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Hausdorff Property of Minimal Rank, Maximal Rank and Non-Rank Preserving Direct Product of Hypergraphs

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Abstract

A hypergraph $H = (V, \mathcal{E})$ is said to be a *Hausdorff hypergraph* if for any two distinct vertices u, v of V there exist hyperedges e_1, $e_2 \in \mathcal{E}$ such that $u \in e_1$, $v \in e_2$ and $e_1 \cap e_2 = \emptyset$. In this paper we derive sufficient conditions for minimal rank, maximal rank, non-rank preserving direct products of two hypergraphs to be Hausdorff.

Mathematics Subject Classification: 05C65

Keywords: Hausdorff hypergraph, Minimal rank preserving direct product, Maximal rank preserving direct product, Non-rank preserving direct product.

1 Introduction

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [9]. A *hypergraph* [2] H is a pair $(V, \mathcal{E})$, where V is a set of elements called nodes or vertices, and $\mathcal{E}$ is a set of nonempty subsets of V called *hyperedges* or *edges*. Therefore, $\mathcal{E}$ is a subset of $P(X)\backslash\{\emptyset\}$, where $P(X)$ is the power set of X. In drawing hypergraphs, each vertex is a point in the plane and each edge is a closed curve separating the respective subset from the remaining vertices. The cardinality of the finite set V, is denoted by $|V|$, is called the *order* [8] of the hypergraph. The number of edges is usually denoted by m or $m(H)$ [8].

A *simple hypergraph* [1] is a hypergraph with the property that if e_i and e_j are hyperedges of H with $e_i \subseteq e_j$, then $i = j$. Two vertices in a hypergraph are *adjacent* [9] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are *incident* [9] if their intersection is nonempty.

A *k-uniform hypergraph* [4] or a k-hypergraph is a hypergraph in which every edge consists of k vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The *rank* [9] $r(H)$ of a hypergraph is the maximum of the cardinalities of the edges in the hypergraph. The *co-rank* [9] $cr(H)$ of a hypergraph is the minimum of the cardinalities of a hyperedge in the hypergraph. If $r(H) = cr(H) = k$, then H is k-uniform. The *degree* [7] $d_H(v)$ of a vertex v in a hypergraph H is the number of edges of H that containing the vertex v. H is k-regular if every vertex has degree k. The *degree* [3], $d(e)$ of a hyperedge, $e \in \mathcal{E}$ is its cardinality $|e|$.

A vertex of a hypergraph which is incident to no edges is called an *isolated vertex.* [9] The degree of an isolated vertex is trivially zero.

A hyperedge e of H with $|e| = 1$ is called a *loop*; more specifically a hyperedge $e = \{v\}$ is a loop at the vertex v. A vertex of degree 1 is called a pendant vertex .

A simple hypergraph H with $|e| = 2$ for each $e \in \mathcal{E}$ is a simple graph.

Let $H = (V, \mathcal{E})$ be a hypergraph. Any hypergraph $H' = (V', \mathcal{E}')$ such that $V \subseteq V'$ and $\mathcal{E} \subseteq \mathcal{E}'$ is called a *subhypergraph* [8] of H.

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Definition 1.1. [6] The *cartesian product* $H_1 \Box H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph $H = (V, \mathcal{E})$ with vertex set $V = V_1 \times V_2$ and edge set $\mathcal{E} = \{\{u\} \times f : u \in V_1, f \in \mathcal{E}_2\} \cup \{e \times \{v\} : e \in \mathcal{E}_1, v \in V_2\}$.

Definition 1.2. A hypergraph $H = (V, \mathcal{E})$ is said to be a Hausdorff hypergraph if for any two distinct vertices u and v of V there exist hyperedges e_1, $e_2 \in \mathcal{E}$ such that $u \in e_1$ and $v \in e_2$; and $e_1 \cap e_2 = \emptyset$.

Theorem 1.3. *Let H_1 and H_2 be two hypergraphs. Then the cartesian product $H_1 \Box H_2$ of H_1 and H_2 is a Hausdorff hypergraph.*

Through out this paper we consider only simple hypergraph with no isolated vertices.

2 Minimal Rank Preserving Direct Product

One of the interesting product of hypergraph is minimal rank preserving direct product.

Definition 2.1. [5] The *Minimal Rank Preserving Direct Product* $H_1 \breve{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \dots (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 \breve{\times} H_2$ if and only if

1. $\{u_1, u_2, \dots, u_r\}$ is an edge of H_1 and $\{v_1, v_2, \dots, v_r\}$ is a subset of an edge of H_2, or
2. $\{u_1, u_2, \dots, u_r\}$ is a subset of an edge of H_1 and $\{v_1, v_2, \dots, v_r\}$ is an edge of H_2.

Let $e_1 = \{u_1, u_2, \dots, u_p\}$ be an edge of H_1 and $e_2 = \{v_1, v_2, \dots, v_q\}$ be an edge of H_2. Then $e = \{(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)\}$ is an edge of $H_1 \breve{\times} H_2$ with cardinality $\min\{|e_1|, |e_2|\}$.

In this paper, we discuss the Hausdorff property, that is the separation of any two distinct vertices by nonadjacent edges of different product of hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$. For the sake of convenience we name the distinct vertices of product hypergraphs by (u_1, v_1) and (u_2, v_2).

Theorem 2.2. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then the minimal rank preserving direct product $H_1 \breve{\times} H_2$ of H_1 and H_2 is Hausdorff, provided the degree of each edge of the hypergraph H_1 (or H_2) is different from 2.*

Proof. Suppose the degree of each edge of the hypergraph H_1 is different from 2. Consider any two distinct vertices of $H_1 \breve{\times} H_2$. Let it be (u_1, v_1) and (u_2, v_2).

Case 1. $u_1 = u_2, v_1 \neq v_2$

Let $e = \{u_1 = u_2, u_3, u_4, u_5, \dots, u_{n+1}\}$. Note that $|e| = n$ and by hypothesis either $n = 1$ or $n \geq 3$.

If $n = 1$, then $e_1 = \{(u_1, v_1)\}$ and $e_2 = \{(u_1, v_2)\}$ are two nonadjacent edges of $H_1 \breve{\times} H_2$.

If $n \geq 3$, then we have the following two subcases.

Subcase 1. There exists an edge f, with $|f| = m$, of H_2 which contains both v_1 and v_2.

Let $f = \{v_1, v_2, \dots, v_m\}$. Suppose $n \geq m$. Then the edges $e_1 = \{(u_1, v_1), (u_3, v_2), (u_4, v_3) \dots, (u_{m+1}, v_m)\}$ and $e_2 = \{(u_1, v_2), (u_3, v_3), (u_4, v_4), \dots, (u_m, v_m), (u_{m+1}, v_1)\}$ of $H_1 \breve{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Subcase 2. There exist no edge of H_2 containing both v_1 and v_2.

