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Stability of finite amplitude Rayleigh Benard Convection in Porous Medium

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Abstract

The stability of finite amplitude Rayleigh-Benard convection in porous medium has been investigated. We have used stress-free boundary conditions and derived a nonlinear two-dimensional Landau-Ginzburg equation with real coefficients by using Newell and Whitehead method [9]. It is observed that steady solutions may be either subcritical or supercritical depending on the choice of physical parameters. The effect of the parameter values on the stability mode is investigated and shown the occurrence of secondary instabilities viz., Eckhaus and Zigzag instabilities. We have studied Nusselt number contribution at the onset of stationary convection.

Key words: Convection, Thermal buoyancy, Bifurcation points, Landau-Ginzburg type equations, Secondary instabilities, Nusselt number.

1. Introduction

Convection in a porous medium uniformly heated from below is of considerable interest in geophysical fluid dynamics, as this phenomenon may occur within the Earth's outer core. The porous medium may be thought at as being composed of closely packed uniform spheres (grains) completely surrounded by a homogeneous fluid (palm et. al [11]). Linear and weakly nonlinear convection in an electrically conducting fluid in a nonporous medium has been studied extensively [1, 2, 3, 4, 6, 7, 8 and 13]. Convection in a porous medium has also received due attention in the literature because of its importance in various fields such as high-quality crystal production, liquid gas storage, migration of moisture in fibrous insulation, transport of contaminants in saturated soil solidification of molten alloys in mushy layer, and geothermally heated lakes and magmas. A review of this field is given by Trevisan and Bejan [14], Shivakumara et.al. [12] and also documented in the books by Vafai [15, 16] and Nield and Bejan [10].

Let $\bar{V}'$ and $\bar{V}'_f$ respectively be the averages of the fluid velocity over a volume element Ω_M of the medium (incorporating both solid and fluid materials) and over a volume element Ω_f of only the fluid. These two velocities are related by the following Dupuit-Forchheimer relation

$$\bar{V}' = \phi \bar{V}'_f \, . \qquad (1.1)$$

Where ϕ is the porosity of the medium and $\bar{V}'$ is called the mean fluid velocity. Equation of continuity is written as

$$\frac{d\rho'}{dt'} + \rho'\left(\nabla'.\bar{V}_f\right) = 0, \qquad (1.2)$$

where

$$\frac{d}{dt'} = \frac{\partial}{\partial t'} + \left(\bar{V}_f.\nabla'\right) = 0,$$

and ρ' is the fluid density. Using Dupuit-Forchheimer relation (1.1) and equation (1.2) becomes

$$\phi\, \frac{d\rho'}{dt'} + \rho'(\nabla'.\bar{V}') = 0. \qquad (1.3)$$

The momentum equation is written as

$$\rho'\left[\frac{\partial \bar{V}'_f}{\partial t'} + (\bar{V}'_f.\nabla')\bar{V}'_f\right] = -\nabla' P' - \frac{\mu}{K}\bar{V}' + \rho'\bar{g} + \mu_e \nabla'^2 \bar{V}'. \qquad (1.4)$$

If we drop inertial term of L.H.S of equation (1.4) and put $\mu_e = 0$ in equation (1.4), we get Darcy's flow model which is relevant to densely packed, low permeability porous media. Here K is the permeability of porous medium, μ_e is the effective fluid viscosity and μ is the fluid viscosity. Using the Dupuit-Forchheimer relation (1.4) becomes

$$\rho'\left[\frac{1}{\phi}\frac{\partial \bar{V}'}{\partial t'} + \frac{1}{\phi^2}(\bar{V}'.\nabla')\bar{V}'\right] = -\nabla' P' - \frac{\mu}{K}\bar{V}' + \rho'\bar{g} + \mu_e \nabla'^2 \bar{V}'. \qquad (1.5)$$

Equation (1.5) is known as Darcy-Lapwood-Brinkman equation and is valid for ϕ as small as *0.8*. Equation (1.5) reduces to a form of Navier-Stokes equation as $K \to \infty$. Earlier μ and μ_e were considered as equal but recent experiments show that the range of the ratio $\Lambda = \left(\frac{\mu_e}{\mu}\right)$ varies from *0.5* to *10.9* (Gilver and Altobelli, [5]).

Since the transport of heat through a porous medium involves two substances, fluid and porous matrix, the process will be characterized by specific parameters of these two substances. Generally properties of these two substances are different and we will consider two equations for heat transport in a porous medium, one for the fluid and other for the porous matrix. We assume that there is a local thermodynamic equilibrium so that

$$T'_s = T'_f = T'. \qquad (1.6)$$

Taking averages over an elemental volume of the medium, we have for the solid phase

$$(1-\phi)(\rho' C)_s \frac{\partial T'_f}{\partial t'} = (1-\phi)(\nabla'.\kappa_s \nabla')T'_s, \qquad (1.7)$$

and for the fluid phase, on using relation (1.1)

$$\phi(\rho' C_p)_f \frac{\partial T'_f}{\partial t'} + (\rho' C_p)_f(\bar{V}'.\nabla')T'_f = \phi(\bar{V}'.\kappa_f \nabla')T'_f, \qquad (1.8)$$

