Int. J. of Mathematical Sciences and Applications,
Vol. 1, No. 3, September 2011

www.journalshub.com

ANNOTATIONS ON DIVISIBILITY TEST

K. RAJA RAMA GANDHI[1] AND B. SREEDHAR[2]
Department of Mathematics,[1]
Fellow, Member of SO MA SS and Resource person in Math - Oxford University Press,
Chaitanya Engineering College, Kommadi, Visakhapatnam, India-41.
Email: rrmath28@gamail.com
And
Department of Mathematics, [2]
Chaitanya Engineering College, Kommadi, Visakhapatnam, India-41.

ABSTRACT

In this paper, we give a new method to test the divisibility of an integer by another. Also, we make some important annotations on divisibility test on 11 with proofs. We will discuss the general divisibility tests by taking advantages of modular arithmetic with polynomials and we generalize our observations in systematic way.

INTRODUCTION

We know that, every number has its own divisibility test/rule. For example, (see the table-1 below)**[1].** I have made some observation on 11, which will produce a quotient. Since all the divisibility rules will describe the perfection, but not quotient including 11. Now, I will show the quotient by modifying the divisibility test of 11. Also, I will discuss the other interesting observation in the next section.

	Divisibility Tests	**Example**
2	A number is divisible by 2 if the last digit is 0, 2, 4, 6 or 8.	168 is divisible by 2 since the last digit is 8.
3	A number is divisible by 3 if the sum of the digits is divisible by 3.	168 is divisible by 3 since the sum of the digits is 15 (1+6+8=15), and 15 is divisible by 3.
4	A number is divisible by 4 if the number formed by the last two digits is divisible by 4.	316 is divisible by 4 since 16 is divisible by 4.
5	A number is divisible by 5 if the last digit is either 0 or 5.	195 is divisible by 5 since the last digit is 5.
6	A number is divisible by 6 if it is divisible by 2 **AND** it is divisible by 3.	168 is divisible by 6 since it is divisible by 2 **AND** it is divisible by 3.
8	A number is divisible by 8 if the number formed by the last three digits is divisible by 8.	7,120 is divisible by 8 since 120 is divisible by 8.
9	A number is divisible by 9 if the sum of the digits is divisible by 9.	549 is divisible by 9 since the sum of the digits is 18 (5+4+9=18), and 18 is divisible by 9.
10	A number is divisible by 10 if the last digit is 0.	1,470 is divisible by 10 since the last digit is 0.
11	A number is divisible by 11 if the difference of the sum of the odd and even digit numbers of number is multiple of 11	$1331 \Rightarrow (1 + 3) - (3 + 1) = 0$

Table-1

One need not memorize motley exotic divisibility tests. There is a universal test that is simpler and much easier recalled. Namely, eventually, the polynomials in nested Horner form, using modular arithmetic. For example, consider evaluating a 3-digit decimal number modulo 7. In Horner form $d_2\, d_1\, d_0$ (d_0 , d_1 ,d_2 stand for reorganization of digits)

$\Rightarrow ((d_2.10+d_1)10+d_0) \equiv ((d_2.3+d_1)3 + d_0 \pmod 7))$ as we know that $10 \equiv 3 \pmod 7$. So, we can find the remainder (mod 7) as follows. Start with the leading digit then repeatedly apply the operation i.e multiply by 3 then add the next digit, doing all of the arithmetic (mod 7). For example, let's use this algorithm to reduce 43211 (mod 7). The algorithm consists of repeatedly replacing the first two leading digits $d_n\, d_{n-1}$ by $d_n(3) + d_{n-1} \pmod 7$, namely 43211.

$$\begin{aligned} &\equiv 1211 \text{ by } 4(3) + 3 \equiv 1 \\ &\equiv 511 \text{ by } 1(3) + 2 \equiv 5 \\ &\equiv 21 \text{ by } 5(3) + 1 \equiv 2 \\ &\equiv 0 \text{ by } 2(3) + 1 \equiv 0 \end{aligned}$$

Hence, $43211 \equiv 0 \pmod 7$, indeed $43211 = 7(6173)$.

