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Composition of fractional integral operators involving the Prathima's multivariable I-function

Frédéric Ayant

Teacher in High School , France
E-mail :fredericayant@gmail.com

ABSTRACT
Jain and Sharma [8] have studied compositions of fractional integral operators involving the product of multivariable H-function, a general class of polynomials of one variable and sequence of functions. In this paper, the author derive compositions of fractional integral operators associated with multivariable I-function defined by Prathima et al [9] and class of multivariable polynomials defined by Srivastava [15] and a sequence of functions. We will quote the particular case concerning the multivariable H-function defined Srivastava et al [17].

Keywords: Multivariable I-function, multivariable class of polynomials, fractional integral operator, multivariable H-function, sequence of functions.

2010 Mathematics Subject Classification. 33C60, 82C31

1. Introduction and preliminaries.

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, it has turned out many phenomena in physics, mechanics, chemistry, biology and other sciences can be described very successfully by models using mathematical tools by models using mathematical tools from fractional calculus.
The generalized polynomials defined by Srivastava [15], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.1}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.
The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I_{p,q:p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.2}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.3}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.4}$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \tag{1.5}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [2] the I-function of r variables is analytic if :

$$U_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\leqslant 0, i=1,\cdots,r \tag{1.6}$$

The integral (2.1) converges absolutely if

$|arg(z_k)|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r$ where

$$\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}>0 \tag{1.7}$$

Agarwal and Chaubey [1], Salim [11] and several others have studied a general sequence of functions. In the present document we shall study the following useful series formula for a general sequence of functions.

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-sx^r}]=\sum_{w,v,u,t,e,k_1,k_2,}\psi(w,v,u,t,e,k_1,k_2)x^R \tag{1.8}$$

where $$\sum_{w,v,u,t,e,k_1,k_2}=\sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{c=0}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \tag{1.9}$$

and the infinite series on the right side (1.8) is absolutely convergent, $R=ln+qv+pt+rw+k_1r+k_2q$ (1.10)

and $$\psi(w,v,u,t,e,k_1,k_2)=\frac{(-)^{t+w+k_2}(-v)_u(-t)_e(\alpha)_tl^n}{w!v!u!t!e!K_nk_1!k_2!}\frac{s^{w+k_1}F^{\gamma n-t}}{(1-\alpha-t)_e}(\alpha-\gamma n)_e$$

$$(-\beta-\delta n)_vg^{v+k_2}h^{\delta n-v-k_2}(v-\delta n)_{k_2}E^t\left(\frac{pe+rw+\lambda+qn}{l}\right)_n \tag{1.11}$$

where K_n is a sequence of constants. This function will note $R_n^{\alpha,\beta}[x]$
By suitably specializing the parameters involving in (2.1), a general sequence of function reduced to generalized polynomials set studied by Raizada [10], a class of polynomials introduced by Fujiwara [5] and several others authors.

2. Definitions

We will note $u=X\left(1-\frac{y^t}{x^t}\right)$ and $v=X\left(1-\frac{x^t}{y^t}\right)$

The pair of new extended fractional integral operators are defined by the following equations :

$$Q_{z_r}^{\eta,\alpha}[f(x)] = tx^{-\eta-t\alpha-1}\int_0^x y^{\eta}(x^t-y^t)^{\alpha} I_{p,q:p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c} \mathrm{z}_1\upsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\upsilon_r \end{array}\middle| \begin{array}{c} (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}: \\ \\ \\ (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) R_n^{\alpha_1,\beta_1}[X\left(1-\frac{y^t}{x^t}\right)^{\mu};E,F,g,h;p,q;\gamma_1;\delta_1;e^{-su^r}]$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r} \end{array}\right) f(y)\mathrm{d}y \tag{2.1}$$

$$R_{\gamma_n}^{\delta,\beta}[f(x)] = tx^{\delta}\int_x^{\infty} y^{-\delta-t\beta-1}(y^t-x^t)^{\beta} I_{p',q':p'_1,q'_1;\cdots;p'_r,q'_r}^{0,n':m'_1,n'_1;\cdots;m'_r,n'_r}\left(\begin{array}{c} \gamma_1\mu_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_r\mu_r \end{array}\middle| \begin{array}{c} (\mathrm{a'}_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(r)}_j;A'_j)_{1,p'}: \\ \\ \\ (\mathrm{b'}_j;\beta'^{(1)}_j,\cdots,\beta'^{(r)}_j;B'_j)_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c'}_j^{(1)},\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(r)}_j,\gamma'^{(r)}_j;C'^{(r)}_j)_{1,p'_r} \\ \\ (\mathrm{d'}_j^{(1)},\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(r)}_j,\delta'^{(r)}_j;D'^{(r)}_j)_{1,q'_r} \end{array}\right) R_{n'}^{\alpha'_1,\beta'_1}[X'\left(1-\frac{x^t}{y^t}\right)^{\mu'};E',F',g',h';p',q';\gamma'_1;\delta'_1;e^{-sv^r}]$$

$$S_{N'_1,\cdots,N'_r}^{M'_1,\cdots,M'_r}\left(\begin{array}{c} \mathrm{y'}_1\left(1-\frac{x^t}{y^t}\right)^{h'_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y'}_r\left(1-\frac{x^t}{y^t}\right)^{h'_r} \end{array}\right) f(y)\mathrm{d}y \tag{2.2}$$

where $\upsilon_i = \left(1-\frac{y^t}{x^t}\right)^{\upsilon_i}$, $\mu_i = \left(1-\frac{x^t}{y^t}\right)^{\upsilon'_i}$; $t,\upsilon_i,\upsilon'_i,h_i,\mu,\mu'$ and h'_i are positive numbers.

