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Certain Submanifolds of Hsu-structure Manifolds

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Abstract

In this paper, authors have studied Hsu-structure manifolds and submanifolds. Submanifolds immersed in Hsu-structure manifold and $(f, g, A_1, A_2, T_1, T_2)$ - Hsu metric structure manifolds have been studied. Certain interesting results have been obtained.

1. Preliminaries

Let V_n be an n-dimensional differentiable manifold of class C^∞. Suppose V_n is equipped with a tensor field F of type (1, 1) such that

(1.1) $$F^2 = a^r I$$

where 'a' is any real or complex number and r is a positive integer. Suppose further that V_n is equipped with a Riemannian metric G such that

(1.2) $$G(FX, FY) = a^r G(X, Y) .$$

Then V_n in view of (1.1) and (1.2) is said to possess a Hsu-metric structure [1]

Let V_m be an m-dimensional differentiable manifold equipped with a (1, 1) tensor field f, vector fields T_1, T_2 and 1-forms A_1, A_2 such that

(i) $$f^2 = a^r I + A_1 \otimes T_1 + A_2 \otimes T_2 \ ,$$

(ii) $f\,T_1 = f\,T_2 = 0$,

(1.3) (iii) $A_1 \circ f = A_2 \circ f$,

(iv) $A_i\,(T_j) + a^r\,\delta_{ij} = 0$, $i, j = 1, 2$

and

(v) Rank $f = m - 2$.

Suppose further that there exists a Riemannian metric 'g' on V_m such that

(1.4) $g\,(fX, fY) = a^r\,g\,(X, Y) + A_1\,(X)\,A_1\,(Y) + A_2\,(X)\,A_2\,(Y)$.

Obviously

(1.5) $g(X, T_1) = A_1\,(X)$ and $g\,(X, T_2) = A_2\,(X)$.

On such a manifold V_m let for the Riemannian connection D

(1.6) $$(D_Xf)Y = \{A_1(Y) + A_2(Y)\}X - 2\,\{A_1(X)A_1(Y) + A_2(X)A_2(Y)\}\,(\lambda T_1 + \mu T_2) + g(X, Y)\,(\lambda T_1 + \mu T_2)$$

is satisfied. Further let

(1.7) $D_XT_1 = -fX + \lambda T_2$, $D_XT_2 = fX + \mu T_1$.

Then we say that the manifold admits $(f, g, A_1, A_2, T_1, T_2)$ - Hsu metric structure [2].

If V_m is submanifold of V_n $(m<n)$, then V_m is called invariant submanifold of V_n if F leaves invariant the tangent space of V_m at all points. If all X tangents to V_n, FX contains tangential and normal points to V_m both then V_m is non invariant submanifold of V_n.

2. Submanifolds of codimension 2 of the Hsu-metric structure manifold

Let V_n be a Hsu-metric structure manifold with structure (F, G) and V_{n-2} the submanifold of codimension 2 of V_n. So

(2.1) $F^2 = a^r\,I_n$ and $G(FX, FY) = a^r\,G(X, Y)$

If 'b' denotes the immersion $V_{n-2} \to V_n$ and $B = db$, a vector field X tangential to V_{n-2} corresponds to a vector field BX tangent to V_n. Let f be (1, 1) tensor field and 'g' the metric induced on the submanifold and let C_1, C_2 be unit normal fields on the submanifold. Then

$$\text{(2.2)}\quad \begin{aligned} &\text{(i)} && G(BX, BY) = g(X, Y)\\ &\text{(ii)} && G(BX, C_1) = G(BX, C_2) = 0\\ &\text{and}\\ &\text{(iii)} && G(C_i, C_j) = \delta_{ij} \text{ for } i, j = 1, 2. \end{aligned}$$

Since V_{n-2} is submanifold of codimension 2 of V_n, the transformation equations can be written as

$$\text{(2.3)}\quad \begin{aligned} &\text{(i)} && FBX = BfX - A_1(X)C_1 - A_2(X)C_2,\\ &\text{(ii)} && FC_1 = BT_1 + \lambda C_2,\\ &\text{(iii)} && FC_2 = BT_2 + \mu C_1, \end{aligned}$$

where A_1, A_2 are 1-forms, T_1, T_2 vector fields on the submanifold and λ, μ scalars.