Here the subscripts *s* and *f* refer to the solid and fluid phases respectively, *C* is the specific heat of the solid, C_p specific heat at constant pressure of the fluid, κ is thermal conductivity. Setting $T'_s = T'_f = T'$ and adding equations (1.7) and (1.8) we have

$$(\rho' C)_M \frac{\partial T'}{\partial t'} + (\rho' C_p)_f(\bar{V}'.\nabla')T' = (\bar{V}'.\kappa_M \nabla')T', \qquad (1.9)$$

where

$$(\rho' C)_M = (1-\phi)(\rho' C)_s + \phi(\rho' C_p)_f,$$

and

$$\kappa_M = (1-\phi)\kappa_s + \phi\kappa_f,$$

are respectively the overall heat capacity per unit volume and the overall thermal conductivity of the porous medium. For constant κ_M equation (1.9) becomes

$$M\frac{\partial T'}{\partial t'} + (\bar{V}'.\nabla')T' = \kappa_T \nabla'^2 T' \qquad (1.10)$$

where $M = (\rho' C)_M/(\rho' C_p)_f$ is dimensionless heat capacity and is defined as the ratio of effective heat capacity of the porous medium to the heat capacity $(\rho' C_p)_f$ of the fluid, κ_T is effective thermal diffusivity of the fluid in a porous medium. Fluid density ρ' is given by

$$\rho' = \rho'_0[1 - \alpha(T' - T'_0)], \quad (1.11)$$

where $\alpha = -\rho'^{-1}_0(\partial\rho'/\partial T')$ is thermal expansion coefficient and ρ'_0 is mean flow density. Equations (1.3), (1.9), (1.10) and (1.11) are basic equations of convection in porous medium.

The object of this paper is to study stability of finite amplitude Rayleigh-Benard convective instability in a sparsely packed porous medium. In Section 2, we write basic dimensionless equations by using Darcy-Lapwood-Brinkman model for momentum equation. In Section 3, we study linear stability analysis. In Section 4, by using the multiple scale analysis of Newell and Whitehead [9], we have derived a two-dimensional time-dependent Landau-Ginzburg equation in complex amplitude *A(X,Y,T)* with real coefficients near the supercritical pitchfork bifurcation, where *X*, *Y* and *T* are slow space and time variables. In Section 4.1, we have shown the occurrence of secondary instabilities such as Eckhaus and Zigzag instabilities and also we have studied Nusselt number contribution at the onset of stationary convection from Landau-Ginzburg equation. In Section 5, we write conclusions of our paper.

2. Basic equations

Consider a horizontal, infinitely extended layer (heated from below) of a fluid in a porous material of depth d. The upper and lower bounding surfaces of the layer are assumed constant, except for the density in the buoyancy term, so that the Boussinesq approximation is valid. The porous medium is considered homogeneous and isotropic. The conduction state is characterized by

$$\bar{V}'_s = 0, \quad T'_s = T'_0 - \left(\frac{\Delta T'}{d}\right)z'.$$

Now the temperature perturbation can be written as $\theta' = T' - T'_s$. We define the dimensionless quantities by

$$x = \frac{x'}{d}, y = \frac{y'}{d}, z = \frac{z'}{d}, t = \frac{\kappa_T t'}{Md^2}, u = \frac{Mdu'}{\kappa_T}, v = \frac{Mdv'}{\kappa_T}, w = \frac{Mdw'}{\kappa_T}, \theta = \frac{\theta'}{\Delta T'},$$

$$P = \frac{P'}{\rho'_0 \kappa_T^2 M^{-1} d^{-2}} \quad (2.1)$$

Here Md^2/κ_T is the thermal diffusion time in porous medium. Using relation (2.1) and equation (1.11), equations (1.3), (1.9) and (1.10) can be written by using Boussinesq approximation in a dimensionless form as

$$\nabla.\bar{V} = 0, \quad (2.2)$$

$$\frac{1}{M^2\phi Pr}\left[\frac{\partial\bar{V}}{\partial t} + \frac{1}{\phi}(\bar{V}.\nabla)\bar{V}\right] = -\frac{\nabla P}{MPr} - \frac{1}{MDa}\bar{V} + \frac{\Lambda}{M}\nabla^2\bar{V} + R\theta\hat{e}_z, \quad (2.3)$$

$$\frac{\partial\theta}{\partial t} + \frac{1}{M}(\bar{V}.\nabla)\theta = \frac{w}{M} + \nabla^2\theta, \quad (2.4)$$

Here dimensionless parameters are: Prandtl number $Pr = \nu/\kappa_T$, Darcy number $Da = \kappa/d^2$ and Rayleigh number $R = g\alpha\Delta T' d^3/\nu\kappa_T$. Darcy number, *Da*, physically represents the scale factor which describes the extent of the division of the structure as compared to the vertical extent of the porous layer. When the scale factor (i.e., permeability *K*) is very high i.e., when the structure is very through blend of solid and fluid phases, the resistance to the flow becomes efficiently controlled by the ordinary viscous resistance $\mu\nabla^2\bar{V}$ and hence $Da \to \infty$ and $\Lambda = 1, M = 1, \phi = 1$. In this case the convection phenomena is similar to that in an ordinary fluid layer in a nonporous medium.