The conditions for the existence of these operators are as follows :

(a) $f(x)\in L_{\ddot{p}}(0,\infty)$, (b) $1\leqslant \ddot{p},\ddot{q}<\infty, \ddot{p}^{-1}+\ddot{q}^{-1}=1$

(b)$Re\left[t\alpha+Rt\mu+t\sum_{i=1}^{r}\left(v_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+K_ih_i\right)\right]>-\ddot{q}^{-1}, Re(\eta)>-\frac{1}{\ddot{q}}$

(c) $Re\left[t\beta+R't\mu'+t\sum_{i=1}^{r}\left(v'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}+K'_ih'_i\right)\right]>-\ddot{q}^{-1}$

(d)$Re\left[\delta+R't\mu'+t\sum_{i=1}^{r}\left(v'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}+K'_ih'_i\right)\right]>-\ddot{p}^{-1}$

(e) $\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}>0$

(f) $\Delta'_k=-\sum_{j=n'+1}^{p'}A'_j\alpha_j'^{(k)}-\sum_{j=1}^{q'}B'_j\beta_j'^{(k)}+\sum_{j=1}^{m'_k}D_j'^{(k)}\delta_j'^{(k)}-\sum_{j=m'_k+1}^{q'_k}D_j'^{(k)}\delta_j'^{(k)}+\sum_{j=1}^{n'_k}C_j'^{(k)}\gamma_j'^{(k)}-\sum_{j=n'_k+1}^{p'_k}C_j'^{(k)}\gamma_j'^{(k)}>0$

The main results of this paper unify the earlier results by S.P. Goyal, R.M. Jain and Neelima Gaur [6], R.K. Saxena and O.P. Dave [12], R.K. Saxena, Y. Singh and A. Ramawat [13], H.M. Srivastava, S.P. Goyal and R.M. Jain [16], Jain and sharma [8] and several others.

3. Required results

The following results are used in next section [7, page 286 (3.197,3)], [3, page 64, (23)], [4 page 201 (8)] and [3 page 62 (15)].

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\upsilon}\mathrm{d}x=B(\lambda,\mu)_2F_1(\upsilon,\lambda,\lambda+\mu;\beta) \tag{3.1}$$

provided that $Re(\lambda)>0, Re(\mu)>0, |\beta|<1$

$$_2F_1(a,b,c;z)=(1-z)^{c-a-b}{}_2F_1(c-a,c-b,c;z), |z|<1 \tag{3.2}$$

$$\int_y^\infty x^{-\lambda}(x+\alpha)^{\upsilon}(x-y)^{\mu-1}\mathrm{d}x=y^{\mu+\upsilon-\lambda}B(\lambda,\lambda-\mu-\upsilon)\left(1+\frac{\alpha}{y}\right)^{\mu+\upsilon}{}_2F_1(\lambda,\mu;\lambda-\upsilon;-\alpha/y) \tag{3.3}$$

$$_2F_1(a,b,c;-z)=\frac{1}{2\pi\omega}\int_{c-\omega\infty}^{c+\omega\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}z^s\mathrm{d}s \tag{3.4}$$

provided that $|arg(-z)|<\pi$

In this paper ,thus $0,\cdots,0$ would me r zeros and so on.

4. Composition of operators of the same nature

Let $X=m_1,n_1;\cdots;m_r,n_r;m_1,n_1;\cdots;m_r,n_r;1,0$ (4.1)

$$Y=p_1,q_1;\cdots;p_r,q_r;p_1,q_1;\cdots;p_r,q_r;0,1 \tag{4.2}$$

$$A=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},0,\cdots,0,0;A_j)_{1,p}\ ;(a_j;0,\cdots,0,\alpha_j^{(1)},\cdots,\alpha_j^{(r)},0;A_j)_{1,p} \tag{4.3}$$

$$B=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)},0,\cdots,0,0;B_j)_{1,q};(b_j;0,\cdots,0,\beta_j^{(1)},\cdots,\beta_j^{(r)},0;B_j)_{1,q} \tag{4.4}$$

$$A_1 =(-\alpha - R\mu - \sum_{i=1}^{r} K_i h_i; v_1, \cdots, v_r, 0, \cdots, 0, 0; 1), (-\beta - R'\mu' - \sum_{i=1}^{r} L_i h_i; 0, \cdots, 0, v_1, \cdots, v_r, 0; 1),$$

$$\left(-\beta - R\mu' - \frac{(\delta-\eta)}{t} - \sum_{i=1}^{r} L_i h_i; 0, \cdots, 0, v_1, \cdots, v_r; 1\right);(-\alpha - R\mu - \sum_{i=1}^{r} K_i h_i; v_1, \cdots, v_r, 0, \cdots, 0, 1; 1) \quad (4.5)$$

$$B_1 = (-1 - \alpha - \beta - R\mu - R'\mu' - \sum_{i=1}^{r}(K_i + L_i)h_i; v_1, \cdots, v_r; v_1, \cdots, v_r, 0; 1);$$

$$(-1 - \alpha - \beta - R\mu - R'\mu' - \sum_{i=1}^{r}(K_i + L_i)h_i; v_1, \cdots, v_r; v_1, \cdots, v_r, 1; 1) \quad (4.6)$$

$$\mathrm{C} = (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r}; (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \quad (4.7)$$

$$\mathrm{D} =(\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r}; (\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r};$$