Let f_1 be an edge of H_2 with $|f_1| = m$ such that $v_1 \in f_1$ and f_2 be an edge of H_2 with $|f_2| = p$ such that $v_2 \in f_2$. Suppose $n \geq m \geq p$ and $|f_1 \cap f_2| = k, 0 \leq k \leq (p-1)$. Let $f_1 = \{v_1, v_3, \dots, v_{k+2}, \dots, v_{m+1}\}$ and $f_2 = \{w_1, w_2, \dots, w_k, w_{k+1} \dots, w_q\}$ with $w_1 = v_2$. If $k \geq 1$, let $w_2 = v_3, w_3 = v_4 \dots, w_{k+1} = v_{k+2}$.

Then the edges

$$e_1 = \begin{cases} \{(u_1, v_1)\} & \text{if } m = 1 \\ \{(u_1, v_1), (u_3, v_3), (u_4, v_4), \dots, (u_{m+1}, v_{m+1})\} & \text{otherwise} \end{cases}$$

and

$$e_2 = \begin{cases} \{(u_1, w_1)\} & \text{if } p = 1 \\ \{(u_1, w_1), (u_4, w_2)\} & \text{if } p = 2 \\ \{(u_1, w_1), (u_3, w_3), (u_4, w_4), (u_5, w_5), \dots, (u_p, w_p), (u_{p+1}, w_2)\} & \text{otherwise} \end{cases}$$

of $H_1 \breve{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Case 2. $u_1 \neq u_2$, $v_1 \neq v_2$

Subcase 1. There exists an edge $e = \{u_1, u_2, u_3, \ldots, u_n\}$ of H_1 containing both u_1 and u_2.
In this case $n \geq 3$.
Suppose there exists an edge $f = \{v_1, v_2, \ldots, v_m\}$ of H_2 containing both v_1 and v_2.
Without loss of generality assume that $n \geq m$. Set

$$e_1 = \begin{cases} \{(u_1, v_1), (u_3, v_2)\} & \text{if } m = 2 \\ \{(u_1, v_1), (u_2, v_3), (u_3, v_4), (u_4, v_5) \ldots, (u_{m-1}, v_m)(u_m, v_2)\} & \text{otherwise} \end{cases}$$

and

$$e_2 = \begin{cases} \{(u_2, v_2), (u_3, v_1)\} & \text{if } m = 2 \\ \{(u_1, v_m), (u_2, v_2), (u_3, v_3), (u_4, v_4), (u_5, v_5), \ldots, (u_{m-1}, v_{m-1}), (u_m, v_1)\} & \text{otherwise} \end{cases}$$

Then e_1 and e_2 are two nonadjacent edges of $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$.
Suppose there exists no edge of H_2 containing both v_1 and v_2.
Let f_1 be an edge of H_2 with $|f_1| = p$, containing v_1 and f_2 be an edge of H_2 with $|f_2| = q$, containing v_2. Suppose $n \geq p \geq q$. Consider a subset A of e containing u_1 and u_2 with cardinality p. Let $A = \{u_1, u_2, \ldots, u_q, \ldots, u_p\}$ and let $B = \{u_1, u_2, \ldots, u_q\}$. Suppose $|f_1 \cap f_2| = k$, where $0 \leq k \leq (q-1)$. Let $f_1 = \{v_1, v_3, \ldots, v_{k+2}, \ldots, v_{p+1}\}$ and $f_2 = \{w_1, w_2, \ldots, w_k, w_{k+1} \ldots, w_q\}$ with $w_1 = v_2$. If $k \geq 1$, let $w_2 = v_3, w_3 = v_4 \ldots, w_{k+1} = v_{k+2}$.
Set

$$e_1 = \begin{cases} \{(u_1, v_1)\} & \text{if } p = 1 \\ \{(u_1, v_1), (u_2, v_3), (u_3, v_4) \ldots, (u_p, v_{p+1})\} & \text{otherwise} \end{cases}$$

and

$$e_2 = \begin{cases} \{(u_2, w_1)\} & \text{if } q = 1 \\ \{(u_2, w_1), (u_3, w_2)\} & \text{if } q = 2 \\ \{(u_1, w_q), (u_2, w_1), (u_3, w_2), (u_4, w_3), \ldots, (u_q, w_{q-1})\} & \text{otherwise} \end{cases}$$

Then e_1 and e_2 are two nonadjacent edges of $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

Subcase 2. There exists no edge of H_1 containing both u_1 and u_2.
Let g_1 be an edge of H_1 with $|g_1| = n$, containing u_1 and g_2 be an edge of H_1 with $|g_2| = m$ containing u_2. Let $n \geq m$ and $|g_1 \cap g_2| = k$, $0 \leq k \leq (m-1)$. Let $g_1 = \{u_1, u_3, \ldots, u_{k+2}, \ldots, u_{n+1}\}$ and $g_2 = \{x_1, x_2, \ldots, x_k, x_{k+1} \ldots, x_m\}$ with $x_1 = u_2$. If $k \geq 1$, let $x_2 = u_3, x_3 = u_4 \ldots, x_{k+1} = u_{k+2}$.
Suppose there exists an edge f of H_2 with $|f| = p$, containing both v_1 and v_2.
Then as in the proof of Subcase 1 of Case 2, we can prove that there exist two nonadjacent edges e_1 and e_2 in $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$.
Suppose there exists no edge of H_2 containing both v_1 and v_2.
Let f_1 be an edge of H_2 with $|f_1| = p$, containing v_1 and f_2 an edge of H_2 with $|f_2| = q$, containing v_2. Assume $n \geq p \geq q$ and $m \geq q$. Let $|f_1 \cap f_2| = l, 0 \leq l \leq (q-1)$. Let $f_1 = \{v_1, v_3, \ldots, v_{l+2}, \ldots, v_{p+1}\}$ and $f_2 = \{y_1, y_2, \ldots, y_l, y_{l+1} \ldots, y_q\}$ with $y_1 = v_2$. If $l \geq 1$, let $y_2 = v_3, y_3 = v_4 \ldots, y_{l+1} = v_{l+2}$.
Set an edge e_1 of $H_1 \breve{\times} H_2$ with cardinality p as,

$$e_1 = \begin{cases} \{(u_1, v_1)\} & \text{if } p = 1 \\ \{(u_1, v_1), (u_3, v_3), (u_4, v_4), \ldots, (u_p, v_p), (u_{p+1}, v_{p+1})\} & \text{otherwise} \end{cases}$$

and an edge e_2 with cardinality q as,

$$e_2 = \begin{cases} \{(x_1, y_1)\} & \text{if } q = 1 \\ \{(x_1, y_1), (x_3, y_2)\} & \text{if } q = 2 \\ \{(x_1, y_1), (x_2, y_3), (x_3, y_4) \ldots, (x_{q-1}, y_q), (x_q, y_2)\} & \text{otherwise} \end{cases}$$

Then e_1 and e_2 are two nonadjacent edges of $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

The other inequalities between n, m, p and q in cases 1 and 2 can be dealt in a similar way. $\square$

Remark 2.3. From the proof of Theorem 2.2 we can conclude the following

For any two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ and for any two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1 \breve{\times} H_2$, if there exists an edge e of H_1 containing u_1 or u_2 or both and an edge f of H_2 containing v_1 or v_2 or both, then there exists two nonadjacent edges e_1 and e_2 in $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$, provided $|e| \neq 2$ or $|f| \neq 2$.