It is convenient to reduce the basic set of dimensionless equations (2.2) to (2.4) to a single equation. To do this we make curl of equation (2.3) which gives

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t}-\frac{\Lambda}{M}\nabla^2+\frac{1}{MDa}\right)\bar{\omega}-R(\nabla\times\theta\hat{e}_z)=-\frac{1}{M^2\phi^2 Pr}[\nabla\times(\bar{V}.\nabla)\bar{V}], \qquad (2.5)$$

where $\bar{\omega}=\nabla\times\bar{V}$ and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. Now taking the curl of the above equation (2.5) and using equation (2.2), we get

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t}-\frac{\Lambda}{M}\nabla^2+\frac{1}{MDa}\right)\nabla^2\bar{V}+R(\nabla\times\nabla\times\theta\hat{e}_z)=\frac{1}{M^2\phi^2 Pr}[\nabla\times\nabla\times(\bar{V}.\nabla)\bar{V}]. \qquad (2.6)$$

The z-component of equation (2.6) is

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t}-\frac{\Lambda}{M}\nabla^2+\frac{1}{MDa}\right)\nabla^2\bar{\omega}+R\nabla_h^2\theta=\frac{1}{M^2\phi^2 Pr}\hat{e}_z[\nabla\times\nabla\times(\bar{V}.\nabla)\bar{V}], \qquad (2.7)$$

where $\nabla_h^2=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$ is horizontal Laplacian operator. Eliminating θ from equations (2.4) and (2.7), we get

$$\mathcal{L}w=N, \qquad (2.8)$$

where

$$\mathcal{L}=\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t}-\frac{\Lambda}{M}\nabla^2+\frac{1}{MDa}\right)\left(\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2-\frac{R}{M}\nabla_h^2, \qquad (2.9)$$

and

$$N=\frac{1}{M^2\phi^2 Pr}\left(\frac{\partial}{\partial t}-\nabla^2\right)\hat{e}_z[\nabla\times\nabla\times(\bar{V}.\nabla)\bar{V}]-\frac{R}{M}\nabla_h^2(\bar{V}.\nabla)\theta. \qquad (2.10)$$

Boundary Conditions

In this paper, we consider stress-free boundary conditions on the surface and vanishing of temperature fluctuations. Thus for stress-free boundaries $W=D^2W=D^4W=0$ at $z=0,\ 1$ and θ at $z=0,\ 1$. W and its even derivatives vanish at $z=0$ and $z=1$ implies that we can assume $W=\sin\pi z$ as a solution of equation (3.1). Since stress-free boundaries cannot be achieved in the laboratory, their relevance can be questioned. However, stress-free boundaries may be appropriate in some geophysical fluid dynamics applications, moreover, they are very commonly used in analytical work, since they allow simple trigonometric eigen functions and also in numerical simulations.

3. Linear Stability Analysis

We study the stability of conduction state ($u=v=w=\theta=0$) by assuming an analytical solution $w=W(z)e^{iqx+pt}$ substituting this solution into $\mathcal{L}w=0$ we get

$$\left[(D^2-q^2)(D^2-q^2-p)\left\{\frac{\Lambda}{M}(D^2-q^2)-\frac{1}{MDa}-\frac{p}{M^2\phi Pr}\right\}\right]W(z)=-\frac{Rq^2}{M}W(z). \qquad (3.1)$$

Substituting $W = \sin \pi z$ into equation (3.1), we get the following eigenvalue relation

$$(\pi^2 + q^2)(\pi^2 + q^2 + p)\left[\frac{\Lambda}{\mathrm{M}}(\pi^2 + q^2) + \frac{1}{MDa} + \frac{p}{M^2 \phi Pr}\right] = \frac{Rq^2}{M}. \tag{3.2}$$

For the onset of stationary convection *p=0* and equation (3.2) gives

$$R = \frac{M\delta^4}{q^2}\left(\frac{\Lambda}{\mathrm{M}}\delta^2 + \frac{1}{MDa}\right). \tag{3.3}$$

Equation (3.3) gives the value of Rayleigh number for Darcy-Lapwood-Brinkman model of porous medium. We define the minimum of *R* from equation (3.3) as $(\partial R/\partial q)_{q=q_{sc}} = 0$ which implies that

$$2\left(\frac{q_{sc}}{\pi}\right)^6 + \left(3\Lambda + \frac{1}{Da\pi^2}\right)\left(\frac{q_{sc}}{\pi}\right)^4 = \Lambda + \frac{1}{Da\pi^2}. \tag{3.4}$$

Solving equation (3.4) for different values of *Da* and Λ, we get the value of critical wave number q_{sc} for the onset of stationary convection. Substituting this value of q_{sc} into equation (3.3), we get the value of the Rayleigh number for the onset of stationary convection as

$$R_{sc} = \frac{M\delta^4{}_{sc}}{q^2}\left(\frac{\Lambda}{\mathrm{M}}\delta^2{}_{sc} + \frac{1}{MDa}\right). \tag{3.5}$$

In Table 1 and Table 2, we give values of R_{sc} and q_{sc} for *Da = 1500* and *50*, M = 0.9 and Λ = 0.5*,* 1 and 10.

Table 1($M = 0.9$, *Da*=1500)

	$\Lambda = 0.5$	$\Lambda = 1$	$\Lambda = 10$
q_{sc}	2.22149	2.22147	2.22144
R_{sc}	328.785	657.541	6575.14

Table 2 ($M = 0.9$, *Da*=50)

	$\Lambda = 0.5$	$\Lambda = 1$	$\Lambda = 10$
q_{sc}	2.22294	2.22219	2.22152
R_{sc}	329.644	658.4	6576

Case1: ($\Lambda = 1, M = 1, Da \to \infty$)

This corresponds to the onset of stationary convection in a non-porous medium, here

$$R_{sc} = \frac{(\pi^2 + q^2{}_{sc})^3}{q^2{}_{sc}} \tag{3.6}$$

and

$$q_{sc} = \pi/\sqrt{2}, \quad R_{sc} = 27\pi^4/4 \approx 657.5 \tag{3.7}$$

Here R_{sc} denotes the critical Rayleigh number. It represents the threshold value at which the convection first sets in and it corresponds to the pitchfork bifurcation. If $R < R_{sc}$, disturbances with the wave number q will be stable, these disturbances become marginally stable when $R = R_{sc}$. If $R > R_{sc}$ then the same disturbance will be unstable. The property that convection with $q > q_{sc}$ requires a high Rayleigh number is mainly a consequence of heat conduction between up and down going fluid particles which diminishes the release of potential energy.