(0,1;1) (4.8)

$$a_r = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_r)_{M_r K_r}}{K_r!} A[N_1, K_1; \cdots; N_r, K_r] \quad (4.9)$$

$$b_r = \frac{(-Q_1)_{P_1 L_1}}{L_1!} \cdots \frac{(-Q_r)_{P_r L_r}}{L_r!} A[Q_1, L_1; \cdots; Q_r, L_r] \quad (4.10)$$

Theorem 1

We have

$$Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}[f(x)] = t x^{-t(\alpha+\beta+1+\frac{\delta+1}{t})} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \sum_{L_1=0}^{[Q_1/P_1]} \cdots \sum_{L_r=0}^{[Q_r/P_r]} \sum_{w,v,u,t,e,k_1,k_2} \sum_{w',v',u',t',e',k_1',k_2'}$$

$$X^R X'^{R'} a_r b_r x^{-t(\sum_{i=1}^{r}(K_i+L_i)+R\mu+R'\mu')} \prod_{i=1}^{r} y_i^{K_i+L_i} \psi(w_1, v_1, u_1, t_1, e_1, k_1, k_2) \psi(w_1', v_1', u_1', t_1', e_1', k_1', k_2')$$

$$\int_0^x u^\delta (x^t - u^t)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^{r}(K_i+L_i)h_i} I_{p+4,q+2;Y}^{0,n+4;X} \left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathrm{A}\ ,\mathrm{A}_1 : C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \left(\frac{u^t}{x^t}-1\right) & \mathrm{B}\ ,\mathrm{B}_1 : D \end{array} \right) f(u)\mathrm{d}u \quad (4.11)$$

where a_r and b_r are defined by (4.9) and (4.10) respectively.
R and $\psi(w, v, u, t, e, k_1, k_2)$ are defined by (1.10) and (1.11) respectively. R' and $\psi(w', v', u', t', e', k_1', k_2')$

stands for the expresssion obtained from it by replacing all the parameters involved therein by the same parameters but having dashes in them.

Provided that

(a) $f(x) \in L_{\ddot{p}}(0,\infty)$, $1 \leqslant \ddot{p} \leqslant 2$ [or $f(x) \in M_{\ddot{p}}(0,\infty), \ddot{p} > 2]$, $1 \leqslant \ddot{p}, \ddot{q} < \infty, \ddot{p}^{-1} + \ddot{q}^{-1} = 1$

(b) $Re\left[t\alpha + Rt\mu + t\sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + K_i h_i\right)\right] > -\ddot{q}^{-1}$

(c) $Re\left[t\beta + R't\mu' + t\sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + L_i h_i\right)\right] > -\ddot{q}^{-1}, Re(\eta) > -\frac{1}{\ddot{q}}$

(d) $Re\left[\alpha + \beta + R\mu + R'\mu' + \sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + (K_i + L_i)h_i\right)\right] > -2, Re(\delta) > -\frac{1}{\ddot{p}}$

(e) $\Delta_k = -\sum_{j=n+1}^{p} A_j\alpha_j^{(k)} - \sum_{j=1}^{q} B_j\beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)}\gamma_j^{(k)} > 0$

(f) $Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] \in L_{\ddot{p}}(0,\infty)$

Proof

$$Q_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] = tx^{-\eta-t\alpha-1}\int_0^x y^{\eta}(x^t - y^t)^{\alpha} I\begin{pmatrix} \mathrm{z}_1 v_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r v_r \end{pmatrix} S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r} \end{pmatrix}$$

$$\Bigg\{ \quad ty^{-\delta-t\beta-1} R_n^{\alpha_1,\beta_1}[X\left(1-\frac{y^t}{x^t}\right); E, F, g, h; p, q; \gamma; \delta; e^{-su^r}]$$

$$\int_0^y u^{\delta}(y^t - u^t)^{\beta} f(u) I\begin{pmatrix} \mathrm{z}_1\left(1-\frac{u^t}{y^t}\right)^{v_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{y^t}\right)^{v_r} \end{pmatrix} R_{n'}^{\alpha_1',\beta_1'}[X'\left(1-\frac{y^t}{x^t}\right); E', F', g', h'; p', q'; \gamma'; \delta'; e^{-su'^r}]$$

$$S_{P_1,\cdots,Q_r}^{P_1,\cdots,Q_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{u^t}{y^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{u^t}{y^t}\right)^{h_r} \end{pmatrix}\Bigg\}\mathrm{d}u\mathrm{d}y \tag{4.12}$$