Operating (2.3)(i) by F and making use of (2.1) and the equation (2.3) itself, we get

$$a^r BX = Bf^2X - A_1(fX)C_1 - A_2(fX)C_2 - A_1(X)\{BT_1 + \lambda C_2\} - A_2(X)\{BT_2 + \mu C_1\}$$

Comparison of tangential and normal vectors yields

$$\text{(2.4)}\quad \begin{aligned} &\text{(i)} && f^2 = a^r I_{n-2} + A_1 \otimes T_1 + A_2 \otimes T_2,\\ &\text{(ii)} && A_1 \circ f + \mu A_2 = 0,\\ &\text{(iii)} && A_2 \circ f + \lambda A_1 = 0, \end{aligned}$$

Operate (2.3)(ii) by F and making use of (2.1) and (2.3) itself, we get

$$a^r C_1 = BfT_1 - A_1(T_1)C_1 - A_2(T_1)C_2 + \lambda\{BT_2 + \mu C_1\}.$$

As $A_i(T_j) + a^r \delta_{ij} = 0$ so above equation takes the form

$$a^r C_1 = BfT_1 + a^r C_1 + \lambda BT_2 + \lambda\mu C_1.$$

Comparison of tangential and normal vector gives

$$\text{(2.5)}\quad \begin{aligned} &\text{(i)} && fT_1 + \lambda T_2 = 0,\\ &\text{(ii)} && \lambda\mu = 0. \end{aligned}$$

In a similar manner, applying F to (2.3)(iii) and proceeding in a similar manner, we get

$$\text{(2.6)}\quad \text{(i)}\quad fT_2 + \mu T_1 = 0,$$

(ii) $\lambda\mu = 0.$

Now operating (2.4)(i) by f and using (2.5) and (2.6) we obtain

(2.7) $$f^3 - a^r f + \lambda A_1 \otimes T_2 + \mu A_2 \otimes T_1 = 0.$$

Also in view of equations (2.1), (2.2) and (2.3), it follows that

(2.8) $$g(fX, fY) = a^r g(X, Y) + A_1(X)A_1(Y) + A_2(X)A_2(Y).$$

Hence we have the theorem :

Theorem (2.1) The non-invariant submanifold V_{n-2} of codimension 2 of the Hsu-metric structure manifold V_n admits the structure given by equation (2.7) and (2.8).

3. Submanifolds of codimension 2 of (f, g, A_1, A_2, T_1, T_2) - Hsu metric manifold.

Now suppose that the enveloping manifold admits (f, g, A_1, A_2, T_1, T_2) - Hsu metric structure. Hence equations (1.3) and (1.4) are satisfied. Suppose T_1, T_2 are no where tangents to the submanifold. The transformation equations can be written as

(3.1) $$fBX = B\tilde{f}X - \alpha_1(X)T_1 - \alpha_2(X)T_2$$

where α_1, α_2 are 1-forms on the submanifold and $\tilde{f}$ the induced (1, 1) tensor field.

Operating (3.1) by f and using (3.1) itself and (1.3), we get

$$a^r BX + A_1(BX)T_1 + A_2(BX)T_2 = B\tilde{f}^2X - \alpha_1(\tilde{f}X)T_1 - \alpha_2(\tilde{f}X)T_2$$

Comparison of tangential vectors gives

$$\tilde{f}^2(X) = a^r X.$$

Hence we have

Theorem (3.1) The non-invariant submanifold of codimension 2 of (f, g, A_1, A_2, T_1, T_2) - Hsu structure manifold where T_1, T_2 are never tangents to the submanifold admits the Hsu-structure.

Let D be the Riemannian connection on the enveloping manifold. Then

(3.2) (i) $$D_{BX}T_1 = -fBX + \lambda T_2 = -B\tilde{f}(X) + \alpha_1(X)T_1 + \alpha_2(X)T_2 + \lambda T_2$$

(ii) $D_{BX}T_2 = fBX + \mu T_1 = B\tilde{f}(X) - \alpha_1(X)T_1 - \alpha_2(X)T_2 + \mu T_2$

Further the Gauss and Wengaster equations are given by

(3.3) $$D_{BX}BY = B(\overline{D}_X Y) + h_1(X,Y)C_1 + h_2(X,Y)C_2,$$

(3.4)
(i) $D_{BX}C_1 = -BH_1X + W_1(X)C_2,$

(ii) $D_{BX}C_2 = BH_2X + W_2(X)C_1.$

where W_1, W_2 are 1-forms on the submanifold and H_1, H_2 tensor fields of type (1, 1) satisfying