Generally the modular arithmetic is simpler if one use a balanced systems of representatives, e.g. $\pm \{0,1,2,3\} \pmod 7$. Observe that, for modulus 11 or 9 the above method reduces to the well-known divisibility tests by 11 or 9.

A positive integer written as $x = d_k....d_1d_0$ (in base 10) is same as $\sum_{j=0}^{k} d_j 10^j$ and consider $10^j \equiv m_j \pmod n$. Then x is divisible by n if and only if $n | \sum_{j=0}^{k} d_j m_j$. Assuming that $(n,10) =1$, 10^j is periodic (mod n), the minimal period being a divisor $\varphi(n)$. For example, in the case $n = 7$, we have $m_0 = 1$, $m_1 = 3$, $m_2 = 2$, $m_3 = -1$, $m_4 = -3$, $m_5 = -2$ and then it repeats. So x is divisible by 7 if and only if $(d_0 - d_3 + d_6 - d_9 +\ldots) + 3(d_1 - d_4 + d_7 - d_{10} +\ldots) + 2(d_2 - d_5 + d_8 - d_{11} +..)$.

In the same way we can develop a new divisibility rule by taking an advantage of modular arithmetic without memorizing the table -1 and or beyond.

Now, we will show our first observation on divisibility test 11.

OBSERVATIONS

If we add "symmetrically" a number with an every number of digits (eg. *ABCD+DCBA*), the sum can be expressed as a sum of numbers consisting of a pair of one of the original digits (such as A,B,C, or D) separated by an even (possibly zero) number of zeros and possibly ending with a few more zeros so ABCD + DCBA = AooA + BBo + CCo + DooD. We know that given a number $a_na_{n-1}\ldots..a_1$ is divisible by 11 if and only if $a_n + a_{n-2} +\ldots.-a_{n-1} -a_{n-3} -\ldots$ divisible by all i.e. 1001 is divisible by all as $1 + 0 – 0 - 1= 0$ is divisible by 11.

Now take $\overline{b_{2n}b_{2n-1}\ldots b_2 b_1}$ and we can see the wonder of the divisibility of $\overline{b = b_{2n}b_{2n-1}\ldots b_2 b_1}$ + $\overline{b_1 b_2 \ldots b_{2n-1} b_{2n}}$ Then we get $b = \sum_{k=1}^{2n} b_k (10^k + 10^{2n+1-k})$ and we only need to prove that $10^k + 10^{2n+1-k}$ is divisible by 11, which we can prove naturally**[2]**. Finally to prove that it only holds for 2n and we can consider 10^2n which has 2n+1 digits and $10^{2n}+1$ is not divisible by 11.

Main ideas

1) It is a consequence of the test for divisibility by 11. The remainder of the division of a number *a* by 11 is equal to the remainder of the difference of the sum of the odd ordered digits of *a* from the sum of the even ordered digits of *a*.
2) And the base 10 properly says for some n the power $10^n \equiv 1 \pmod{11}$.
3) $10 \equiv (-(-10)) \equiv -1 \pmod{11}$.