If we interchange the order of integration, which is permissible under the conditions stated, we have

$$Q^{\eta,\alpha}_{z_r}Q^{\delta,\beta}_{z_r}\left[f(x)\right]=t^2x^{-\eta-t\alpha-1}\int_0^x I_1u^{\delta}f(u)\mathrm{d}u \tag{4.13}$$

where $I_1=\int_u^x y^{\eta-\delta-t\beta-1}(x^t-y^t)^{\alpha}(y^t-u^t)^{\beta}\, I\begin{pmatrix} \mathrm{z}_1 v_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r v_r\end{pmatrix} S^{M_1,\cdots,M_r}_{N_1,\cdots,N_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r}\end{pmatrix}$

$$R^{\alpha_1,\beta_1}_n[X\left(1-\frac{y^t}{x^t}\right);E,F,g,h;p,q;\gamma;\delta;e^{-su^r}]I\begin{pmatrix} \mathrm{z}_1\left(1-\frac{u^t}{y^t}\right)^{v_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\left(1-\frac{u^t}{y^t}\right)^{v_r}\end{pmatrix} S^{P_1,\cdots,Q_r}_{P_1,\cdots,Q_r}\begin{pmatrix} \mathrm{y}_1\left(1-\frac{u^t}{y^t}\right)^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_r\left(1-\frac{u^t}{y^t}\right)^{h_r}\end{pmatrix}$$

$$R^{\alpha'_1,\beta'_1}_{n'}[X'\left(1-\frac{y^t}{x^t}\right);E',F',g',h';p',q';\gamma';\delta';e^{-su'^r}]\ \mathrm{d}y \tag{4.14}$$

Now, expressing in series the classes of multivariable polynomials $S^{M_1,\cdots,M_r}_{N_1,\cdots,N_r}[.]$ and $S^{P_1,\cdots,Q_r}_{P_1,\cdots,Q_r}[.]$, the sequence of functions $R^{\alpha_1,\beta_1}_n[.;E,F,g,h;p,q;\gamma;\delta;.]$ and $R^{\alpha'_1,\beta'_1}_n[.;E',F',g',h';p',q';\gamma';\delta';.]$ with the help of (1.1) and (1.8) respectively, expressing the multivariable I-functions defined by Prathima et al [9] in Mellin-Barnes contour integrals with the help of (1.3). Interchange the order of summations and integrations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process), then

$$I_1=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}\sum_{w,v,u,t,e,k_1,k_2}\sum_{w',v',u',t',e',k'_1,k'_2}\psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)$$

$$\psi(w'_1,v'_1,u'_1,t'_1,e'_1,k'_1,k'_2)a_rb_r\prod_{i=1}^r y_i^{K_i+L_i}\,X^RX'^{R'}\ \frac{1}{(2\pi\omega)^{2r}}\int_{L_1}\cdots\int_{L_{2r}}\phi(s_1,\cdots,s_r)\phi(s'_1,\cdots,s'_r)$$

$$\prod_{i=1}^r\phi_i(s_i)\phi_i(s'_i)z_i^{s_i+s'_i}\Bigg\{\int_u^x y^{\eta-\delta-t\mu-t\beta-tR\mu-tR'\mu'-t\sum_{i=1}^r(v_is'_i+h_iL_i)-1}\,(y^t-u^t)^{R\mu+R'\mu'+\beta+\sum_{i=1}^r(v_is'_i+h_iL_i)}$$

$$(x^t-u^t)^{R\mu+R'\mu'+\alpha+\sum_{i=1}^r(v_is_i+h_iK_i)}x^{-t(R\mu+\sum_{i=1}^r(v_is_i+h_iK_i)}\,\mathrm{d}y\Bigg)\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s'_1\cdots\mathrm{d}s'_r \tag{4.15}$$

The y-integral can be evaluated fairly easily by the substitution $w=\dfrac{x^t-y^t}{x^t-u^t}$

The w-integral can be evaluated with the help of the integral (3.1) and then on using (3.4), the equation (4.15) transforms into the desired form (4.11), when we apply the definition (1.3).

Let $X=m'_1,n'_1;\cdots;m'_r,n'_r;m'_1,n'_1;\cdots;m'_r,n'_r;1,0$ (4.16)

$$Y=p'_1,q'_1;\cdots;p'_r,q'_r;p'_1,q'_1;\cdots;p'_r,q'_r;0,1 \tag{4.17}$$

$$A=(a'_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(r)}_j,0,\cdots,0,0;A'_j)_{1,p'};(a'_j;0,\cdots,0,\alpha'^{(1)}_j,\cdots,\alpha'^{(r)}_j,0;A'_j)_{1,p'} \tag{4.18}$$

$$B=(b'_j;\beta'^{(1)}_j,\cdots,\beta'^{(r)}_j,0,\cdots,0,0;B'_j)_{1,q'};(b'_j;0,\cdots,0,\beta'^{(1)}_j,\cdots,\beta'^{(r)}_j,0;B'_j)_{1,q'} \tag{4.19}$$

$$A_2=(-\alpha-R\mu-\sum_{i=1}^{r}K_ih'_i;v'_1,\cdots,v'_r,0,\cdots,0,0;1),(-\beta-R'\mu'-\sum_{i=1}^{r}L_ih'_i;0,\cdots,0,v'_1,\cdots,v'_r,0;1),$$

