0 < x \le 1$,

The leading order uniform asymptotic approximation is given by
$$y(x) = 2e^{-2(x+1)/\varepsilon} + e^{-2(1-x)/\varepsilon}, \quad \text{for } -1 \le x \le 1$$
Step 1: $y(0) = \dfrac{f(0)}{b(0)} = 0 = \mu$
Step 2:
In the interval $[-1,0]$ we have left layer.
The problem is:
$$\varepsilon y''(x) - 2xy'(x) + (1+x^2)y(x) = 0; -1 \le x \le 0,$$
with
$$y(-1) = 2 \quad \text{and} \quad y(0) = 0 .$$

and in the interval $[0,1]$ we have right layer.
The problem is:
$$\varepsilon y''(x) - 2xy'(x) + (1+x^2)y(x) = 0; 0 \le x \le 1,$$
with
$$y(0) = 0 \quad \text{and} \quad y(1) = 1$$

Step 3: We now apply DQM to solve these problems to obtain solutions over the intervals $[-1,0]$ and $[0,1]$ respectively.

The computational results are presented in Table 4.3.2(a), 4.3.2(b), 4.3.2(c), 4.3.2(d), 4.3.2(e), and 4.3.2(f) for different values of N and ε.

Table 4.3.2(a), Computational results for example-4.3.2

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over [-1,0] $N = 82, \varepsilon = 0.01$	DQ solution- $y(x)$, over [-1,0] $N = 132, \varepsilon = 0.01$
-1.00	2.0000000	2.0000000	2.0000000
-.98	.0366313	.0440724	.0410362
-.96	.0006709	.0010519	.0009214
-.94	.0000123	.0000274	.0000228
-.92	.0000002	.0000007	.0000005
-.90	.0000000	.0000000	.0000000
-.70	.0000000	.0000000	.0000000
-.50	.0000000	.0000000	.0000000
-.30	.0000000	.0000000	.0000000
-.10	.0000000	.0000000	.0000000
.00	.0000000	.0000000	.0000000

Table 4.3.2(b), Computational results for example-4.3.2

x	Asymptotic Solution- $y(x)$	DQ solution- $y(x)$, over[0,1] $N = 82, \varepsilon = 0.01$	DQ solution- $y(x)$, over[0,1] $N = 132, \varepsilon = 0.01$
.00	.0000000	.0000000	.0000000
.10	.0000000	.0000000	.0000000
.30	.0000000	.0000000	.0000000
.50	.0000000	.0000000	.0000000
.70	.0000000	.0000000	.0000000
.90	.0000000	.0000000	.0000000
.92	.0000001	.0000004	.0000003
.94	.0000061	.0000137	.0000114
.96	.0003355	.0005259	.0004607
.98	.0183156	.0220361	.0205181
1.00	1.0000000	.9999957	.9999956

Table 4.3.2(c), Computational results for example-4.3.2

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[-1,0] $N=151, \varepsilon=0.001$N	DQ solution- $y(x)$, over[-1,0] $N=182, \varepsilon=0.001$
-1.00	2.0000000	2.0000000	2.0000000
-.98	.0000000	.0000008	.0000000
-.96	.0000000	-.0000018	.0000000
-.94	.0000000	.0000015	.0000000
-.92	.0000000	-.0000006	.0000000
-.90	.0000000	-.0000016	.0000000
-.70	.0000000	-.0000011	.0000000
-.50	.0000000	-.0000046	.0000000
-.30	.0000000	-.0000012	.0000000
-.10	.0000000	-.0000006	.0000000
.00	.0000000	.0000000	.0000000

Table 4.3.2(d), Computational results for example-4.3.2

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[0,1] $N=151, \varepsilon=0.001$	DQ solution- $y(x)$, over[0,1] $N=182, \varepsilon=0.001$
.00	.0000000	.0000000	.0000000
.30	.0000000	-.0000006	.0000000
.50	.0000000	-.0000023	.0000000
.70	.0000000	-.0000005	.0000000
.90	.0000000	-.0000008	.0000000
.92	.0000000	-.0000003	.0000000
.94	.0000000	.0000007	.0000000
.96	.0000000	-.0000009	.0000000
.98	.0000000	.0000004	.0000000
1.00	1.0000000	.9999584	.9999574

Table 4.3.2(e), Computational results for example-4.3.2

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[-1,0] $N = 311, \varepsilon = 0.0001$	DQ solution- $y(x)$, over[-1,0] $N = 364, \varepsilon = 0.0001$
-1.00	2.0000000	2.0000000	2.0000000
-.98	.0000000	.0047871	-.0004215
-.96	.0000000	.0026163	-.0001113
-.94	.0000000	-.0022184	-.0004523
-.92	.0000000	-.0001537	.0002121
-.90	.0000000	.0005671	.0001085
-.70	.0000000	.0014118	.0005022
-.50	.0000000	-.0040606	-.0000455
-.30	.0000000	.0013092	-.0006512
-.10	.0000000	.0001436	-.0002236
.00	.0000000	.0000001	.0000000

Table 4.3.2(f), Computational results for example-4.3.2

x	Asymptotic solution- $y(x)$	DQ solution- $y(x)$, over[0,1] $N = 311, \varepsilon = 0.0001$	DQ solution- $y(x)$, over[0,1] $N = 364, \varepsilon = 0.0001$
.00	.0000000	.0000000	.0000000
.10	.0000000	.0000718	-.0001118
.30	.0000000	.0006546	-.0003256
.50	.0000000	-.0020303	-.0000227
.70	.0000000	.0007059	.0002511
.90	.0000000	.0002836	.0000542
.92	.0000000	-.0000768	.0001060
.94	.0000000	-.0011092	-.0002262
.96	.0000000	.0013082	-.0000557
.98	.0000000	.0023935	-.0002108
1.00	1.0000000	.9996241	.9996054

5 Discussion and Conclusion

In this paper, we have applied the DQM to solve the two singular perturbation problems with right end boundary layer, one singular perturbation problem with an internal layer, and two singular perturbation problem with two boundary layers. The applications presented here showed that the DQM has the capability of solving general singularly perturbed two point boundary value problems and of producing accurate results with minimal computational effort. We have given here only a few values although the solutions can be computed at desired number of uniform points. It can be observed from the tables that the DQM approximates the exact or asymptotic solution very well with small number of sampling points. This shows the efficiency and accuracy of the present method. This method provides an alternative technique to the conventional ways of solving singular perturbation problems.

It has been observed that an increase in the number of grid points gives rise to an increase in the accuracy of the DQM solution, as in the most numerical techniques. However a small number of grid points in the DQM produces highly accurate results with the use of non-uniform grids.

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