Remark 2.4. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If both H_1 and H_2 contain edges of degree 2, then the minimal rank preserving direct product $H_1 \breve{\times} H_2$ of H_1 and H_2 need not be Hausdorff.(See Figure 1.)

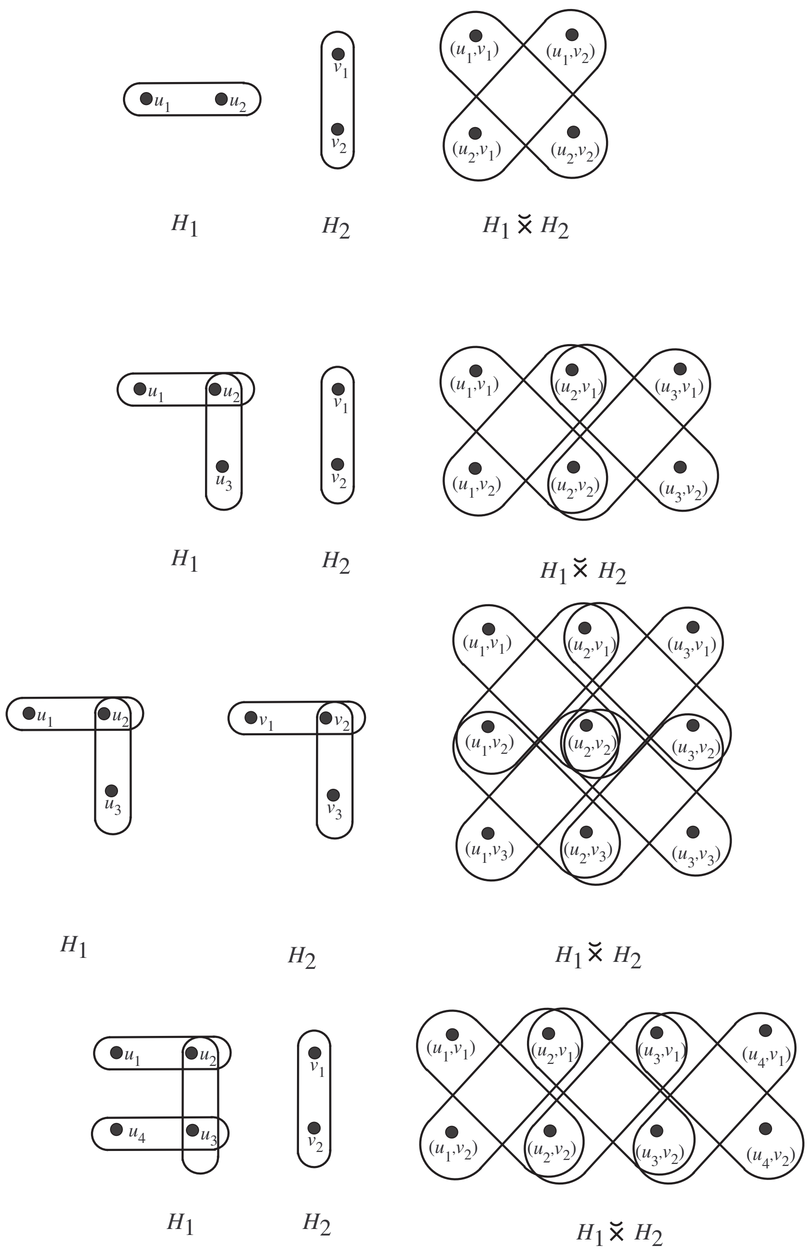

Figure 1: The minimal rank preserving direct product of H_1 and H_2.

Remark 2.5. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If the degree of each vertex in any edge of degree 2 of the hypergraph H_1(or H_2) is different from 1, then the minimal rank preserving direct product $H_1 \breve{\times} H_2$ of H_1 and H_2 is Hausdorff. (See Figure 2).

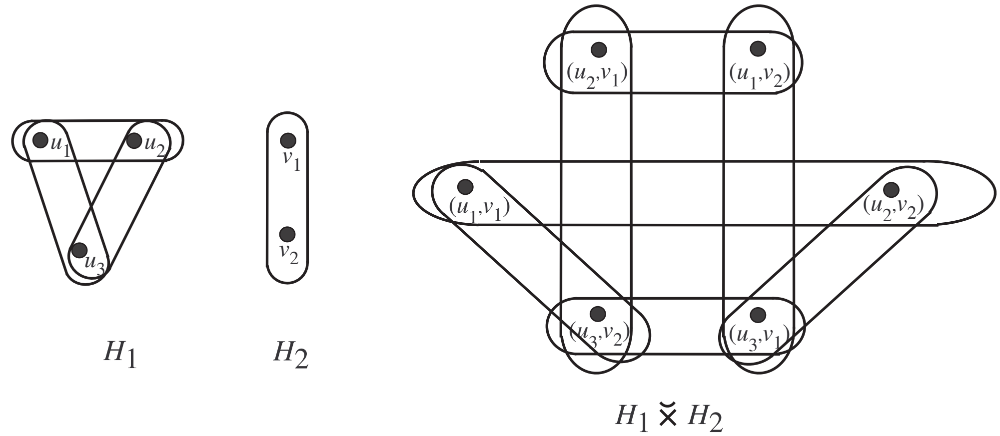

Figure 2: The minimal rank preserving direct product of H_1 and H_2.

Theorem 2.6. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then the minimal rank preserving direct product $H_1 \breve{\times} H_2$ of H_1 and H_2 is Hausdorff provided degree of each vertex in any edge of degree 2 of the hypergraph H_1(or H_2) is different from 1.*

Proof. Suppose the degree of each vertex of degree 2 of the hypergraph H_1 is different from 1. Let(u_1, v_1) and (u_2, v_2), be two distinct vertices of $H_1 \breve{\times} H_2$..

By remark 2.3 we need only to consider the cases where the edges considered are of degree 2.

Case 1. $u_1 = u_2, v_1 \neq v_2$

Let $e = \{u_1 = u_2, u_3\}$ be an edge of H_1 and f be an edge of H_2 containing v_1. By hypothesis of the theorem there exists another edge h containing u_1 and a vertex x different from u_3.