$$(\frac{(\delta-\eta)}{t}-\alpha-R'\mu'-\sum_{i=1}^{r}L_ih'_i;v'_1,\cdots,v'_r;0,\cdots,0,1;1);(-\beta-R\mu-\sum_{i=1}^{r}K_ih'_i;0,\cdots,0;v'_1,\cdots,v'_r,1;1) \tag{4.20}$$

$$B_2=(-1-\alpha-\beta-R\mu-R'\mu'-\sum_{i=1}^{r}(K_i+L_i)h'_i;v'_1,\cdots,v'_r;v'_1,\cdots,v'_r,0;1);$$

$$(-1-\alpha-\beta-R\mu-R'\mu'-\sum_{i=1}^{r}(K_i+L_i)h'_i;v'_1,\cdots,v'_r;v'_1,\cdots,v'_r,1;1) \tag{4.21}$$

$$\mathrm{C}=(\mathrm{c}'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(r)}_j,\gamma'^{(r)}_j;C'^{(r)}_j)_{1,p'_r};$$

$$(\mathrm{c}'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(r)}_j,\gamma'^{(r)}_j;C'^{(r)}_j)_{1,p'_r} \tag{4.22}$$

$$\mathrm{D}=(\mathrm{d}'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(r)}_j,\delta'^{(r)}_j;D'^{(r)}_j)_{1,q'_r};$$

$$(\mathrm{d}'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(r)}_j,\delta'^{(r)}_j;D'^{(r)}_j)_{1,q'_r};(0,1;1) \tag{4.23}$$

a_r and b_r are defined by (4.13) and (4.14) respectively

Theorem 2

We have

$$R^{\eta,\alpha}_{z_r}R^{\delta,\beta}_{z_r}\left[f(x)\right]=tx^{\eta}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}\sum_{w,v,u,t,e,k_1,k_2}\sum_{w',v',u',t',e',k'_1,k'_2}$$

$$X^RX'^{R'}a_rb_rx^{-t(\sum_{i=1}^{r}(K_i+L_i)+R\mu+R'\mu')}\prod_{i=1}^{r}y'^{K_i+L_i}_i\psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)\psi(w'_1,v'_1,u'_1,t'_1,e'_1,k'_1,k'_2)$$

$$\int_x^{\infty}u^{-(\eta+1+t(\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^{r}h'_i(K_i+L_i)))}(u^t-x^t)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^{r}(K_i+L_i)h'_i}$$

$$I_{p'+4,q'+2;Y}^{0,n'+4;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v'_1} & \mathrm{A}\ ,\mathrm{A}_2:C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v'_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v'_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v'_r} & \mathrm{B}\ ,\mathrm{B}_2:D \\ \left(\frac{x^t}{u^t}-1\right) & \end{array}\right) f(u)\mathrm{d}u \tag{4.24}$$

Provided that

(a) $f(x)\in L_{\ddot{p}}(0,\infty)$, $1\leqslant\ddot{p}\leqslant 2$ [or $f(x)\in M_{\ddot{p}}(0,\infty),\ddot{p}>2$], $1\leqslant\ddot{p},\ddot{q}<\infty,\ddot{p}^{-1}+\ddot{q}^{-1}=1$

(b)$Re\left[t\alpha+tR\mu+t\sum_{i=1}^{r}\left(v'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}+K_ih'_i\right)\right]>-\ddot{q}^{-1}$

(c) $Re\left[t\beta+tR'\mu'+t\sum_{i=1}^{r}\left(v'_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}+L_ih'_i\right)\right]>-\ddot{p}^{-1}$

(d) $\Delta'_k=-\sum_{j=n'+1}^{p'}A'_j\alpha_j'^{(k)}-\sum_{j=1}^{q'}B'_j\beta_j'^{(k)}+\sum_{j=1}^{m'_k}D_j'^{(k)}\delta_j'^{(k)}-\sum_{j=m'_k+1}^{q'_k}D_j'^{(k)}\delta_j'^{(k)}+\sum_{j=1}^{n'_k}C_j'^{(k)}\gamma_j'^{(k)}-\sum_{j=n'_k+1}^{p'_k}C_j'^{(k)}\gamma_j'^{(k)}>0$

(e) $R_{z_r}^{\eta,\alpha}R_{z_r}^{\delta,\beta}\left[f(x)\right]\in L_{\ddot{p}}(0,\infty)$

The proof is similar that (4.11)

5. Composition of mixed type operators

Let $X=m_1,n_1;\cdots;m_r,n_r;m'_1,n'_1;\cdots;m'_r,n'_r;1,0$ (5.1)

$Y=p_1,q_1;\cdots;p_r,q_r;p'_1,q'_1;\cdots;p'_r,q'_r;0,1$ (5.2)

$A=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},0,\cdots,0,0;A_j)_{1,p};(a'_j;0,\cdots,0,\alpha_j'^{(1)},\cdots,\alpha_j'^{(r)},0;A'_j)_{1,p'}$ (5.3)

$B=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)},0,\cdots,0,0;B_j)_{1,q};(b'_j;0,\cdots,0,\beta_j'^{(1)},\cdots,\beta_j'^{(r)},0;B'_j)_{1,q'}$ (5.4)

$\mathrm{C}=(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r};$

$(\mathrm{c}'^{(1)}_j,\gamma_j'^{(1)};C_j'^{(1)})_{1,p'_1};\cdots;(c_j'^{(r)},\gamma_j'^{(r)};C_j'^{(r)})_{1,p'_r}$ (5.5)