Case 2: ($\Lambda = 0$)

This corresponds to the onset of stationary convection in Darcy's model of porous medium. Here

$$R = \frac{1}{Da}\frac{(\pi^2 + q^2{}_{sc})^3}{q^2{}_{sc}} \tag{3.8}$$

In Darcy's model $q_{sc} = \pi$, $R_{sc} = 4\pi^2/Da$ and hence $R_{D_{sc}} = 4\pi^2$ where we define $R_D = DaR$ as Rayleigh number in Darcy's model.

In Table 1 and Table 2, we have shown that critical wave number q_{sc} does not depend on Λ but q_{sc} of Darcy-Lapwood-Brinkman model is always less than $q_{sc} = \pi$ of Darcy's model (Palm et.al. [11]) and is always greater than $q_{sc} = \pi/\sqrt{2}$ of non-porous medium (Chandrasekhar [4]). We have also shown that R_{sc} (for $\Lambda = 1$) in Darcy-Lapwood-Brinkman model is less than $R_{sc} = 27\pi^4/4$ for non-porous medium (Chandrasekhar [4]) but is larger than $R_{sc} = 27\pi^4/Da$ for Darcy's model. As Figure 1, shows, the system is stable when *p<0* and unstable when *p>0*. The minimum in Figure 1, determines the critical Rayleigh number and the unique critical wave number. According to linear theory a continuum of unstable wave numbers exists for any supercritical Rayleigh number. This feature is of major importance for supercritical convection.

4. Two Dimensional Landau-Ginzburg equation near the onset of stationary convection

A Small amplitude convection cell is imposed on the basic flow. If this amplitude is of size $O(\epsilon)$ then the intersections of the cell with itself forces a second harmonic and mean state correction of size $O(\epsilon^2)$ and these in turn drives a $O(\epsilon^3)$ correction to the fundamental component of the imposed roll. A solvability criterion for this last correction yields an equation of the complex valued amplitude *A(X, Y, T)* of the imposed disturbance, the two-dimensional Landau-Ginzburg equation. To simplify the problem we assume the formation of cylindrical rolls with axis parallel to *y*-axis, so that *y*-dependence disappears from equation (2.8). The *z*-dependence is contained entirely in the *sine* and *cosine* functions, which ensure that the free boundary conditions are satisfied. We use the expansion parameter ϵ as

$$\epsilon^2 = \frac{R - R_{sc}}{R_{sc}}. \tag{4.1}$$

For the values of R close to threshold value R_{sc}, i.e., $\epsilon^2 \ll 1$ the structure of the slow length scales will be insensitive to ϵ, but a slow modulation in a space and time is possible by making use of the band of the unstable solutions, and linear growth is likely to saturate due to nonlinear effects. This behavior can be analyzed by writing solutions of equations (2.2) - (2.4) in powers of ϵ as

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 \ldots \ldots, \tag{4.2}$$

where $f = f(u, v, w, \theta)$ with the first approximation is given by the eigenvector of the linearized problem:

$$u_0 = \frac{i\pi}{q_{sc}}\left[A(X, Y, T)e^{iq_{sc}x}\cos\pi z - c.c.\right], \tag{4.3}$$

$$v_0 = 0, \tag{4.4}$$

$$w_0 = \left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c.c.\right], \tag{4.5}$$

$$\theta_0 = \frac{1}{M\delta^2{}_{sc}}\left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c.c.\right], \tag{4.6}$$

where $\delta^2 = \pi^2 + q^2{}_{sc}$. Here c.c. stands complex conjugate, $e^{iq_{sc}x}\sin\pi z$ is the critical mode for the linear problem at $R = R_{sc}$ and $q = q_{sc}$ complex amplitude *A(X, Y, T)* depend on the slow variable *X, Y, Z* and *T* are scaled by introducing multiple scales

$$X = \epsilon x, Y = \epsilon^{\frac{1}{2}} y, Z = z \text{ and } T = \epsilon^2 t, \tag{4.7}$$

and these formally separate the fast and slow dependent variables in *f*. It should be noted that the difference in scaling in the two directions reflects the inherent symmetry breaking of instability which was chosen here with wave vector in *x*-direction. The differential operators can be expressed as

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial X},\ \ \frac{\partial}{\partial y} \to \epsilon^{\frac{1}{2}}\frac{\partial}{\partial Y},\ \ \frac{\partial}{\partial z} \to \frac{\partial}{\partial Z},\ \ \frac{\partial}{\partial t} \to \epsilon\frac{\partial}{\partial T}\,. \tag{4.8}$$

The solvability condition for the later yields the amplitude equation. Using relation (4.8), the operators (2.9) and (2.10) can be written as