If $v_2 \in f$, then $f = \{v_1, v_2\}$. In this case the edges $e_1 = \{(u_1, v_1), (u_3, v_2)\}$ and $e_2 = \{(u_1, v_2), (x, v_1)\}$ of $H_1 \breve{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$ and $(u_1, v_2) \in e_2$.

If $v_2 \notin f$, then let $f = \{v_1, v_3\}$, where $v_3 \neq v_2$ and let $g = \{w_1 = v_2, w_2\}$ be an edge of H_2 containing v_2. Then the edges $e_1 = \{(u_1, v_1), (u_3, v_3)\}$ and $e_2 = \{(u_1, w_1), (x, w_2)\}$ of $H_1 \breve{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$ and $(u_1, v_2) \in e_2$.

Case 2. $u_1 \neq u_2$, $v_1 \neq v_2$

Subcase 1. There exists an edge $e = \{u_1, u_2\}$ of H_1 containing both u_1 and u_2.

Suppose there exists an edge $f = \{v_1, v_2\}$ of H_2 containing both v_1 and v_2.

By hypothesis of the theorem there exists an edge h_1 containing u_1 and a vertex x different from u_2 and another edge h_2 containing u_2 and a vertex y different from u_1. Then $e_1 = \{(u_1, v_1), (x, v_2)\}$ and $e_2 = \{(y, v_1), (u_2, v_2)\}$ are two nonadjacent edges of $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$.

Suppose there exist no edge of H_2 containing both v_1 and v_2.

Let $f = \{v_1, v_3\}$ and $g = \{w_1 = v_2, w_2\}$ be two edges of H_2. Set $e_1 = \{(u_1, v_1), (u_2, v_3)\}$ and $e_2 = \{(u_1, w_2), (u_2, w_1)\}$. Then e_1 and e_2 are two nonadjacent edges of $H_1 \breve{\times} H_2$ and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

Subcase 2. There exists no edge of H_1 containing both u_1 and u_2.

Let $e = \{u_1, u_3\}$ and $g = \{x_1 = u_2, x_2\}$ be two edges of H_1

Suppose there exists an edge of H_2 containing both v_1 and v_2.

Then as in the proof of Subcase 1 of Case 2, we can prove that there exist two nonadjacent edges e_1 and e_2 in $H_1 \breve{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$.

Suppose there exist no edge of H_2 containing both v_1 and v_2.

Let $f = \{v_1, v_3\}$ and $h = \{y_1 = v_2, y_2\}$ be two edges of H_2.

Suppose $e \cap g \neq \emptyset$, then $u_3 = x_2$. By hypothesis of the theorem there exists an edge g_1 containing u_1 and a vertex x different from u_3. Then $e_1 = \{(u_1, v_1), (x, v_3)\}$ and $e_2 = \{(x_1, y_1), (x_2, y_2)\}$ are two nonadjacent edges of $H_1 \breve{\times} H_2$ and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$. Suppose $e \cap g = \emptyset$, then

$e_1 = \{(u_1, v_1), (u_3, v_3)\}$ and $e_2 = \{(x_1, y_1), (x_2, y_2)\}$ are two nonadjacent edges of $H_1 \check{\times} H_2$ and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

Hence the theorem. $\square$

Let H_1 and H_2 be two hypergraphs, if all the edges of H_1 or H_2 are loops, then all the edges of $H_1 \check{\times} H_2$ are loops. As a consequence we have the following proposition.

Proposition 2.7. *Let H_1 and H_2 be two hypergraphs. If all the edges of one of them are loops, then the minimal rank preserving direct product $H_1 \check{\times} H_2$ of H_1 and H_2 is Hausdorff.*

Definition 2.8. The *Normal product* [5] $H_1 \check{\boxtimes} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \check{\boxtimes} H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $v_1 = v_2 = \ldots = v_n \in V_2$, or
2. $\{v_1, v_2, \ldots, v_n\}$ is a subset of an edge of H_2 and $u_1 = u_2 = \ldots = u_n \in V_1$, or
3. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $\{v_1, v_2, \ldots, v_n\}$ is a subset of an edge of H_2, or
4. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $\{u_1, u_2, \ldots, u_n\}$ is a subset of an edge of H_1.

Remark 2.9. Cartesian product $H_1 \Box H_2$ of two hypergraphs H_1 and H_2 is a subhypergraph of their normal product $H_1 \check{\boxtimes} H_2$ with $V(H_1 \Box H_2) = V(H_1 \check{\boxtimes} H_2)$.

Theorem 2.10. *Let H_1 and H_2 be two hypergraphs. Then the normal product $H_1 \check{\boxtimes} H_2$ of H_1 and H_2 is Hausdorff.*

3 Maximal Rank Preserving Direct Product

Definition 3.1. [5] The *Maximal Rank Preserving Direct Product* $H_1 \hat{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 \hat{\times} H_2$ if,

1. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and there is an edge $f \in \mathcal{E}_2$ of H_2 such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset[2] of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or
2. $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2 and there is an edge $e \in \mathcal{E}_1$ of H_1 such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of e, and $e \subseteq \{u_1, u_2, \ldots, u_r\}$.

Let $e_1 = \{u_1, u_2, \ldots, u_p\}$ be an edge of H_1 and $e_2 = \{v_1, v_2, \ldots, v_q\}$ be an edge of H_2. Then $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ is an edge of $H_1 \hat{\times} H_2$ with cardinality $\max |e_1|, |e_2|$.

Remark 3.2. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then the maximal rank preserving direct product $H_1 \hat{\times} H_2$ of H_1 and H_2 is need not be Hausdorff if one of the hypergraph contains a loop.(See Figure 3.)

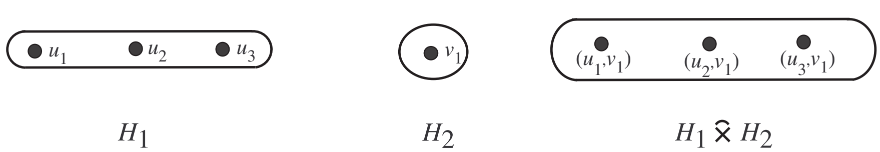

Figure 3: The maximal rank preserving direct product of H_1 and H_2.

[2] A multiset is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words a multiset is a set in which elements may belong more than once. $\{1, 1, 1, 2, 3, 3\}$ is a multiset.

Theorem 3.3. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. Then the maximal rank preserving direct product $H_1 \hat{\times} H_2$ of H_1 and H_2 is Hausdorff provided degree of each edge of the hypergraph H_1(or H_2) is different from 2.*

Proof. Suppose the degree of each edge of the hypergraph H_1 is different from 2.

Consider two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1 \hat{\times} H_2$.

Case 1. $u_1 = u_2, v_1 \neq v_2$

Let $e = \{u_1, u_3, u_4 \ldots u_{n+1}\}$ be an edge of H_1 containing u_1 with $|e| = n$.