$\mathrm{D}=(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r};$

$(\mathrm{d}'^{(1)}_j,\delta_j'^{(1)};D_j'^{(1)})_{1,q'_1};\cdots;(d_j'^{(r)},\delta_j'^{(r)};D_j'^{(r)})_{1,q'_r}$;(0,1;1) (5.6)

$$A_3 = \big(-\alpha-\beta-R\mu-R'\mu'-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0;1\big);$$

$$\big(-\alpha-\beta-R\mu-R'\mu'-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,1;1\big);$$

$$\big(-\beta-R'\mu'-\sum_{i=1}^{r}L_ih'_i;0,\cdots,0;v'_1,\cdots,v'_r,1;1\big) \tag{5.7}$$

$$B_3 = \big(-\alpha-\beta-R\mu-R'\mu'-\frac{2(\eta+\delta+1)}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0;1\big);$$

$$\big(\beta-R'\mu'-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}L_ih'_i;0,\cdots,0;v'_1,\cdots,v'_r,1;1\big) \tag{5.8}$$

$$A_4 = \big(-\alpha-\beta-R\mu-R'\mu'-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0;1\big);$$

$$\big(-\alpha-\beta-R\mu-R'\mu'-\frac{\eta+\delta+1}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,1;1\big);$$

$$\big(-\alpha-R\mu-\sum_{i=1}^{r}K_ih_i;v_1,\cdots,v_r;0,\cdots,0,0;1\big) \tag{5.9}$$

$$B_4 = \big(-\alpha-\beta-R\mu-R'\mu'-\frac{2(\eta+\delta+1)}{t}-\sum_{i=1}^{r}(K_ih_i+L_ih'_i);v_1,\cdots,v_r;v'_1,\cdots,v'_r,0;1\big);$$

$$\big(-\alpha-R\mu-\sum_{i=1}^{r}K_ih_i;v_1,\cdots,v_r;0,\cdots,0,0;1\big) \tag{5.10}$$

a_r and b_r are defined by (4.13) and (4.14) respectively

We have the following result

Theorem 3

$$Q_{z_r}^{\eta,\alpha}R_{z_r}^{\delta,\beta}\left[f(x)\right] = R_{z_r}^{\eta,\alpha}Q_{z_r}^{\delta,\beta}\left[f(x)\right] = tx^{-\eta-1}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}a_rb_r\prod_{i=1}^{r}y_i^{K_i}{y'_i}^{L_i}$$

$$\sum_{w,v,u,t,e,k_1,k_2}\ \sum_{w',v',u',t',e',k'_1,k'_2}\Gamma\left(\frac{\eta+\delta+1}{t}\right)\ \psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)$$

$$\psi(w'_1,v'_1,u'_1,t'_1,e'_1,k'_1,k'_2)\quad X^RX'^{R'}\int_0^x u^\eta\left(1-\frac{u^t}{x^t}\right)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^{r}(K_ih_i+L_ih'_i)}$$

$$I^{0,n+n'+3;X}_{p+p'+3,q+q'+2;Y}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathrm{A},\mathrm{A}_3:C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \left(\frac{u^t}{x^t}-1\right) & \mathrm{B},\mathrm{B}_3:D \end{array}\right) f(u)\mathrm{d}u \; + \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}$$

$$a_r b_r \prod_{i=1}^{r} y_i^{K_i} y_i'^{L_i} \sum_{w,v,u,t,e,k_1,k_2} \; \sum_{w',v',u',t',e',k_1',k_2'} \Gamma\left(\frac{\eta+\delta+1}{t}\right) \psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)$$

$$\psi(w_1',v_1',u_1',t_1',e_1',k_1',k_2') \; X^R X'^{R'} \int_x^\infty u^{-\delta-1}\left(1-\frac{x^t}{u^t}\right)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^{r}(K_i h_i+L_i h_i')}$$

$$I^{0,n+n'+3;X}_{p+p'+3,q+q'+2;Y}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \mathrm{A}\;,\mathrm{A}_4;C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \cdot \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \cdot \\ \left(\frac{x^t}{u^t}-1\right) & \mathrm{B}\;,\mathrm{B}_4;D \end{array}\right) f(u)\mathrm{d}u \qquad (5.11)$$

Provided that

(a) $f(x)\in L_{\ddot{p}}(0,\infty), 1\leqslant \ddot{p}\leqslant 2$ [or $f(x)\in M_{\ddot{p}}(0,\infty), \ddot{p}>2],\; 1\leqslant \ddot{p},\ddot{q}<\infty, \ddot{p}^{-1}+\ddot{q}^{-1}=1$

(b)$Re\left[t\alpha+tR\mu+t\sum_{i=1}^{r}\left(v_i \min_{1\leqslant j\leqslant m_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}+K_i h_i\right)\right]>-\ddot{q}^{-1}$

(c)$Re\left[t\beta+tR'\mu'+t\sum_{i=1}^{r}\left(v_i' \min_{1\leqslant j\leqslant m_i'}\frac{b_j^{(i)}}{\beta_j^{(i)}}+L_i h_i'\right)\right]>-\ddot{q}^{-1}$