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 \ \ldots\ldots, \tag{4.9}$$

$$N = \epsilon^2 N_0 + \epsilon^3 N_1 \ldots\ldots, \tag{4.10}$$

where

$$\mathcal{L}_0 = \left(\frac{\Lambda}{\mathrm{M}}\nabla^2 - \frac{1}{MDa}\right)\nabla^4 - \frac{\mathrm{R}_{sc}}{\mathrm{M}}\frac{\partial^2}{\partial x^2}, \tag{4.11}$$

$$\mathcal{L}_1 = \left(2\frac{\partial^2}{\partial x\partial X} + \frac{\partial^2}{\partial Y^2}\right)\left(3\frac{\Lambda}{\mathrm{M}}\nabla^4 - \frac{2}{MDa}\nabla^2 - \frac{\mathrm{R}_{sc}}{\mathrm{M}}\right), \tag{4.12}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial T}\left(-\frac{\Lambda}{\mathrm{M}}\nabla^4 - \frac{\nabla^4}{M^2\phi Pr} + \frac{\nabla^2}{MDa}\right) + \frac{\partial^2}{\partial X^2}\left(3\frac{\Lambda}{\mathrm{M}}\nabla^4 - \frac{2}{MDa}\nabla^2 - \frac{\mathrm{R}_{sc}}{\mathrm{M}}\right)$$
$$+ \left(2\frac{\partial^2}{\partial x\partial X} + \frac{\partial^2}{\partial Y^2}\right)^2\left(3\frac{\Lambda}{\mathrm{M}}\nabla^2 - \frac{1}{Da}\right) - \frac{\mathrm{R}_{sc}}{\mathrm{M}}\frac{\partial^2}{\partial x^2}, \tag{4.13}$$

substituting equations (4.2), (4.9) and (4.10) into equation (2.8), we get by equating the coefficients of $\epsilon, \epsilon^2, \epsilon^3$

$$\mathcal{L}_0 w_0 = 0, \tag{4.14}$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = N_0, \tag{4.15}$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = N_1, \tag{4.16}$$

Putting the value of w_0 from equation (4.3) into equation (4.14), we get

$$R_{sc} = \frac{M\delta^4{}_{sc}}{q^2}\left(\frac{\Lambda}{\mathrm{M}}\delta^2{}_{sc} + \frac{1}{MDa}\right). \tag{4.17}$$

substituting the value of w_0 into $\mathcal{L}_1 w_0 = 0$, we get, $q_{sc} = \pi/\sqrt{2}$. In equation (4.15),$N_0 = 0$, $\mathcal{L}_1 w_0 = 0$ implies that equation (4.15) reduce to $w_1 = 0$. Similarly$u_1 = 0$, $v_1 = 0$, from equation (2.4) we get,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta_1 = u_0\frac{\partial\theta_0}{\partial x} + w_0\frac{\partial\theta_0}{\partial z} \tag{4.18}$$

$$\theta_1 = -\frac{1}{2M\pi\delta^2}|A|^2\sin 2\pi z \tag{4.19}$$

To determine the variation of A we turn to equation (4.16) and rewrite it as $\mathcal{L}_0 w_2 = N_1 - \mathcal{L}_2 w_0$. In order that this equation is solvable in the presence of $\mathcal{L}_0 w_0 = 0$, one requires that r.h.s be orthogonal to w_0, which is ensured that if the coefficient of $\sin\pi z$ is zero. For the term N_1 we notice that a contribution of the form $\sin\pi z$ comes from$R_{sc}\frac{\partial^2}{\partial x^2}(w_0\frac{\partial}{\partial z})\theta_1$. Hence equating coefficient of $\sin\pi z$ in $N_1 - \mathcal{L}_2 w_0$to zero, we get,

$$\lambda_0\frac{\partial A}{\partial T} - \lambda_1\left(\frac{\partial}{\partial X} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial Y^2}\right)^2 A - \lambda_2 A + \lambda_3|A|^2 A = 0, \tag{4.20}$$

where

$$\lambda_0 = \left(\frac{\Lambda}{\mathrm{M}} + \frac{1}{M^2\phi Pr}\right)\delta^4{}_{sc} + \frac{\delta^2{}_{sc}}{MDa},$$

$$\lambda_1 = 4q^2{}_{sc}\left(3\frac{\Lambda}{\mathrm{M}}\delta^2{}_{sc} + \frac{1}{MDa}\right)\delta^4{}_{sc},$$

$$\lambda_2 = \frac{R_{sc}q^2{}_{sc}}{M},$$

$$\lambda_3 = \frac{R_{sc}q^2{}_{sc}}{2M^3\delta^2{}_{sc}}.$$

Equation (4.20) is two-dimensional, nonlinear time dependent Landau-Ginzburg equation. Dropping the time dependence from equation (4.20), we get

$$\frac{d^2A}{dX^2} + \frac{\lambda_2}{\lambda_1}\left(1 - \frac{\lambda_3}{\lambda_1}|A|^2\right)A = 0, \tag{4.21}$$

since $\lambda_1 > 0$, the solution of equation (4.21) is given by

$$A(X) = A_0\tan h\left(\frac{X}{\Lambda_1}\right), \tag{4.22}$$

where

$$A_0 = (\lambda_2/\lambda_3)^{\frac{1}{2}} \text{ and } \Lambda_1 = (2\lambda_1/\lambda_2)^{\frac{1}{2}}. \tag{4.23}$$

4.1 Long wave-length Instabilities (Secondary Instabilities)

The two-dimensional Landau-Ginzburg equation (4.20), can be written in fast variables *x, y, t* and $A(X,Y,T) = A(x,y,t)/\epsilon$, as

$$\lambda_0 \frac{\partial A}{\partial T} - \lambda_1 \left(\frac{\partial}{\partial x} - \frac{i}{2q_{sc}} \frac{\partial^2}{\partial y^2} \right)^2 A - \epsilon^2 \lambda_2 A + \lambda_3 |A|^2 A = 0, \tag{4.24}$$

In order to study the properties of a structure with a given phase winding number δk, we substitute

$$A(x,y,t) = A_1(x,y,t)\, e^{i\delta kx}, \tag{4.25}$$

Into the equation (4.24) and we obtain

$$\lambda_0 \frac{\partial A_1}{\partial t} = (\epsilon^2 \lambda_2 - \lambda_1(\delta k)^2) A_1 + 2i\lambda_1 \delta k \left(\frac{\partial}{\partial x} - \frac{i}{2q_{sc}} \frac{\partial^2}{\partial y^2} \right) A_1 + \lambda_1 \left(\frac{\partial}{\partial x} - \frac{i}{2q_{sc}} \frac{\partial^2}{\partial y^2} \right)^2 A_1 - \lambda_3 |A_1|^2 A_1 = 0. \tag{4.26}$$