(3.5) $$g(H_1X,Y) = h_1(X,Y), \qquad g(H_2X,Y) = h_2(X,Y)$$

and $\overline{D}$ the Riemannian connection induced on the submanifold. As T_1, T_2 are never tangents to the submanifold, so comparison of (3.2) and (3.4) gives

(3.6) (i) $-B\tilde{f}(X) + \alpha_1(X)T_1 + \alpha_2(X)T_2 + \lambda T_2 = -BH_1X + W_1(X)C_2$

(ii) $B\tilde{f}(X) - \alpha_1(X)T_1 - \alpha_2(X)T_2 + \mu T_1 = BH_2X + W_2(X)C_1$

Comparing tangential and normal vectors, we get

$$H_1 = H_2 = \tilde{f} \text{ and } W_1(X) = +\alpha_2(X) + \lambda, \quad W_2(X) = -\alpha_1(X) + \mu$$

Hence we have

Theorem (3.2) For a submanifold of codimension 2 of a (f, g, A_1, A_2, T_1, T_2) - Hsu metric structure manifold, we have $H_1 = H_2 = \tilde{f}$ and $W_1(X) = +\alpha_2(X) + \lambda$, $W_2(X) = -\alpha_1(X) + \mu$ where $\tilde{f}$ is induced (1, 1) tensor field on the submanifold.

Since A_1, A_2 are 1-forms on the enveloping manifold and α_1, α_2 induced 1-forms on the submanifold, hence $A_1(BX) = \alpha_1(X)$, $A_2(BX) = \alpha_2(X)$

If $\tilde{g}$ be the metric induced on the submanifold

(3.7) $$g(BX,BY) = \tilde{g}(X,Y)$$

Now

$$g(fBX,fBY) = g(B\tilde{f}(X) - \alpha_1(X)T_1 - \alpha_2(X)T_2, B\tilde{f}(Y) - \alpha_1(Y)T_1 - \alpha_2(X)T_2)$$

or

(3.8) $$g(fBX,fBY) = g(B\tilde{f}X, B\tilde{f}Y) + \alpha_1(X)\alpha_1(Y) + \alpha_2(X)\alpha_2(Y).$$

In view of the equations (2.8) and (3.7), we get

$$a^r g(BX, BY) + A_1(BX)A_1(BY) + A_2(BX)A_2(BY)$$
$$= g(B\tilde{f}X, B\tilde{f}Y) + \alpha_1(X)\alpha_1(Y) + \alpha_2(X)\alpha_2(Y)$$

or

(3.9) $$\tilde{g}(\tilde{f}X, \tilde{f}Y) = a^r \tilde{g}(X, Y)$$

It has already been shown that

$$\tilde{f}^2(X) = a^r X$$

Thus we have

Theorem (3.3) A non-invariant submanifold of codimension 2 of a (f, g, A_1, A_2, T_1, T_2) - Hsu metric structure manifold admits the Hsu metric structure.

References

1. Goldberg, S. : Invariant submanifolds of codimension two of almost contact manifold
   Annali Scuola Norm. Sup. Di Pisa 25(3), 1971, pp. 377 – 388.
2. Kanemak, K. : Non-invariant submanifolds of codimension 2 in an almost contact manifold
   Tensor, N.S. 23(1972), pp. 233 – 239.
3. Mijazawa, T. : An invariant hypersurface immersed in an almost Riemannian manifold.
   TRU Mathematics 14(1978), pp. 27 – 30.
4. Mishra, R.S. : Structures on a differentiable manifold and their applications.
   Chandrama Prakashan, 50A Balrampur House, Allahabad, India.
5. Nivas, Ram and Agnihotri, Rupali : On a certain class of submanifolds (To appear).
6. Das, LJSK : Invariant submanifolds of the manifolds with (K,$(-)^{K+1}$) – structure.
   Tensor, N.S., vol. 64, 2003.
7. Ram Nivas, LJSK Das and V.N. Pathak : On $f_{\lambda,\mu}(3, 1)$ – structure manifold.
   Math. Science Research Journal, USA 6(12), 2002, pp. 582 – 588.