(d) $Re(\delta) > -\frac{1}{\ddot{p}}, Re(\eta) > -\frac{1}{\ddot{q}}$

(e) $Q_{z_r}^{\eta,\alpha} R_{z_r}^{\delta,\beta}\,[f(x)] = R_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\,[f(x)] \in L_{\ddot{p}}(0,\infty)$

(f) $\Delta_k = -\sum_{j=n+1}^{p} A_j\alpha_j^{(k)} - \sum_{j=1}^{q} B_j\beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)}\gamma_j^{(k)} > 0$

(g) $\Delta'_k = -\sum_{j=n'+1}^{p'} A'_j{\alpha'}_j^{(k)} - \sum_{j=1}^{q'} B'_j{\beta'}_j^{(k)} + \sum_{j=1}^{m'_k} {D'}_j^{(k)}{\delta'}_j^{(k)} - \sum_{j=m'_k+1}^{q'_k} {D'}_j^{(k)}{\delta'}_j^{(k)} + \sum_{j=1}^{n'_k} {C'}_j^{(k)}{\gamma'}_j^{(k)} - \sum_{j=n'_k+1}^{p'_k} {C'}_j^{(k)}{\gamma'}_j^{(k)} > 0$

The proof is similar that (4.11)

6. Multivariable H-function

a) If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, the multivariable I-function defined by Nambisan et al [9] reduces to multivariable H-function defined by Srivastava et al [17]. We obtain the following fractional integral operators and we have three following relations.

$$Q_{z_r}^{\eta,\alpha}\,[f(x)] = tx^{-\eta-t\alpha-1}\int_0^x y^{\eta}(x^t-y^t)^{\alpha} H_{p,q:p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c} \mathrm{z}_1 \upsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \upsilon_r \end{array}\middle| \begin{array}{l} (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,p} : \\ \\ \\ \\ (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{l} (\mathrm{c}_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots,\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{array}\right) R_n^{\alpha_1,\beta_1}[X\left(1-\frac{y^t}{x^t}\right)^{\mu};E,F,g,h;p,q;\gamma_1;\delta_1;e^{-su^r}]$$

$$S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}\left(\begin{array}{c} \mathrm{y}_1\left(1-\frac{y^t}{x^t}\right)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{y}_r\left(1-\frac{y^t}{x^t}\right)^{h_r} \end{array}\right) f(y)\mathrm{d}y \tag{6.1}$$

and

$$R_{\gamma_n}^{\delta,\beta}\,[f(x)] = tx^{\delta}\int_x^{\infty} y^{-\delta-t\beta-1}(y^t-x^t)^{\beta} H_{p',q':p'_1,q'_1;\cdots;p'_r,q'_r}^{0,n':m'_1,n'_1;\cdots;m'_r,n'_r}\left(\begin{array}{c} \gamma_1\mu_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_r\mu_r \end{array}\middle| \begin{array}{l} (\mathrm{a'}_j;{\alpha'}_j^{(1)},\cdots,{\alpha'}_j^{(r)})_{1,p'} : \\ \\ \\ \\ (\mathrm{b'}_j;{\beta'}_j^{(1)},\cdots,{\beta'}_j^{(r)})_{1,q'} : \end{array}\right.$$

$$\left.\begin{matrix}(\text{c'}_j^{(1)},\gamma_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(r)},\gamma_j'^{(r)})_{1,p_r'}\\ \\ (\text{d'}_j^{(1)},\delta_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(r)},\delta_j'^{(r)})_{1,q_r'}\end{matrix}\right) R_{n'}^{\alpha_1',\beta_1'}[X'\left(1-\frac{x^t}{y^t}\right)^{\mu'};E',F',g',h';p',q';\gamma_1';\delta_1';e^{-sv^r}]$$

$$S_{N_1',\cdots,N_r'}^{M_1',\cdots,M_r'}\left(\begin{matrix}\text{y'}_1\left(1-\frac{x^t}{y^t}\right)^{h_1'}\\ \cdot\\ \cdot\\ \cdot\\ \text{y'}_r\left(1-\frac{x^t}{y^t}\right)^{h_r'}\end{matrix}\right)f(y)\mathrm{d}y \tag{6.2}$$

with the same notations that (2.1) and (2.2)

Corollary 1

$$Q_{z_r}^{\eta,\alpha}Q_{z_r}^{\delta,\beta}\left[f(x)\right]=tx^{-t(\alpha+\beta+1+\frac{\delta+1}{t})}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}\sum_{w,v,u,t,e,k_1,k_2}\sum_{w',v',u',t',e',k_1',k_2'}$$

$$X^R X'^{R'}a_r b_r x^{-t(\sum_{i=1}^r(K_i+L_i)+R\mu+R'\mu')}\prod_{i=1}^r y_i^{K_i+L_i}\psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)\psi(w_1',v_1',u_1',t_1',e_1',k_1',k_2')$$

$$\int_0^x u^\delta(x^t-u^t)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^r(K_i+L_i)h_i}H_{p+4,q+2;Y}^{0,n+4;X}\left(\begin{matrix}\text{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1}\\ \vdots\\ \text{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r}\\ \text{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1}\\ \vdots\\ \text{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r}\\ \left(\frac{u^t}{x^t}-1\right)\end{matrix}\ \middle|\ \begin{matrix}\text{A },\text{A}_1:C\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \text{B },\text{B}_1:D\end{matrix}\right)f(u)\mathrm{d}u \tag{6.1}$$

under the same notations and conditions that (4.11) with $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$

