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HYPERBOLIC OPERATOR ON BANACH SPACE

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ABSTRACT. In this paper hyperbolic operator on a Banach space is defined in terms of its spectrum and characterized. Examples are given. Further tensor product of hyperbolic operators are discussed.

1. INTRODUCTION.

An operator T on a complex Banach space is said to be hyperbolic if its spectrum is the disjoint union of two non empty closed sets, of which one lies completely inside the unit circle and the other outside. This spectral definition shows that the spectrum of the operator has no points on the unit circle. In this paper, equivalent condition for an operator to have spectrum inside a circle of radius r centered at the origin is obtained. Using this, the hyperbolic operators are characterized and its stability properties are discussed. Also some examples of non hyperbolic operators on Hilbert spaces are provided. In the last section tensor product of hyperbolic operators are characterized and an example is given to illustrate the result obtained.

2. PRELIMINARIES

Definition 2.1. [1] Let T be an operator on a complex Banach space X. The spectrum $\sigma(T)$ of T is defined as $\sigma(T)$=$\{\lambda \in C : \lambda I - T$is not invertible$\}$

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Definition 2.2. [1] Let T be an operator on a complex Banach space X. The spectral radius $r(T)$ of T is defined as $r(T)=Sup_{\lambda\in\sigma(T)}|\lambda|$.

Definition 2.3. [2] A bounded linear operator T on a Banach space X is said to be uniformly exponentially stable if there exist constants $M \geq 0$ and $\epsilon > 0$ such that $\|T^n\| \leq Me^{-\epsilon n}$ for all $n \in N$.

Theorem 2.4. *(Riesz decomposition theorem)* [3] *Suppose the spectrum $\sigma(T)$ splits into two disjoint non empty closed subsets σ_1 and σ_2, $\sigma(T)=\sigma_1 \cup \sigma_2$. Then there are non trivial T-invariant closed subspaces X_s and X_u of X such that $X = X_s \oplus X_u$, $\sigma(T/X_s) = \sigma_1$ and $\sigma(T/X_u) = \sigma_2$.*

Theorem 2.5. *(Spectral mapping theorem)* [5] *Let f be a holomorphic function on a neighbourhood of $\sigma(T)$. Then $\sigma(f(T))=f(\sigma(T))$*

Proposition 2.6. *Let T be a bounded linear operator on a Banach space X. The following assertions are equivalent.*
(a) $r(T) < 1$.
(b) $lim_{n\to\infty}\|T^n\| = 0$.
(c) T is uniformly exponentially stable.

3. RESULTS AND DISCUSSIONS

Let X be a Banach space and $T \in B(H)$ with $\sigma(T) = \sigma_1 \cup \sigma_2$ where $\sigma_1 = \sigma(T) \cap D$, $\sigma_2 = \sigma(T) \cap (C/\overline{D})$. If $\sigma(T) \cap \Gamma = \phi$, $\sigma_1 \neq \phi$ and $\sigma_2 \neq \phi$ then T is called hyperbolic operator, where $D = \{z \in C/|z| < 1\}$ is the open unit disc and $\Gamma=\{z \in C/|z| = 1\}$ is the unit circle.
The spectral Riesz projection P for T is given by $P = \frac{1}{2\pi i}\int_\Gamma (z - T)^{-1}dz$. This spectrum corresponds to the part σ_1 of the projection. It is to be noted that P commutes with T. Using this fact, we define the restriction $T_s = T/X_s : X_s \to X_s$

where $X_s = ImP$ and the restriction of T to the image of the complementary projection $T_u = T/X_u : X_u \to X_u$ where $X_u = Im(I - P)$. This complementary projection corresponds to the part σ_2 of the spectrum.

Example 3.1. Let $T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $0 < \alpha < 1$ and $\beta > 1$. Here $\sigma(T) = \{\alpha, \beta\}$. Clearly $\sigma(T) \cap \Gamma = \phi$. Hence T is a hyperbolic operator.