The steady state uniform solution of equation (4.26) is

$$A_1 = A_{1o} = [(\epsilon^2 \lambda_2 - \lambda_1(\delta k)^2)\lambda_3^{-1}]^{\frac{1}{2}}. \tag{4.27}$$

Let $\tilde{u}(x,y,t) + i\tilde{v}(x,y,t)$ be an infinitesimal perturbation from a uniform steady state solution A_{1o} given by equation (4.27). Now substituting

$$A_1 = A_{1o} = [(\epsilon^2 \lambda_2 - \lambda_1(\delta k)^2)\lambda_3^{-1}]^{\frac{1}{2}} + \tilde{u} + i\tilde{v}$$

Into equation (4.26) and equating real and imaginary parts, we obtain

$$\lambda_0 \frac{\partial \tilde{u}}{\partial t} = \left[-2(\epsilon^2 \lambda_2 - \lambda_1(\delta k)^2) + \lambda_1 \left(\frac{\partial^2}{\partial x^2} + \frac{\delta k}{q_{sc}} \frac{\partial^2}{\partial y^2} - \frac{1}{4q^2_{sc}} \frac{\partial^4}{\partial y^4} \right) \right] \tilde{u} - \left(2\lambda_1 \delta k - \frac{\lambda_1}{q_{sc}} \frac{\partial^2}{\partial y^2} \right) \frac{\partial \tilde{v}}{\partial x}, \tag{4.28a}$$

$$\lambda_0 \frac{\partial \tilde{v}}{\partial t} = \left(2\lambda_1 \delta k - \frac{\lambda_1}{q_{sc}} \frac{\partial^2}{\partial y^2} \right) \frac{\partial \tilde{u}}{\partial x} + \lambda_1 \left(\frac{\partial^2}{\partial x^2} + \frac{\delta k}{q_{sc}} \frac{\partial^2}{\partial y^2} - \frac{1}{4q^2_{sc}} \frac{\partial^4}{\partial y^4} \right) \tilde{v}. \tag{4.28b}$$

We analyze equations (4.28a) and (4.28b) by using normal modes of the form

$$\tilde{u} = Ue^{St} \cos(q_x x) \cos(q_y y),\ \tilde{v} = Ve^{St} \sin(q_x x) \cos q_y y, \tag{4.29}$$

Putting equation (4.29) into equation (4.8a) and (4.8c) we get,

$$[\lambda_0 S + 2(\epsilon^2 \lambda_2 - \lambda_1(\delta k)^2) + \chi_1] U + \lambda_1 q_x \chi_2 V = 0, \tag{4.30a}$$

$$\lambda_1 q_x \chi_2 U + (\lambda_0 S + \chi_1) V = 0 \tag{4.30b}$$

Here $\chi_1 = \lambda_1\left[q^2{}_x + \left(q^2{}_y \delta k\right)/q_{sc} + q^4{}_y/4q^2{}_{sc}\right]$, $\chi_2 = (2\delta k + q^2{}_y/q_{sc})$. On solving equation (4.30a) and equation (4.30b) we get,

$$\lambda^2{}_0 S^2 + 2S[2\lambda_0(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) + \lambda_0\chi_1] + [2(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) + \chi_1]\psi_1 - q^2{}_x\lambda_1\chi_2 = 0,$$

Whose roots ($S\pm$) are real. Here $(S\pm)$ is defined as

$$(S\pm) = -\frac{1}{\lambda^2{}_0}\left\{(2\lambda_0(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) + \lambda_0\chi_1) \pm \left(2\lambda_0(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2)^2 + \lambda^2{}_1 q^2{}_x \chi^2{}_2\right)^{\frac{1}{2}}\right\}. \qquad (4.31)$$

Solution $S(-)$ is clearly negative, thus the corresponding mode is stable and if $S(+)$ is positive then rolls can be unstable. Symmetry considerations help us to restrict the study of $S(+)$ to a domain $q_x \geq 0, q_y \geq 0$.

4.1.1 Longitudinal perturbations and Eckhaus instability

Inserting $q_y = 0$ into equation (4.31), we get

$$\lambda^2{}_0 S^2 + 2S\left[2\lambda_0(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) + \lambda_0\lambda_1 q^2{}_x\right] + \lambda_1 q^2{}_x\left[2(\epsilon^2\lambda_2 - 3\lambda_1(\delta k)^2) + q^2{}_x\right] = 0,$$

since the roots are real and their sum always negative, the pattern is stable as long as both roots are negative, i.e., their product is positive. The cell pattern becomes unstable when the product is negative, i.e., when

$$q^2{}_x \leq 2(3\lambda_1(\delta k)^2 - \epsilon^2\lambda_2),$$

For this requires $|\delta k| \geq \sqrt{(\epsilon^2\lambda_2/3\lambda_1)}$, this condition defines the domain of Eckhaus Instability. The above condition implies that the most unstable wave vector tends to zero, when

$$|\delta k| \to \sqrt{(\epsilon^2\lambda_2/3\lambda_1)}.$$

4.1.2 Transverse perturbations and zigzag instability

Let us consider $q_x = 0$ into equation (4.31), we get

$$\lambda^2{}_0 S^2 + 2S\left[2\lambda_0(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) + \lambda_0\chi^y{}_1\right] + \left[2(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) + \chi^y{}_1\right]\chi^y{}_1 = 0,$$

where $\chi^y{}_1 = \lambda_1(q^2{}_y\delta k/q_{sc} + q^4{}_y/4q^2{}_{sc})$. The two eigenmodes are uncoupled and we have $S(-)$,

$$S(-) = -2(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2) - \frac{\lambda_1}{q_{sc}}\delta k q^2{}_y - \frac{\lambda_1}{4q^2{}_{sc}}q^2{}_y < 0,$$

for one of them. The other is amplified when

$$S(+) = -\lambda_1 q^2{}_y\left(\delta k + \frac{q^2{}_y}{4q_{sc}}\right) > 0.$$