Subcase 1. There exists an edge f, with $|f| = m$, of H_2 which contains both v_1 and v_2.

Let $f = \{v_1, v_2, \ldots, v_m\}$. Without loss of generality assume that $n \geq m$.

If $n = m$, then $e_1 = \{(u_1, v_1), (u_3, v_2), (u_4, v_3) \ldots, (u_{n+1}, v_m)\}$ and $e_2 = \{(u_1, v_2), (u_3, v_3), (u_4, v_4) \ldots, (u_n, v_m), (u_{n+1}, v_1)\}$ are nonadjacent edges of $H_1 \hat{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_1, v_2) \in e_2$.

If $n > m$, then the edges $e_1 = \{(u_1, v_1), (u_3, v_2), (u_4, v_3) \ldots, (u_{m+1}, v_m), (u_{m+2}, v_m), \ldots, (u_{n+1}, v_m)\}$ and $e_2 = \{(u_1, v_2), (u_3, v_3), (u_4, v_4), \ldots, (u_m, v_m), (u_{m+1}, v_1), (u_{m+2}, v_1) \ldots, (u_{n+1}, v_1)\}$ of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Subcase 2. There exist no edge of H_2 containing both v_1 and v_2.

Let f_1 be an edge of H_2 with $|f_1| = p$, containing v_1 and f_2 be an edge of H_2 with $|f_2| = q$, containing v_2.

Assume $p \geq q$ and $|f_1 \cap f_2| = k$, $0 \leq k \leq (q-1)$. Let $f_1 = \{v_1, v_3, \ldots, v_{k+2}, \ldots, v_{p+1}\}$ and $f_2 = \{w_1, w_2, \ldots, w_{k+1}, \ldots, w_q\}$ with $w_1 = v_2$. If $k \geq 1$, let $w_2 = v_3, w_3 = v_4, \ldots, w_{k+1} = v_{k+2}$.

If $n = p = q$, then $e_1 = \{(u_1, v_1), (u_3, v_3), (u_4, v_4) \ldots, (u_{n+1}, v_{n+1})\}$ and $e_2 = \{(u_1, w_1), (u_3, w_3), (u_4, w_4), (u_4, w_4), \ldots, (u_n, w_n), (u_{n+1}, w_2)\}$ are nonadjacent edges of $H_1 \hat{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_1, v_2) \in e_2$.

If $n = p > q$, then the edges $e_1 = \{(u_1, v_1), (u_3, v_3), (u_4, v_4) \ldots, (u_{n+1}, v_{n+1})\}$ and

$$e_2 = \begin{cases} \{(u_1, w_1), (u_3, w_1), (u_4, w_2), (u_5, w_1), (u_6, w_1), \ldots, (u_{n+1}, w_1)\} & \text{if } q = 2 \\ \{(u_1, w_1), (u_3, w_3), (u_4, w_4), (u_5, w_5), \ldots, (u_q, w_q), (u_{q+1}, w_2), (u_{q+2}, w_1), & \\ (u_{q+3}, w_1), \ldots, (u_{n+1}, w_1)\} & \text{if } q \neq 2 \end{cases}$$

of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

If $n > p$, then the edges $e_1 = \{(u_1, v_1), (u_3, v_3), (u_4, v_4) \ldots, (u_{p+1}, v_{p+1}), (u_{p+2}, v_1), \ldots, (u_{n+1}, v_1)\}$ and

$$e_2 = \begin{cases} \{(u_1, w_1), (u_3, w_1), (u_4, w_2), (u_5, w_1), (u_6, w_1), \ldots, (u_{n+1}, w_1)\} & \text{if } q = 2 \\ \{(u_1, w_1), (u_3, w_3), (u_4, w_4), (u_5, w_5), \ldots, (u_q, w_q), (u_{q+1}, w_2), (u_{q+2}, w_1), & \\ (u_{q+3}, w_1), \ldots, (u_{n+1}, w_1)\} & \text{if } q \neq 2 \end{cases}$$

are nonadjacent edges of $H_1 \hat{\times} H_2$ and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Case 2. $u_1 \neq u_2$,$v_1 \neq v_2$

Subcase 1. There exists an edge e of H_1 with $|e| = n$, containing both u_1 and u_2.

Let $e = \{u_1, u_2, \ldots, u_n\}$

Suppose there exists an edge f of H_2 with $|f| = m$, containing both v_1 and v_2. Let $f = \{v_1, v_2, \ldots, v_m\}$.

If $n = m$, then the edges $e_1 = \{(u_1, v_1), (u_2, v_3), (u_3, v_4), (u_4, v_5), \ldots, (u_{n-1}, v_n), (u_n, v_2)\}$ and

$$e_2 = \begin{cases} \{(u_1, v_3), (u_2, v_2), (u_3, v_1)\} & \text{if } m = 3 \\ \{(u_2, v_2), (u_3, v_1), (u_4, v_3), (u_5, v_4), (u_6, v_5), \ldots, (u_n, v_{m-1}), (u_1, v_m)\} & \text{if } m \neq 3 \end{cases}$$

of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

If $n > m$, then the edges

$$e_1 = \begin{cases} \{(u_1, v_1), (u_2, v_1), (u_3, v_2), (u_4, v_2), \ldots, (u_n, v_2)\} & \text{if } m = 2 \\ \{(u_1, v_1), (u_2, v_3), (u_3, v_4), (u_4, v_5), \ldots, (u_{m-1}, v_m), & \\ (u_m, v_2), (u_{m+1}, v_2), (u_{m+2}, v_2), \ldots, (u_n, v_2)\} & \text{if } m \neq 2 \end{cases}$$

and

$$e_2 = \begin{cases} \{(u_1,v_2),(u_2,v_2),(u_3,v_1),(u_4,v_1),(u_5,v_1)\ldots,(u_n,v_1)\} & \text{if } m=2 \\ \{(u_1,v_3),(u_2,v_2),(u_3,v_1),(u_4,v_1),(u_5,v_1)\ldots,(u_n,v_1)\} & \text{if } m=3 \\ \{(u_2,v_2),(u_3,v_1),(u_4,v_3),(u_5,v_4),(u_6,v_5),\ldots,(u_m,v_{m-1}), & \\ (u_1,v_m),(u_{m+1},v_1),(u_{m+2},v_1)\ldots,(u_n,v_1)\} & \text{otherwise} \end{cases}$$

of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1,v_1) \in e_1$; and $(u_2,v_2) \in e_2$.

Suppose there exists no edge of H_2 containing both v_1 and v_2.

Let f_1 be an edge of H_2 with $|f_1| = p$ such that $v_1 \in f_1$ and f_2 be an edge of H_2 with $|f_2| = q$ such that $v_2 \in f_2$.