Theorem 3.2. *Let $r > 0$. Then*
(i) $\sigma(T) \subset \{z \in C/|z| < r\}$ if and only if there are $\epsilon > 0$ and $M > 0$ such that $\|T^n x\| \leq M(r - \epsilon)^n \|x\|$ for all $x \in X$ and $n \in N_0$.
(ii) $\sigma(T) \subset \{z \in C/|z| > r\}$ if and only if there are $\epsilon > 0$ and $M > 0$ such that $\|T^n x\| \geq M(r + \epsilon)^n \|x\|$ for all $x \in X$ and $n \in N_0$.

Proof. Let $\sigma(T) \subset \{z \in C/|z| < r\}$. Then the spectral radius formula imply that $lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T) < r$. Hence, if $\in < r - r(T)$ then there is some $M > 0$ such that $\|T^n\| \leq M(r - \epsilon)^n$ for all $n \in N_0$. Conversely, for some $\epsilon > 0$ and $\eta > 0$ assume $\|T^n x\| \leq M(r - \epsilon)^n \|x\|$ for all $x \in X$ and $n \in N_0$.
The spectral radius formula implies that
$r(T) = lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ and so
$r(T) \leq lim_{n\to\infty} M^{\frac{1}{n}}(r - \epsilon)$
$< r$
Therefore $\sigma(T)$ is a compact set such that $\sigma(T) \subset \{z \in C/|z| < r\}$
(ii) By assumption $0 \notin \sigma(T)$, so that T is invertible. Since $\sigma(T^{-1}) = \sigma(T)^{-1}$, we have $\sigma(T^{-1}) \subset \left\{z \in C/|z| < \frac{1}{r}\right\}$. By (i) there some $\eta > 0$ with $\eta < \frac{1}{r}$ and $M > 0$ such that $\|(T^{-1})^n y\| \leq M(\frac{1}{r} - \eta)^n \|y\|$ for all $y \in X$ and $n \in N_0$. Setting $y = T^n x$ and defining ϵ by $\frac{1}{r} - \eta = \frac{1}{r+\epsilon}$ we obtain the result. Conversely, assume $\|T^n x\| \geq M(r + \epsilon)^n \|x\|$. Then
$r(T) \geq lim_{n\to\infty} M^{\frac{1}{n}}(r + \epsilon)$

$> r$

Hence $\sigma(T) \subset \{z \in C/|z| > r\}$ □

Corollary 3.3. *T is hyperbolic if and only if*
(i) $\|T_s^n x\| \leq M(1-\epsilon)^n\|x\|$ and
(ii) $\|T_u^n x\| \geq M(1+\epsilon)^n\|x\|$ where T_s and T_u are the restricted operators on the subspaces X_s and X_u.

Corollary 3.4. *T is hyperbolic if and only if there exist constants $\beta > 0$ and $M > 0$ such that, for all integers $n \geq 0$*
(i) $\|T_s^n x\| \leq Me^{-\beta n}\|x\|$ and
(ii) $\|T_u^n x\| \geq M^{-1}e^{\beta n}\|x\|$

Definition 3.5. [6] A linear operator T on a Banach space X is called
(i) Power bounded if $Sup_{n\geq 0}\|T^n\| < \infty$
(ii) a contraction if $\|T\| \leq 1$
(iii) quasinilpotent if $lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} = 0$

Proposition 3.6. *No power bounded operator is hyperbolic. In particular no contraction, quasinilpotent and volterra operator is hyperbolic.*

Corollary 3.7. *T is hyperbolic if and only if*
(i) The operator T_s is uniformly exponentially stable on X_s.
(ii) The operator $T_u$is invertible on X_u and T_u^{-1} uniformly exponentially stable on X_u.

