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ON NON METRIZABILITY OF CONE METRIC SPACES

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Abstract

In this paper, we show that the result of Mehdi Asadi, S. Mansour Vaezpour, Hossien Soleimani, [1] on renorming a Banach space do not hold.

Key words: *Equivalent norm, metrizability, normal cone, non normal cone, cone metric space.*

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1 Introduction

In 2007, H. L. Guang and Z. Xain [2] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings which satisfy various contractive conditions. After that many authors are working on this concept. But the basic question is " Are those spaces a real generalization of metric spaces?". Haghi et. al. [3] have shown that some fixed point generalizations are not real generalizations.

Recently, Mehdi Asadi, S. Mansour Vaezpour, Hossien Soleimani [1] investigated about this question. They claimed that a cone metric space is metrizable via renorming the Banach spac. It is in the way, they have claimed to have answered the question in the negative. The purpose of this paper is to show that their contention does not hold. For all definitions and notations we come across in this paper, we refer to [1] and [2].

Notation : In what follows, unless otherwise specified E is a real Banach space with cone P.

First we state the result of Mehdi Asadi et. al.

Theorem 1.1 [1]: *Let $(E, || \, . \, ||)$ be a real Banach space with a positive cone P. Then there exists an equivalent norm on E such that P is a normal cone with normal constant $K = 1$ with respect to this norm.*

In fact, they defined for $x \in E$

$$||| \, x \, ||| = inf\{|| \, u \, ||: x \leq u\} + inf\{|| \, v \, ||: v \leq x\} + || \, x \, || \tag{1.1.1}$$

and claimed that $|||\ .\ |||$ *is an equivalent norm to* $||\ .\ ||$ *with normal constant 1.*

That is there exists $\lambda, \mu > 0$ such that

$$\lambda\ |||\ x\ ||| \ \leq\ ||\ x\ || \ \leq\ \mu\ |||\ x\ ||| \tag{1.1.2}$$

$$\text{and } 0 \leq x \leq y \Rightarrow \ |||\ x\ ||| \ \leq\ |||\ y\ ||| \tag{1.1.3}$$

Note : If $x \in P$ then $inf\{||\ v\ ||: v \leq x\} = 0$, since $0 \in P$.
Thus, for $x \in P$, we have $|||\ x\ ||| = inf\{||\ u\ ||: x \leq u\} + ||\ x\ ||$ (1.1.4)

In 2008, Sh. Rezapour and R . Hamlbarani [4] showed that there are non normal cones. Motivated with this concept of non normal cones we show that the above result does not hold.

2 Main Results

We first prove a lemma followed by two examples. The first example [4] is a non normal cone. The second example shows that theorem 1.1 does not hold.

Lemma 2.1 *Let* $E = C^2_R([0,1])$ *the set of all real valued continuously differentiable mappings on* $[0,1]$ *with norm*

$$||\ f\ || \ = \ ||\ f\ ||_\infty + ||\ f^{'}\ ||_\infty$$

where $||\ f\ ||_\infty = Sup\ \{|\ f(x)\ |: x \in [0,1]\}$. *Consider the cone* $P = \{f \in E : f \geq 0\}$. *Suppose* $f \in P$. *Then* $inf\{||\ u\ ||: f \leq u\} = ||\ f\ ||_\infty$

Proof : Since $0 \leq f \leq u$

$$\begin{aligned}
&\Rightarrow \ Sup\ \{f(x) : x \in [0,1]\} \leq Sup\ \{u(x) : x \in [0,1]\} \\
&\Rightarrow \ ||\ f\ ||_\infty \leq ||\ u\ ||_\infty \leq ||\ u\ ||_\infty + ||\ u^{'}\ ||_\infty = ||\ u\ || \\
&\Rightarrow \ ||\ f\ ||_\infty \leq inf\ \{||\ u\ ||: f \leq u\} = \alpha(say) \\
&\Rightarrow \ ||\ f\ ||_\infty \leq \alpha \qquad (2.1.1)
\end{aligned}$$

Define the constant function $\lambda(x) = ||\ f\ ||_\infty\ \forall\ x \in [0,1]$.
Then clearly $0 \leq f \leq \lambda$ and $||\ \lambda\ || = ||\ f\ ||_\infty$.
Therefore $\alpha = inf\ \{||\ u\ ||: f \leq u\} \leq ||\ \lambda\ || = ||\ f\ ||_\infty \ \Rightarrow \alpha \leq ||\ f\ ||_\infty$ (2.1.2)
From (2.1.1)*and*(2.1.2) we get $||\ f\ ||_\infty = \alpha$.
Thus $inf\ \{||\ u\ ||: f \leq u\} = ||\ f\ ||_\infty$ for $f \in P$.

To illustrate the existence of non normal cones the following example is given in Rezapour and Hamlbarani [4].

Example 2.2 [4] Let $E = C^2_R([0,1])$ with the norm

$$||\ f\ || = ||\ f\ ||_\infty + ||\ f^{'}\ ||_\infty$$

and consider the cone $P = \{f \in E : f \geq 0\}$. Then P is a non normal cone.

The following example shows that theorem 1.1 does not hold.

Example 2.3 : Let E and P be as in example 2.2.
Let $f(x) = x$ and $g(x) = x^2 \ \forall \ x \in [0, 1]$. Then clearly $0 \leq g \leq f$.
Further $\| f \| = \| f \|_\infty + \| f^{'} \|_\infty = 1 + 1 = 2$
and $\| g \| = \| g \|_\infty + \| g^{'} \|_\infty = 1 + 2 = 3$
Since $f \in P$, from (1.1.4)

$$\begin{aligned} ||| f ||| &= inf \ \{\| u \|: f \leq u\} + \| f \| \\ &= \| f \|_\infty + 2 \qquad (byLemma 2.1) \\ &= 1 + 2 \\ &= 3 \\ Also, ||| g ||| &= inf \ \{\| u \|: g \leq u\} + \| g \| \\ &= \| g \|_\infty + \| g \| \qquad (byLemma 2.1) \\ &= 1 + 3 \\ &= 4 \end{aligned}$$

According to theorem 1.1, P is a normal cone with normal constant $K = 1$.

Hence $0 \leq g \leq f \ \Rightarrow ||| g ||| \leq ||| f |||$
$\Rightarrow 4 \leq 3$ a contradiction.

That is the norm $||| x |||$ is not an equivalent norm on E such that P is a normal cone with normal constant $K = 1$, with respect to $||| . |||$. Thus theorem 1.1 does not hold.

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