This implies that $\delta k < 0$ the above condition defines the domain of the Zigzag Instability. When $\delta k \to 0$ from below the wave vector q_y of the instability also tends to zero, while the growth rate varies as $q^2{}_y$. For this system λ_1 is positive and λ_2 depends on *M*. We have observed that Eckhaus Instability and Zigzag Instability regions increases for some fixed parameters $Pr, \phi,\ M, \Lambda$ and *Da* (see Figures 2a and 2b).

4.1.3 Heat transport by convection

The maximum of steady amplitude *A* is denoted by $|A_{max}|$which is given as

$$|A_{max}| = (\frac{\epsilon^2 \lambda_2}{\lambda_3})^{\frac{1}{2}}, \tag{4.32}$$

equation (4.32) is obtained from equation (4.22) with $\tan h(X/\Lambda_1) = 1$. We use $|A_{max}|$to calculate Nusselt number *Nu*. To discuss the heat transfer near the neutral region, we express it through the Nusselt number is defined as $Nu = Hd/\kappa\Delta T$,which the ratio of the heat transported across any layer to the heat which would be transported by conduction alone. Here *H* is the rate of heat transfer per unit area and is defined as

$$H = -\langle\frac{\partial T_{total}}{\partial z'}\rangle_{z'=0}. \tag{4.33}$$

In equation (4.33), angular brackets correspond to a horizontal average. The Nusselt number can be calculated in terms of amplitude *A* and is given as

$$Nu = 1 + \frac{\epsilon^2}{\delta^2{}_{sc}} |A_{max}|^2. \tag{4.34}$$

From equation (4.34) we get conduction for $R_1 < R_{1sc}$ and convection for $R_1 > R_{1sc}$, since the amplitude equation is valid for $\lambda_3 > 0$, which is possible for $R_1 > R_{1sc}$ (supercritical pitchfork bifurcation), we get $Nu > 1$for $R_1 > R_{1sc}$. Thus we get convection for $Nu > 1$ and conduction for $Nu = 1$. In stationary convection *Nu* increases implies that heat conducted by steady mode increases. In the problem of Stability of finite amplitude Rayleigh Benard Convection in Porous Medium, *Nu* depends on M, Λ and *Da*. We have computed *Nu* for some fixed values of other parameters and observed that *Nu* increases for some fixed parameters M, Λ and *Da* (see Figure 3a-3d). This implies that Stability of finite amplitude Rayleigh Benard Convection in Porous Medium inhibits the heat transport.

5. Conclusions

In this paper we have considered stability of finite amplitude Rayleigh Benard Convection in Porous Medium by using free-free (stress-free) boundary conditions. Our goal is to identify the region of parameter values, for which rolls emerge at the onset of convection. Rayleigh-Benard convection in a porous medium is an example of single diffusive system and hence principle of exchange of stabilities is valid for this system which we have shown in the linear stability analysis. According to linear theory a continuum of unstable wave numbers exists for any supercritical Rayleigh number. This feature is of major importance for supercritical convection. By using multiple scale perturbation theory we have obtained two dimensional Landau-Ginzburg equation at the onset of stationary convection for free-free boundary conditions. From this amplitude equation we have obtained conditions for long wave length instabilities viz., Eckhaus and Zigzag instabilities. In equation (4.20) with fast variables *x, y* and *t* gives condition for Eckhaus Instability $\sqrt{\epsilon^2\lambda_2/3\lambda_1} \leq \delta q_s \leq \sqrt{\epsilon^2\lambda_2/\lambda_1}$ and Zigzag Instability $\delta q_s < 0$. The equation (4.20) with fast variables gives $\delta q_s \to 0$, as$q \to q_s$. From equation (4.20) which is valid for $\lambda_3 > 0$ and pitchfork bifurcation is possible for a certain choice of physical parameters, and it always occurs at a lower value of the Rayleigh number than the one obtained for simple bifurcation. We have calculated Nusselt number *Nu* and studied heat transport by convection.

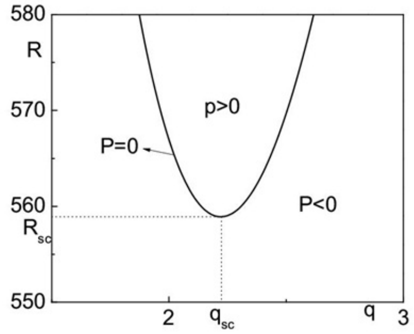

Figure 1: The neutral stability curve for a layer of motionless fluid. The system is stable when $p < 0$ and unstable when $p > 0$. This graphs is plotted for fixed values of $\Lambda = 0.85, M = 0.9$ and $Da = 1500$.