Remark 3.8. T is a hyperbolic operator and $X = X_s \oplus X_u$ is its corresponding decomposition then $X_u = \{x \in X/lim_{n\to\infty}T^n x = 0\}$

Theorem 3.9. *An operator T on a Banach space is hyperbolic if and only if $\sigma(logT) \cap iR = \phi$*

Proof. T is a hyperbolic operator

$\Leftrightarrow \sigma(T) \cap \Gamma = \phi$

$\Leftrightarrow e^{i\lambda} \notin \sigma(T)$

$\Leftrightarrow i\lambda \notin log\sigma(T)$

$\Leftrightarrow i\lambda \notin \sigma(logT)$

$\Leftrightarrow \sigma(logT) \cap iR = \phi$ ☐

Theorem 3.10. *An operator T on a Hilbert space is hyperbolic if and only if $\sigma(logT) \cap iR = \phi$ and $\|R(\lambda, logT)\| \leq M$ for all $\lambda \in iR$*

4. EXAMPLES OF NON HYPERBOLIC OPERATORS ON HILBERT SPACES

We give examples to show that many of the known classes need not be hyperbolic.

Example 4.1. If U is unitary, then $\sigma(U) \subset \{z \in C/|z| = 1\}$. So any unitary operator can never be hyperbolic.

Example 4.2. $T=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is normal but not hyperbolic.

Example 4.3. Paranormal operators need not be hyperbolic

Let U and P be infinite matrices as follows

$$U=\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ and } P=\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Put $T=\begin{pmatrix} U+I & P \\ 0 & 0 \end{pmatrix}$. Then T is an operator on $H \oplus H$, where $H = l^2$. Then T is paranormal with $\sigma(T) \subset \{\lambda \in C/|\lambda - 1| \leq 1\}$ and so T is not hyperbolic.

Example 4.4. Normaloid operators need not be hyperbolic.

Let T be an infinite matrix of the form

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & M & 0 & 0 & \ldots \\ 0 & 0 & M & 0 & \ldots \\ 0 & 0 & 0 & M & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ where } M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Here $\|T^n\| = \|T\|^n = 1$. Hence T is normaloid. Also $\sigma(T) = \{0\} \cup \{1\}$. Clearly $\sigma(T) \cap \Gamma \neq \phi$. Hence T is not a hyperbolic operator.

Example 4.5. Convexoid operators need not be hyperbolic.

Let $T = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$ where $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and N be the normal operator, whose spectrum is the closed unit disc $\overline{D}$. Here T is convexoid with $\sigma(T) = \{0\} \cup \overline{D} = \overline{D}$ and so T is not hyperbolic.

Example 4.6. Any unilateral and bilateral shifts are never be hyperbolic.

If U is the unilateral shift then $\sigma(U) = \bar{D}$ and if W is the bilateral shift then $\sigma(W) = \Gamma$ [4]. Hence it is not hyperbolic.

5. TENSOR PRODUCT OF HYPERBOLIC OPERATORS

Let $X \overset{\alpha}{\otimes} Y$ denote the tensor product of Banach spaces X and Y with respect to uniform cross norm α. Also it denotes either one of the tensor product is the projective or the injective.[7]

Lemma 5.1. *Let $a, b \in R^+$. If $ab < 1$ then there exists $k > 0$ such that $ka < 1$ and $\frac{1}{k}b < 1$.*

Proof. Let $a > 1$ and $b < 1$. Choose $k = \frac{ab+1}{2a}$. $\square$

Theorem 5.2. *The tensor product $T \otimes S$ is uniformly exponentially stable on $X \overset{\alpha}{\otimes} Y$ if and only if βT and $\frac{1}{\beta}S$ are uniformly exponentially stable on X and Y respectively for an unique $\beta > 0$.*

Proof. If both βT and $\frac{1}{\beta}S$ are uniformly exponentially stable on X and Y respectively, then there exists constants $M_1, M_2 \geq 1$, $\epsilon_1, \epsilon_2 > 0$ so that $||\beta T^{n_1}|| \leq M_1 e^{-\epsilon_1 n_1}$ and $||\frac{1}{\beta}S^{n_2}|| \leq M_2 e^{-\epsilon_2 n_2}$ for all $n_1, n_2 \in N$. Then $||(T^{n_1} \otimes S^{n_2})|| \leq Me^{-\epsilon(n_1+n_2)}$ where $M = M_1 M_2$ and $\epsilon = min(\epsilon_1, \epsilon_2)$ showing that $T \otimes S$ is uniformly exponentially stable on $X \overset{\alpha}{\otimes} Y$.
Conversely, let $T \otimes S$ be uniformly exponentially stable. Then by Proposition 2.6, $r(T \otimes S) < 1$ and so $r(T).r(S) < 1$. By Lemma, there exists $\beta > 0$ such that $\beta r(T) < 1$ and $\frac{1}{\beta}r(S) < 1$ showing that βT and $\frac{1}{\beta}S$ are uniformly exponentially stable. □