Assume $p \geq q$ and $|f_1 \cap f_2| = k, 0 \leq k \leq (q-1)$. Let $f_1 = \{v_1,v_3,v_4\ldots,v_{k+2},\ldots,v_{p+1}\}$ and $f_2 = \{w_1,w_2,\ldots,w_{k+1},\ldots,w_q\}$ with $w_1 = v_2$. If $k \geq 1$, let $w_2 = v_3, w_3 = v_4,\ldots,w_{k+1} = v_{k+2}$.

If $n = p = q$, then the edges $e_1 = \{(u_1,v_1),(u_2,v_3),(u_3,v_4)\ldots,(u_n,v_{n+1})\}$ and $e_2 = \{(u_2,w_1),(u_3,w_2),(u_4,w_3),\ldots,(u_n,w_{n-1}),(u_1,w_n)\}$ of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1,v_1) \in e_1$ and $(u_2,v_2) \in e_2$.

If $n = p > q$, then $e_1 = \{(u_1,v_1),(u_2,v_3),(u_3,v_4)\ldots,(u_n,v_{n+1})\}$ and $e_2 = \{(u_2,w_1),(u_3,w_2),(u_4,w_3),\ldots,(u_q,w_{q-1})(u_1,w_q),(u_{q+1},w_1),(u_{q+2},w_1),\ldots,(u_n,w_1)\}$ are nonadjacent edges of $H_1 \hat{\times} H_2$ such that $(u_1,v_1) \in e_1$ and $(u_2,v_2) \in e_2$.

If $n > p$, then the edges $e_1 = \{(u_1,v_1),(u_2,v_3),(u_3,v_4)\ldots,(u_p,v_{p+1}),(u_{p+1},v_1),(u_{p+2},v_1),\ldots,(u_n,v_1)\}$ and $e_2 = \{(u_2,w_1),(u_3,w_2),(u_4,w_3),\ldots,(u_q,w_{q-1}),(u_1,w_q),(u_{q+1},w_1),(u_{q+2},w_1),\ldots,(u_n,w_1)\}$ of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1,v_1) \in e_1$; and $(u_2,v_2) \in e_2$.

Subcase 2. There exists no edge of H_1 containing both u_1 and u_2.

Let g_1 be an edge of H_1 with $|g_1| = n$, containing u_1 and g_2 be an edge of H_1 with $|g_2| = m$ containing u_2. Without loss of generality assume that $n \geq m$. Let $|g_1 \cap g_2| = k$, $0 \leq k \leq (m-1)$. Let $g_1 = \{u_1,u_3,u_4,\ldots,u_{k+2},\ldots,u_{n+1}\}$ and $g_2 = \{x_1,x_2,x_3,\ldots,x_{k+1},\ldots,x_m\}$ with $x_1 = u_2$. If $k \geq 1$, let $x_2 = u_3, x_3 = u_4,\ldots,x_{k+1} = u_{k+2}$.

Suppose there exists an edge f of H_2 with $|f| = p$, containing both v_1 and v_2.

Then as in the proof of Subcase 1 of Case 2 , we can prove that there exist two nonadjacent edges e_1 and e_2 of $H_1 \hat{\times} H_2$ such that $(u_1,v_1) \in e_1$ and $(u_2,v_2) \in e_2$.

Suppose there exists no edge of H_2 containing both v_1 and v_2.

Let f_1 be an edge of H_2 with $|f_1| = p$ such that $v_1 \in f_1$ and f_2 be an edge of H_2 with $|f_2| = q$ such that $v_2 \in f_2$.

Assume $p \geq q$ and $|f_1 \cap f_2| = t, 0 \leq t \leq (q-1)$. Let $f_1 = \{v_1,v_3,v_4,\ldots,v_{t+2},\ldots,v_{p+1}\}$ and $f_2 = \{y_1,y_2,\ldots,y_{t+1},\ldots,y_q\}$ with $y_1 = v_2$. If $t \geq 1$, let $y_2 = v_3, y_3 = v_4,\ldots,y_{t+1} = v_{t+2}$. Then $(u_2,v_2) = (x_1,y_1)$.

If $n = m$, then we have to consider four cases $n = p$, $m = q$; $n > p$, $m = q$; $n = p$, $m > q$ and $n > p$, $m > q$.

Suppose $n = p$, $m = q$. Then the edges $e_1 = \{(u_1,v_1),(u_3,v_3),(u_4,v_4),\ldots,(u_{n+1},v_{n+1})\}$ and $e_2 = \{(x_1,y_1),(x_2,y_3),(x_3,y_4),(x_4,y_5),\ldots,(x_{m-1},y_m),(x_m,y_2)\}$ of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1,v_1) \in e_1$, $(x_1,y_1) \in e_2$.

Suppose $n = p$, $m > q$. Then $e_1 = \{(u_1,v_1),(u_3,v_3),(u_4,v_4),\ldots,(u_{n+1},v_{n+1})\}$ and

$$e_2 = \begin{cases} \{(x_1,y_1),(x_2,y_1),(x_3,y_2),(x_4,y_2),\ldots,(x_{m-1},y_2),(x_m,y_2)\} & \text{if } q=2 \\ \{(x_1,y_1),(x_2,y_3),(x_3,y_4),(x_4,y_5),\ldots,(x_{q-1},y_q),(x_q,y_2), & \\ (x_{q+1},y_2),(x_{q+2},y_2),\ldots,(x_m,y_2)\} & \text{if } q \neq 2 \end{cases}$$

Suppose $n > p$, $m = q$. Then the edges $e_1 = \{(u_1,v_1),(u_3,v_3),\ldots,(u_p,v_p),(u_{p+1},v_{p+1}),(u_{p+2},v_1),(u_{p+3},v_1)\ldots,(u_{n+1},v_1)\}$ and $e_2 = \{(x_1,y_1),(x_2,y_3),(x_3,y_4),(x_4,y_5),\ldots,(x_{m-1},y_m),(x_m,y_2)\}$ of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1,v_1) \in e_1$, $(x_1,y_1) \in e_2$.

Suppose $n > p$, $m > q$. Then the edges

$$e_1 = \begin{cases} \{(u_1,v_1),(u_3,v_3),(u_4,v_1),(u_5,v_1),\ldots,(u_{n+1},v_1)\} & \text{if } p=2 \\ \{(u_1,v_1),(u_3,v_3),\ldots,(u_p,v_p),(u_{p+1},v_{p+1}),(u_{p+2},v_1), & \\ (u_{p+3},v_1)\ldots,(u_{n+1},v_1)\} & \text{if } p \neq 2 \end{cases}$$

$$e_2 = \begin{cases} \{(x_1,y_1),(x_2,y_1),(x_3,y_2),(x_4,y_2),\ldots,(x_m,y_2)\} & \text{if } q=2 \\ \{(x_1,y_1),(x_2,y_3),(x_3,y_4),(x_4,y_5),\ldots,(x_{q-1},y_q), & \\ (x_q,y_2),(x_{q+1},y_2),(x_{q+2},y_2),\ldots,(x_m,y_2)\} & \text{if } q \neq 2 \end{cases}$$

of $H_1 \hat{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(x_1, y_1) \in e_2$.