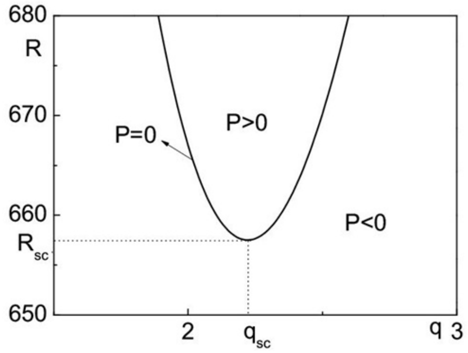

Figure 2: The neutral stability curve for a layer of motionless fluid. The system is stable when $p < 0$ and unstable when $p > 0$. This graphs is plotted for fixed values of $\Lambda = 1, M = 1$ and $Da \to \infty$.

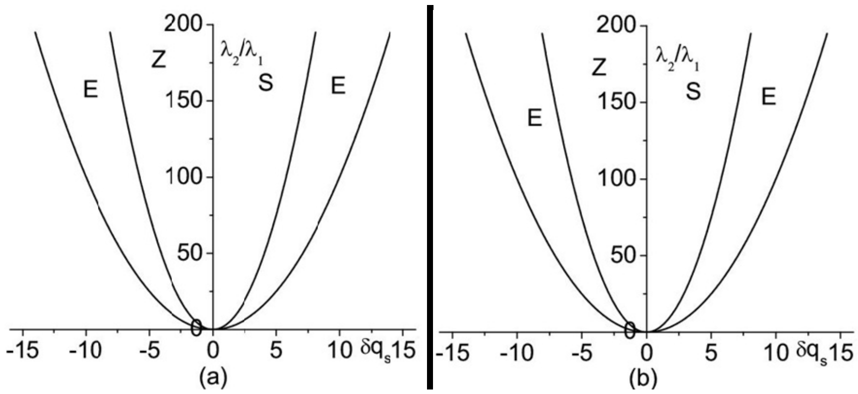

Figure 3: Graph (a) is plotted for $Da = 1500$ and Graph (b) is plotted for $Da \to \infty$ for the fixed values of $\Lambda = 0.85$ and $\phi = 0.9$ Numerically computed secondary instability regions of Eckhaus Instability (E), Zigzag Instability (Z) and Stable regions (S) are plotted in $(\lambda_2/\lambda_1,\ \delta q_s)$-plane.As $|\delta q_s|$ increases then the secondary instability regions increases.

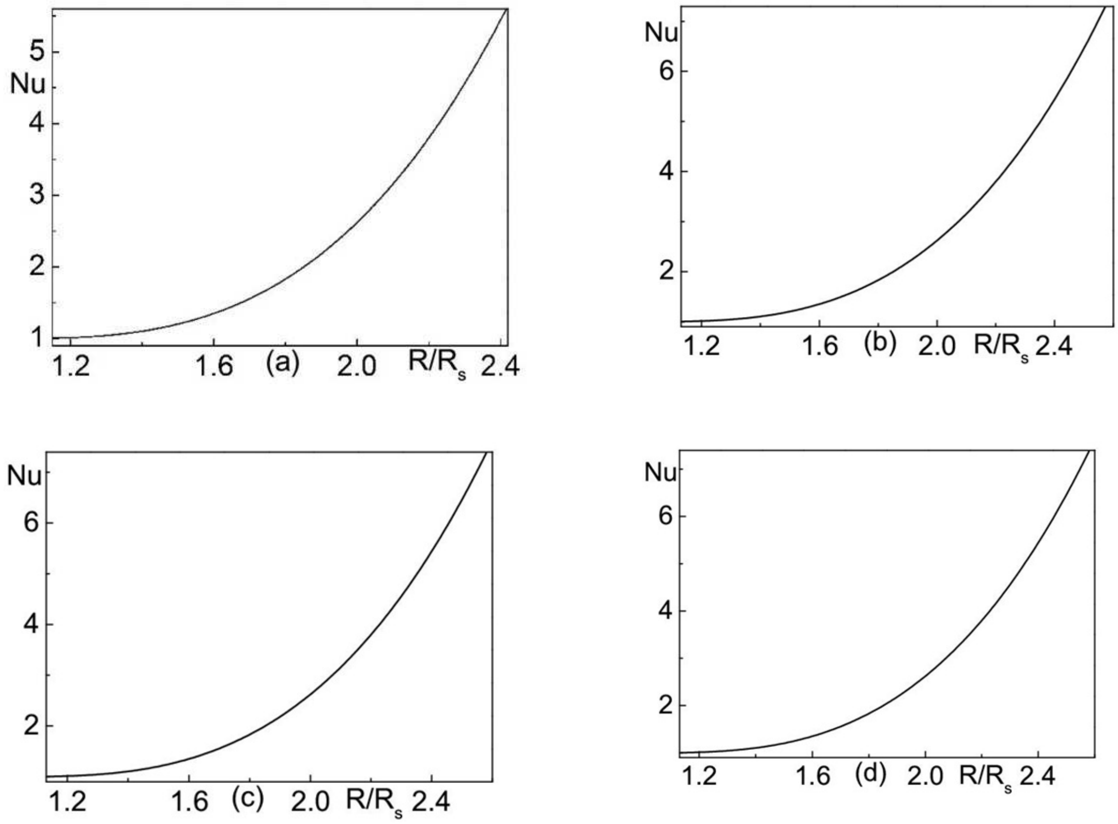

Figure 3: Graph (a) is plotted for $Da = 1$, Graph (b) is plotted for $Da = 15$, Graph (c) is plotted for $Da = 1500.$ and Graph (d) is plotted for $Da \to \infty$ for the fixed values of $\Lambda = 0.85$ and $M = 0.9$. In $(Nu, R/R_{sc})$-plane. In Graphs (a)-(d), as $, R/R_{sc}$ increases then *Nu* increases.

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