Let $a = a_n a_{n-1} \ldots a_2 a_1 a_0 = a_n 10^n + a_{n-1} 10 a^{n-1} + \ldots + a_2 10^2 + a_1 10 + a_0$, For (mod 11), we have, $10^0 \equiv 1$, $10^1 \equiv 10$, $10^2 \equiv 1$, and if n is even $10^n \equiv 1$, if n is odd $10^n \equiv 10$. Thus $a \equiv (a_0 + a_2 + a_4 + \ldots) + 10\,(a_1 + a_3 + a_5 + \ldots) \pmod{11}$ since $-10 \equiv 1 \pmod{11}$, we have; $10 \equiv (-(-10)) \equiv -1 \pmod{11}$ and $a \equiv (a_0 + a_2 + a_4 + \ldots) - (a_1 + a_3 + a_5 + \ldots) \pmod{11}$. For our case, for $a = A10^3 + B10^2 + C10 + D$, $a \equiv (D+B) - (C+A) \pmod{11}$ and for $b \equiv D10^3 + C10^2 + B10 + A$, $b \equiv (A+C) - (B+D) \pmod{11}$ and $a + b \equiv (D+B) - ((C+A) + (A+C) - (B+D)) \equiv 0 \pmod{11}$. For $a = A10^4 + B10^3 + C10^2 + D10 + E$, and $b = E10^4 + D10^3 + C10^2 + B10 + A$, we have, $a + b \equiv (A+C+E) - (B+D) + ((E+C+A) - (D+B))$ not congruent to 0 (mod 11). Of course, we can generalize this part. The question arrives that, why it is applicable only for 11 and why it is valid for even digit of numbers? Here is a possible generalization of the proof above (radix10) for the radix r.

We know that $r^0 \equiv 1 \pmod{r+1}$, $r^1 \equiv r \pmod{r+1}$, $r^2 \equiv 1 \pmod{r+1}$, $r^3 \equiv r \pmod{r+1}$,…and $ABCD_r \equiv (D+B)_r - (C+A)_r \pmod{r+1}$, $DCBA_r \equiv (A+C)_r - (B+D)_r \pmod{r+1}$, we obtain $ABCD_r + DCBA_r \equiv 0 \pmod{r+1}$.**[3]**

Take ***abcde***, If we subtract ***e*** from ***d***, getting ***d¹***, and we apply the method to ***abcd¹***, if the method works for ***abcd¹***, it yields ***n*** such that ***abcd¹ = 11×n***. But ***abcd¹0*** is ***abcd¹×10 = 11 × 10 × n*** and ***abcde = abcd¹ × 10 + e × 11*** hence ***abcde = 11 × 10 × n + 11 × e*** is ***11 (10 × n + e)***. This process that ***10 × n + e*** are ended the correct answer for ***abcde***.

Take $N = a_k a_{k-1} \ldots a_2 a_1 a_0$ with $k \geq 1$. If we subtract a_0 from a_1, getting b_1 and we apply the method to $a_k a_{k-1} \ldots a_2 b_1$. If the method works for $a_k a_{k-1} \ldots a_2 b_1$, it yields n such that $a_k a_{k-1} \ldots a_2 b_1 = 11n$; but $a_k a_{k-1} \ldots a_2 b_1 0 = a_k a_{k-1} \ldots a_2 b_1 \times 10 = 11 \times 10 \times n$ and $N = a_k a_{k-1} \ldots a_2 b_1 \times 10 + a_0 \times 11$.

Hence $N = 11 \times 10 \times n + 11 \times a_0 = 11 \times (10 \times n + a_0)$

This process that $10 \times n + a_0$ is indeed the correct answer for N if n was the correct answer for $a_k a_{k-1} \ldots a_2 b_1$. A recursion on the number of digits of N yields the result.

Remark: we can prove the same by following hint.

$$(x + 1)(a_n x^n + \ldots + a_1 x + a_0) = a_n x^{n+1} + (a_n + a_{n-1})x^n + \ldots + (a_1 + a_0)x + a_0 \text{ where } x=10.$$

(Which is very easy to prove and I am leaving the same to readers)

ACKNOWLEDGMENT

I am heartily thankful to Mathematician Prof. K. Raja Rama Gandhi, whose encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of the subject.

REFERENCES

[1] Bassarear, Mathematics for Elementary School Teachers, 5th edition, cengage Publishers, USA.

[2] Christof paar jan pelzl, Understanding Cryptography, Springer Publication, Germany.

[3] Joseph A. gallian, Contemporary Abstract Algebra, Seventh edition, Cengage Learning Publishers,USA.