Similarly, if $n > m$, we can show that there exists two nonadjacent edges e_1 and e_2 in $H_1 \hat{\times} H_2$ such that $(u_1, v_1) \in e_1$, $(x_1, y_1) \in e_2$.

The other inequalities between n, m, p and q in cases 1 and 2 can be dealt in a similar way. $\square$

Remark 3.4. As in the minimal rank preserving direct product here also we have, for any two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ with no loops and for any two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1 \hat{\times} H_2$, if there exists an edge e of H_1 containing u_1 or u_2 or both and an edge f of H_2 containing v_1 or v_2 or both, then there exists two nonadjacent edges e_1 and e_2 in $H_1 \hat{\times} H_2$ such that $(u_1, v_1) \in e_1$ and $(u_2, v_2) \in e_2$, provided $|e| \neq 2$ or $|f| \neq 2$.

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. The edges of $H_1 \hat{\times} H_2$ and $H_1 \check{\times} H_2$ corresponding to the edges of degree 2 in H_1 and H_2 are same. Hence as in the case of minimal rank preserving direct product, $H_1 \hat{\times} H_2$ need not be Hausdorff if both the graphs contains edges of degree 2 (See Figure 1) and a similar result of Theorem 2.6 also holds in the case of maximal rank preserving direct product.

Theorem 3.5. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. Then the maximal rank preserving direct product $H_1 \hat{\times} H_2$ of H_1 and H_2 is Hausdorff provided degree of each vertex in any edge of degree 2 of the hypergraph H_1(or H_2) is different from 1.*

Let H_1 and H_2 be two hypergraphs, if all the edges of both H_1 and H_2 are loops, then all the edges of $H_1 \hat{\times} H_2$ are loops. As a consequence we have the following proposition.

Proposition 3.6. *Let H_1 and H_2 be two hypergraphs. If all the edges of both of them are loops, then the maximal rank preserving direct product $H_1 \hat{\times} H_2$ of H_1 and H_2 is Hausdorff.*

Definition 3.7. The *Strong product* [5] $H_1 \widehat{\boxtimes} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \widehat{\boxtimes} H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $v_1 = v_2 = \ldots = v_r \in V_2$, or
2. $\{v_1, v_2, \ldots, v_n\}$ is a subset of an edge of H_2 and $u_1 = u_2 = \ldots = u_n \in V_1$, or
3. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and there is an edge $f \in \mathcal{E}_2$ of H_2 such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or
4. $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2 and there is an edge $f \in \mathcal{E}_1$ of H_1 such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of f, and $f \subseteq \{u_1, u_2, \ldots, u_r\}$.

Remark 3.8. $E(H_1 \widehat{\boxtimes} H_2) = E(H_1 \square H_1) \cup E(H_1 \hat{\times} H_2)$. Thus it is immediate that if H_1 and H_2 are two Hausdorff hypergraphs then their strong product is Hausdorff.

4 Non-rank Preserving Direct Product

Definition 4.1. [5] The *Non-rank preserving direct product* $H_1 \tilde{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and edge set $\big\{\{(u, v)\} \cup ((e - \{u\}) \times (f - \{v\}))/u \in e \in \mathcal{E}_1, v \in f \in \mathcal{E}_2\big\}$.

Remark 4.2. If H_1 is a hypergraph with all of its edges are loops then for any hypergraph H_2, the edges of $H_1 \tilde{\times} H_2$ are loops. Hence it is Haudorff.

Theorem 4.3. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Suppose $|e| \neq 2$ for any $e \in \mathcal{E}_1$ and H_2 is 2-uniform, then $H_1 \tilde{\times} H_2$ is Hausdorff.*

Proof. Consider two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1 \tilde{\times} H_2$

Case 1. $u_1 = u_2, v_1 \neq v_2$

Let e be an edge of H_1 with $|e| = n$, such that $u_1 \in e$. If e is the loop $\{u_1\}$, then $\{(u_1, v_1)\}$ and $\{(u_2, v_2)\}$ are nonadjacent edges of $H_1 \tilde{\times} H_2$. Otherwise let $e = \{u_1, u_3, u_4, \ldots u_{n+1}\}$.

Subcase 1. $f = \{v_1, v_2\}$ is an edge of H_2.

Now, the edges $e_1 = \{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{n+1}\} \times \{v_2\}\big)$ and $e_2 = \{(u_1, v_2)\} \cup \big(\{u_3, u_4, \ldots, u_{n+1}\} \times \{v_1\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Subcase 2. There exists no edge of H_2 containing both v_1 and v_2.

Let f_1 be an edge of H_2 such that $v_1 \in f_1$ and f_2 be an edge of H_2 such that $v_2 \in f_2$. Let $f_1 = \{v_1, v\}$ and $f_2 = \{v_2, w\}$. Then the edges $e_1 = \{(u_4, v)\} \cup \big(\{u_1, u_3, u_5, u_6 \ldots, u_{n+1}\} \times \{v_1\}\big)$ and $e_2 = \{(u_3, w)\} \cup \big(\{u_1, u_4, u_5 \ldots, u_{n+1}\} \times \{v_2\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Case 2. $u_1 \neq u_2$, $v_1 \neq v_2$

Subcase 1. There exists an edge e of H_1 with $|e| = n$, containing both u_1 and u_2, where $n \geq 3$.

Let $e = \{u_1, u_2, u_3 \ldots, u_n\}$

Assume that $f = \{v_1, v_2\}$ is an edge of H_2. Then the edges $e_1 = \{(u_n, v_2)\} \cup \big(\{u_1, u_2, \ldots, u_{n-1}\} \times \{v_1\}\big)$ and $e_2 = \{(u_n, v_1)\} \cup \big(\{u_1, u_2, \ldots, u_{n-1}\} \times \{v_2\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1, (u_2, v_2) \in e_2$.

Suppose there exists no edge of H_2 containing both v_1 and v_2. Let f_1 be an edge of H_2 such that $v_1 \in f_1$ and f_2 be an edge of H_2 such that $v_2 \in f_2$. Suppose $f_1 = \{v_1, v\}$ and $f_2 = \{v_2, w\}$. Then the edges $e_1 = \{(u_1, v_1)\} \cup \big(\{u_2, u_3, \ldots, u_n\} \times \{v\}\big)$ and $e_2 = \{(u_1, w)\} \cup \big(\{u_2, u_3, \ldots, u_n\} \times \{v_2\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

Subcase 2. There exists no edge of H_1 containing both u_1 and u_2.