Theorem 5.3. *Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ be Banach spaces. Then $T \otimes S$ is a hyperbolic operator on*
$(X_1 \overset{\alpha}{\otimes} Y_1) \oplus (X_2 \overset{\alpha}{\otimes} Y_2)$ if and only if there exists a unique $\beta > 0$ so that βT and $\frac{1}{\beta}S$ are hyperbolic operators on X and Y respectively.

Proof. Let T and S be hyperbolic on X and Y respectively. Then by Theorem 5.2, $(T_1 \otimes S_1)$ is uniformly exponentially stable on $(X_1 \overset{\alpha}{\otimes} Y_1)$ where T_1 and S_1 are the restricted operators of T and S on X_1 and Y_1 respectively.

Further let the operators T_2 and S_2 denote the restrictions of T and S on X_2 and Y_2 respectively. By assumption T_2 and S_2 are invertible on X_2 and Y_2 respectively and so $T_2 \otimes S_2$ is invertible on $(X_2 \overset{\alpha}{\otimes} Y_2)$. Moreover T_2^{-1} and S_2^{-1} are uniformly exponentially stable on X_2 and Y_2 respectively. Again by Theorem 5.2 $(T_2 \otimes S_2)^{-1}$ is uniformly exponentially stable on $(X_2 \overset{\alpha}{\otimes} Y_2)$.

Conversely let $(T \otimes S)$ be a hyperbolic operator on
$(X_1 \overset{\alpha}{\otimes} Y_1) \oplus (X_2 \overset{\alpha}{\otimes} Y_2)$ so that $(T \otimes S)$ is uniformly exponentially stable on

$X_1 \overset{\alpha}{\otimes} Y_1$. Then by Theorem 5.2, βT and $\frac{1}{\beta}S$ are uniformly exponentially stable on X_1 and Y_1 respectively for unique $\beta > 0$.

Further invertibility of the operator $T \otimes S$ on $(X_2 \overset{\alpha}{\otimes} Y_2)$ shows the invertibility of T and S on X_2 and Y_2 respectively. Moreover
$(T \otimes S)^{-1} = T^{-1} \otimes S^{-1}$ is uniformly exponentially stable on $X_2 \overset{\alpha}{\otimes} Y_2$ shows that $(\beta T)^{-1}$ and $(\frac{1}{\beta}S)^{-1}$ are uniformly exponentially stable on X_2 and Y_2 respectively. Thus (βT) and $(\frac{1}{\beta}S)$ are hyperbolic on X and Y respectively. $\square$

To illustrate the theorem following example is given.

Example 5.4. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is not a hyperbolic operator on X and

$S = \begin{pmatrix} e^{-k} & 0 \\ 0 & e^{k} \end{pmatrix}$ where $k > 0$ is a hyperbolic operator on Y.

Then $T \otimes S = \begin{pmatrix} e^{-k} & 0 & 0 & 0 \\ 0 & e^{k} & 0 & 0 \\ 0 & 0 & e^{-k} & 0 \\ 0 & 0 & 0 & e^{k} \end{pmatrix}$

is a hyperbolic operator on $X \otimes Y = R^4$ with
$X_u = \{(x_1, x_2, x_3, 0)/x_1, x_2, x_3 \in R\}$
$X_v = \{(0, 0, 0, x_4)/x_4 \in R\}$.

Taking $\beta = \frac{e^{-k}+1}{2}$ it can be observed that (βT) and $(\frac{1}{\beta}S)$ are hyperbolic.

Conclusion 5.5. In this paper we give a new definition for hyperbolic operator on Banach space in terms of its spectrum and characterized. Examples for non hyperbolic operators are given. Further tensor product of hyperbolic operators are discussed. This will be quite useful for many researchers for further development of hyperbolic operator on other classes of operators.

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