Corollary 2

We have

$$R_{z_r}^{\eta,\alpha}R_{z_r}^{\delta,\beta}\left[f(x)\right]=tx^{\eta}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_r=0}^{[N_r/M_r]}\sum_{L_1=0}^{[Q_1/P_1]}\cdots\sum_{L_r=0}^{[Q_r/P_r]}\sum_{w,v,u,t,e,k_1,k_2}\sum_{w',v',u',t',e',k_1',k_2'}$$

$$X^R X'^{R'} a_r b_r x^{-t(\sum_{i=1}^r (K_i+L_i)+R\mu+R'\mu')} \prod_{i=1}^r y_i'^{K_i+L_i} \psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)\psi(w_1',v_1',u_1',t_1',e_1',k_1',k_2')$$

$$\int_x^\infty u^{-(\eta+1+t(\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^r h_i'(K_i+L_i)))} (u^t-x^t)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^r (K_i+L_i)h_i'}$$

$$H_{p'+4,q'+2;Y}^{0,n'+4;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \mathrm{A}\ ,\mathrm{A}_2 : C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \cdot \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \mathrm{B}\ ,\mathrm{B}_2 : D \\ \left(\frac{x^t}{u^t}-1\right) & \end{array}\right) f(u)\mathrm{d}u \tag{6.2}$$

under the same notations and conditions that (4.24) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$

Corollary 3

$$Q_{z_r}^{\eta,\alpha} R_{z_r}^{\delta,\beta}\left[f(x)\right] = R_{z_r}^{\eta,\alpha} Q_{z_r}^{\delta,\beta}\left[f(x)\right] = t x^{-\eta-1} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \sum_{L_1=0}^{[Q_1/P_1]} \cdots \sum_{L_r=0}^{[Q_r/P_r]} a_r b_r \prod_{i=1}^r y_i^{K_i} y_i'^{L_i}$$

$$\sum_{w,v,u,t,e,k_1,k_2} \sum_{w',v',u',t',e',k_1',k_2'} \Gamma\left(\frac{\eta+\delta+R\mu+R'\mu'+1}{t}\right) \psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)$$

$$\psi(w_1',v_1',u_1',t_1',e_1',k_1',k_2') \quad X^R X'^{R'} \int_0^x u^\eta \left(1-\frac{u^t}{x^t}\right)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^r (K_i h_i + L_i h_i')}$$

$$H_{p+p'+3,q+q'+2;Y}^{0,n+n'+3;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \mathrm{A},\mathrm{A}_3 : C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \cdot \\ \mathrm{z}_1\left(1-\frac{u^t}{x^t}\right)^{v_1} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{u^t}{x^t}\right)^{v_r} & \mathrm{B},\mathrm{B}_3 : D \\ \left(\frac{u^t}{x^t}-1\right) & \end{array}\right) f(u)\mathrm{d}u \ + \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \sum_{L_1=0}^{[Q_1/P_1]} \cdots \sum_{L_r=0}^{[Q_r/P_r]}$$

$$a_r b_r \prod_{i=1}^{r} y_i^{K_i} y_i'^{L_i} \sum_{w,v,u,t,e,k_1,k_2} \sum_{w',v',u',t',e',k_1',k_2'} \Gamma\left(\frac{\eta+\delta+R\mu+R'\mu'+1}{t}\right) \psi(w_1,v_1,u_1,t_1,e_1,k_1,k_2)$$

$$\psi(w_1',v_1',u_1',t_1',e_1',k_1',k_2') \ X^R X'^{R'} \int_x^\infty u^{-\delta-1}\left(1-\frac{x^t}{u^t}\right)^{\alpha+\beta+1+R\mu+R'\mu'+\sum_{i=1}^{r}(K_i h_i+L_i h_i')}$$

$$H_{p+p'+3,q+q'+2;Y}^{0,n+n'+3;X}\left(\begin{array}{c|c} \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \mathrm{A}\ ,\mathrm{A}_4;C \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \cdot \\ \mathrm{z}_1\left(1-\frac{x^t}{u^t}\right)^{v_1'} & \cdot \\ \vdots & \cdot \\ \mathrm{z}_r\left(1-\frac{x^t}{u^t}\right)^{v_r'} & \mathrm{B}\ ,\mathrm{B}_4;D \\ \left(\frac{x^t}{u^t}-1\right) & \end{array}\right) f(u)\mathrm{d}u \tag{6.3}$$

under the same notations and conditions that (5.11) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$

Remark. If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$and $S_{N_1,\cdots,N_r}^{M_1,\cdots,M_r}[.] \rightarrow S_N^M[.]$, the multivariable I-function reduces to multivariable H-function and the class of multivariable polynomials reduce to class of polynomials of one variable defined by Srivastava [14], for more details see Jain et al [8].

7. Conclusion

In this paper we have evaluated the compositions of fractional integral operator concerning the multivariable I-functions defined by Prathima et al [1], a sequence of functions and a class of multivariable polynomials defined by Srivastava [15] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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