Let g_1 be an edge of H_1 with $|g_1| = n$, containing u_1 and g_2 be an edge of H_1 with $|g_2| = m$, containing u_2. Let us suppose that $n \geq m$. If $n = 1$, then there is nothing to prove. So assume $n > 1$. Suppose $|g_1 \cap g_2| = k$, $0 \leq k \leq (m-1)$. Let $g_1 = \{u_1, u_3, u_4, \ldots, u_{n+1}\}$ and $g_2 = \{w_1, w_2, \ldots w_k \ldots, w_m\}$ with $w_1 = u_2$. If $k \geq 1$, let $w_2 = u_3, w_3 = u_4, \ldots w_{k+1} = u_{k+2}$.

Assume that $f = \{v_1, v_2\}$ is an edge of H_2. Note that, the edges $e_1 = \{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{n+1}\} \times \{v_2\}\big)$ and $e_2 = \{(w_1, v_2)\} \cup \big(\{w_2, w_3, \ldots, w_m\} \times \{v_1\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

Suppose there exists no edge of H_2 containing both v_1 and v_2.

Suppose $f_1 = \{v_1, v\}$ and $f_2 = \{v_2, w\}$ are two edges of H_2. Then the edges $e_1 = \{(u_4, v)\} \cup \big(\{u_1, u_3, u_4, \ldots, u_{n+1}\} \times \{v_1\}\big)$ and $e_2 = \{(w_2, w)\} \cup \big(\{w_1, w_3, w_4, \ldots, w_m\} \times \{v_2\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_2, v_2) \in e_2$.

Hence the theorem. □

Theorem 4.4. *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If H_1 is Hausdorff and for any vertex $v \in V_1$, if there exists distinct edges e and f containing v such that $e \cap f = \{v\}$, then $H_1 \tilde{\times} H_2$ is Hausdorff.*

Proof. Consider two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1 \tilde{\times} H_2$.

Case 1. $v_1 = v_2$

In this case $u_1 \neq u_2$. As H_1 is Hausdorff, there exist nonadjacent edges e_1 and e_2 in H such that $u_1 \in e_1$ and $u_2 \in e_2$. Let $|e_1| = n$ and $|e_2| = m$. If $n = 1$ or $m = 1$, then there is nothing to prove. So assume that $n, m \geq 2$. Let $e_1 = \{u_1, u_3, \ldots, u_{n+1}\}$ and $e_2 = \{w_1, w_2, \ldots, w_m\}$ with $w_1 = u_2$. Let g be an edge of H_2 with $|g| = p$, containing v_1. Suppose $g = \{v_1, v_3, v_4 \ldots, v_{p+1}\}$. Such an edge exists by hypothesis. Then the edges $e = \{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{n+1}\} \times \{v_3, v_4, \ldots, v_{p+1}\}\big)$ and $f = \{(w_1, v_1)\} \cup \big(\{w_2, w_3, \ldots, w_m\} \times \{v_3, v_4, \ldots, v_{p+1}\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e$, $(u_2, v_1) \in f$.

Case 2. $v_1 \neq v_2$

Subcase 1. $u_1 = u_2$

By hypothesis there exist edges e and f containing u_1 such that $e \cap f = \{u_1\}$. Let $|e| = n, |f| = m$. Suppose $e = \{x_1, x_2, \ldots, x_{n-1}, u_1\}$, $f = \{y_1, y_2, \ldots, y_{m-1}, u_1\}$.

Suppose there exists an edge g with $|g| = p$ of H_2 containing both v_1 and v_2. Let us suppose that $g = \{v_1, v_2, \ldots, v_p\}$. Now the edges $e_1 = \{(u_1, v_1)\} \cup \big(\{x_1, x_2, \ldots, x_{n-1}\} \times \{v_2, v_3, \ldots, v_p\}\big)$ and

$e_2 = \{(u_1, v_2)\} \cup \big(\{y_1, y_2, \ldots, y_{m-1}\} \times \{v_1, v_3, \ldots, v_p\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Suppose there exist no edge of H_2 containing both v_1 and v_2. Let f_1 be an edge of H_2 with $|f_1| = p$ such that $v_1 \in f_1$ and f_2 be an edge of H_2 with $|f_2| = q$ such that $v_2 \in f_2$. Suppose $f_1 = \{v_1, v_3, v_4 \ldots, v_{p+1}\}$ and $f_2 = \{w_1, w_2, \ldots, w_q\}$ with $w_1 = v_2$. Now the edges $e_1 = \{(u_1, v_1)\} \cup \big(\{x_1, x_2, \ldots, x_{n-1}\} \times \{v_3, v_4, \ldots, v_{p+1}\}\big)$ and $e_2 = \{(u_1, w_1)\} \cup \big(\{y_1, y_2, \ldots, y_{m-1}\} \times \{w_2, w_3, \ldots, w_q\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e_1$, $(u_1, v_2) \in e_2$.

Subcase 2. $u_1 \neq u_2$

As H_1 is Hausdorff, there exist nonadjacent edges e_1 and e_2 such that $u_1 \in e_1$ and $u_2 \in e_2$. Let $|e_1| = n$ and $|e_2| = m$. Suppose $e_1 = \{u_1, u_3, u_4 \ldots, u_{n+1}\}$ and $e_2 = \{w_1, w_2, \ldots, w_m\}$ with $w_1 = u_2$.

Suppose there exists an edge g with $|g| = p$ of H_2 containing both v_1 and v_2. Let us suppose $g = \{v_1, v_2, \ldots, v_p\}$. Then the edges $e = \{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{n+1}\} \times \{v_2, v_3, \ldots, v_p\}\big)$ and $f = \{(w_1, v_2)\} \cup \big(\{w_2, w_3, \ldots, w_m\} \times \{v_1, v_3, \ldots, v_p\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e$, $(u_2, v_2) \in f$.

Suppose there exist no edge of H_2 containing both v_1 and v_2. Let g_1 be an edge of H_2 with $|g_1| = p$ such that $v_1 \in g_1$ and g_2 be an edge of H_2 with $|g_2| = q$ such that $v_2 \in g_2$. Let $g_1 = \{v_1, v_3, v_4 \ldots, v_{p+1}\}$ and $g_2 = \{y_1, y_2, \ldots, y_q\}$ with $y_1 = v_2$. Then the edges $e = \{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{n+1}\} \times \{v_3, v_4, \ldots, v_{p+1}\}\big)$ and $f = \{(w_1, y_1)\} \cup \big(\{w_2, w_3, \ldots, w_m\} \times \{y_2, y_3, \ldots, y_q\}\big)$ of $H_1 \tilde{\times} H_2$ are nonadjacent and $(u_1, v_1) \in e$, $(u_2, v_2) \in f$.

$\square$

5 Conclusion

In this paper we have discussed conditions under which minimal rank, maximal rank , non-rank, preserving direct product of two hypergraphs to be Hausdorff. It is proved that normal product and strong product of any two hypergraphs is always Hausdorff.

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